The Equation: $y'' + p(x) y' + q(x) y = 0$

- Review of Power Series
- Regular Point Equations
- Solutions Using Power Series
- Examples of the Power Series Method
Review of Power Series

Definition
A function $y$ is analytic at $x_0$ iff there exists $\rho > 0$ such that

$$ y(x) = \sum_{n=0}^{\infty} y_n (x - x_0)^n, \quad \text{for } x \text{ such that } 0 \leq |x - x_0| < \rho. $$

Example (Analytic Functions)

$\cdot$ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$. Here $x_0 = 0$ and $|x| < 1$.

$\cdot$ $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$. Here $x_0 = 0$ and $x \in \mathbb{R}$.

$\cdot$ The Taylor series of $y : \mathbb{R} \to \mathbb{R}$ centered at $x_0 \in \mathbb{R}$ is

$$ y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!} (x - x_0)^n = y(x_0) + y'(x_0)(x - x_0) + \cdots. $$

Review of Power Series

Example (Taylor Series)
Find the Taylor series of $y(x) = \sin(x)$ centered at $x_0 = 0$.

Solution: $y(x) = \sin(x), \ y(0) = 0. \ y'(x) = \cos(x), \ y'(0) = 1.$

$y''(x) = -\sin(x), \ y''(0) = 0. \ y'''(x) = -\cos(x), \ y'''(0) = -1.$

$$ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \Rightarrow \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1}. $$

Remark: The Taylor series of $y(x) = \cos(x)$ centered at $x_0 = 0$ is

$$ \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}. $$
Review of Power Series

Remark: The power series of a function may not be defined on the whole domain of the function.

Example (Convergence of Power Series)
The function \( y(x) = \frac{1}{1-x} \) is defined for \( x \in \mathbb{R} - \{1\} \).

\[
y(x) = \sum_{n=0}^{\infty} x^n
\]

The power series
\[
y(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]
converges only for \( |x| < 1 \).

\[\triangle\]

Review of Power Series

Definition
The radius of convergence of a power series
\[
y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n
\]
is the number \( \rho \geq 0 \) that satisfies both
(a) the series converges absolutely for \( |x - x_0| < \rho \);
(b) the series diverges for \( |x - x_0| > \rho \).
Definition
The power series \( y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \) converges absolutely at \( x \) iff the series \( \sum_{n=0}^{\infty} |a_n| |x - x_0|^n \) converges.

Example (Alternating Harmonic Series)
The series \( s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges, but it does not converge absolutely, since \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

Example (Radius of Convergence)

1. \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) has radius of convergence \( \rho = 1 \).
2. \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) has radius of convergence \( \rho = \infty \).
3. \( \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1} \) has radius \( \rho = \infty \).
4. \( \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \) has radius of convergence \( \rho = \infty \).
Review of Power Series

Remark: One way to compute the radius of convergence of a power series is with the ratio test.

Theorem (Ratio Test)

Given the power series \( y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \), introduce the number \( L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \). Then, the following statements hold:

1. The power series converges in the domain \( |x - x_0| L < 1 \).
2. The power series diverges in the domain \( |x - x_0| L > 1 \).
3. The power series may or may not converge at \( |x - x_0| L = 1 \).

Therefore, if \( L \neq 0 \), then \( \rho = \frac{1}{L} \) is the series radius of convergence; if \( L = 0 \), then the radius of convergence is \( \rho = \infty \).

Review of Power Series

Remarks: On summation indices:

\[
y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots
\]

\[
y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k = \sum_{m=-3}^{\infty} a_{m+3} (x - x_0)^{m+3}.
\]

\[
y'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} = a_1 + 2a_2(x - x_0) + \cdots
\]

\[
y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{m=0}^{\infty} (m + 1) a_{m+1} (x - x_0)^m
\]

where \( m = n - 1 \), that is, \( n = m + 1 \).
The Equation: \( y'' + p(x) y' + q(x) y = 0 \)

- Review of Power Series
- **Regular Point Equations**
- Solutions Using Power Series
- Examples of the Power Series Method

### Regular Point Equations

**Definition**
A point \( x_0 \in \mathbb{R} \) is a regular point of the equation
\[
y'' + p(x) y' + q(x) y = 0,
\]
iff \( p, q \) are analytic functions at \( x_0 \). Otherwise \( x_0 \) is a singular point.

**Example**
Find all the singular points of the equation \( x y'' + y' + x^2 y = 0 \).

**Solution:** Rewrite the equation in the standard form
\[
y'' + \frac{1}{x} y' + x y = 0 \quad \Rightarrow \quad p(x) = \frac{1}{x}, \quad q(x) = x.
\]

Then \( q \) is analytic for all \( x \in \mathbb{R} \), but \( p \) is analytic for \( x \in \mathbb{R} - \{0\} \).
So \( x_0 = 0 \) is the only singular point of the equation. \( \triangleleft \)
Regular Point Equations

Recall:
A point \( x_0 \in \mathbb{R} \) is a regular point of the equation
\[ y'' + p(x) y' + q(x) y = 0, \]
iff \( p, q \) are analytic functions at \( x_0 \). Otherwise \( x_0 \) is a singular point.

Theorem (Solutions Centered at Regular Points)
If \( p, q \) are analytic on an open interval \((x_0 - \rho, x_0 + \rho)\), then
\[ y'' + p(x) y' + q(x) y = 0 \]
has two independent solutions \( y_1, y_2 \), which are analytic on the same interval.

Power Series Solutions Near Regular Points (§ 3.1)

The Equation: \( y'' + p(x) y' + q(x) y = 0 \)

- Review of Power Series
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Solutions Using Power Series

Summary for regular points:

(1) Propose a power series representation of the solution centered at $x_0$, given by

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n; \quad (1)$$

(2) Introduce Eq. (1) into the differential equation

$$y'' + p(x)y' + q(x)y = 0.$$  

(3) Find a recurrence relation among the coefficients $a_n$;

(4) Solve the recurrence relation in terms of free coefficients;

(5) If possible, add up the resulting power series for the solution $y$.

Power Series Solutions Near Regular Points ($\S$ 3.1)

The Equation: $y'' + p(x)y' + q(x)y = 0$

- Review of Power Series
- Regular Point Equations
- Solutions Using Power Series
- Examples of the Power Series Method
Examples of the Power Series Method

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[
y' + cy = 0, \quad c \in \mathbb{R}.
\]

Solution: Recall: The solution is \( y(x) = a_0 e^{-cx} \).

We now use the power series method. We propose a power series centered at \( x_0 = 0 \):
\[
y(x) = \sum_{n=0}^{\infty} a_n x^n \implies y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}.
\]

Change the summation index: \( m = n - 1 \), so \( n = m + 1 \).
\[
y'(x) = \sum_{m=0}^{\infty} (m + 1) a_{m+1} x^m = \sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n.
\]

Introduce \( y \) and \( y' \) into the differential equation,
\[
\sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n + \sum_{n=0}^{\infty} c a_n x^n = 0
\]
\[
\sum_{n=0}^{\infty} \left[(n + 1) a_{n+1} + c a_n\right] x^n = 0
\]

The recurrence relation is \( (n + 1) a_{n+1} + c a_n = 0 \) for all \( n \geq 0 \).
Examples of the Power Series Method

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + cy = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recurrence relation: $(n + 1)a_{n+1} + ca_n = 0, \quad n \geq 0$.

Equivalently: $a_{n+1} = -\frac{c}{n+1}a_n$. That is,

- $n = 0, \quad a_1 = -ca_0 \Rightarrow a_1 = -ca_0$,
- $n = 1, \quad 2a_2 = -ca_1 \Rightarrow a_2 = \frac{c^2}{2!}a_0$,
- $n = 2, \quad 3a_3 = -ca_2 \Rightarrow a_3 = -\frac{c^3}{3!}a_0$,
- $n = 3, \quad 4a_4 = -ca_3 \Rightarrow a_4 = \frac{c^4}{4!}a_0$.

Examples of the Power Series Method

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + cy = 0, \quad c \in \mathbb{R}.$$ 

Solution: Solved recurrence relation: $a_n = \frac{(-c)^n}{n!}a_0$.

The solution $y$ of the differential equation is given by

$$y(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!}a_0 x^n \quad \Rightarrow \quad y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-cx)^n}{n!}.$$ 

If we recall the power series $e^{ax} = \sum_{n=0}^{\infty} \frac{(ax)^n}{n!}$,

then, we conclude that the solution is $y(x) = a_0 e^{-cx}$.  \hspace{1cm} \triangledown
Examples of the power series method.

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[ y'' + y = 0. \]

Solution: Recall: The characteristic polynomial is \( r^2 + 1 = 0 \), hence the general solution is \( y(x) = a_0 \cos(x) + a_1 \sin(x) \).

We re-obtain this solution using the power series method:
\[
y = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1)a_{m+1} x^m,
\]
where \( m = n - 1 \), so \( n = m + 1 \);
\[
y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m.
\]
where \( m = n - 2 \), so \( n = m + 2 \).

Examples of the Power Series Method

Example
Find a power series solution \( y(x) \) around the point \( x_0 = 0 \) of the equation
\[ y'' + y = 0. \]

Solution: Introduce \( y \) and \( y'' \) into the differential equation,
\[
\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0
\]
\[
\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n] x^n = 0.
\]
The recurrence relation is \( (n + 2)(n + 1)a_{n+2} + a_n = 0, \quad n \geq 0 \).
Equivalently: \( (n + 2)(n + 1) a_{n+2} = -a_n \).
Examples of the Power Series Method

Example
Find a power series solution \(y(x)\) around the point \(x_0 = 0\) of the equation
\[ y'' + y = 0. \]

Solution: Recall: \((n + 2)(n + 1) a_{n+2} = -a_n, \quad n \geq 0.\)

For \(n\) even: \(n = 0, \quad (2)(1)a_2 = -a_0 \quad \Rightarrow \quad a_2 = -\frac{1}{2!} a_0,\)
\[ n = 2, \quad (4)(3)a_4 = -a_2 \quad \Rightarrow \quad a_4 = \frac{1}{4!} a_0, \]
\[ n = 4, \quad (6)(5)a_6 = -a_4 \quad \Rightarrow \quad a_6 = -\frac{1}{6!} a_0. \]

We obtain: \(a_{2k} = \frac{(-1)^k}{(2k)!} a_0, \quad \text{for} \quad k \geq 0.\)

Examples of the Power Series Method

Example
Find a power series solution \(y(x)\) around the point \(x_0 = 0\) of the equation
\[ y'' + y = 0. \]

Solution: Recall: \(a_{2k} = \frac{(-1)^k}{(2k)!} a_0 \) and \((n + 2)(n + 1) a_{n+2} = -a_n.\)

For \(n\) odd: \(n = 1, \quad (3)(2)a_3 = -a_1 \quad \Rightarrow \quad a_3 = -\frac{1}{3!} a_1,\)
\[ n = 3, \quad (5)(4)a_5 = -a_3 \quad \Rightarrow \quad a_5 = \frac{1}{5!} a_1, \]
\[ n = 5, \quad (7)(6)a_7 = -a_5 \quad \Rightarrow \quad a_7 = -\frac{1}{7!} a_1. \]

We obtain \(a_{2k+1} = \frac{(-1)^k}{(2k + 1)!} a_1, \quad \text{for} \quad k \geq 0.\)
Examples of the Power Series Method

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$

Solution: Recall: $a_{2k} = \frac{(-1)^k}{(2k)!} a_0$ and $a_{2k+1} = \frac{(-1)^k}{(2k + 1)!} a_1$.

Therefore, the solution of the differential equation is given by

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1}.$$

One can check that these are precisely the power series representations of the cosine and sine functions, respectively,

$$y(x) = a_0 \cos(x) + a_1 \sin(x).$$

Examples of the Power Series Method

Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x y = 0.$$

Solution: We propose: $y = \sum_{n=0}^{\infty} a_n (x - 2)^n$.

It is convenient to rewrite the function $xy$ as follows,

$$xy = \sum_{n=0}^{\infty} a_n x (x - 2)^n = \sum_{n=0}^{\infty} a_n [(x - 2) + 2] (x - 2)^n,$$

$$xy = \sum_{n=0}^{\infty} a_n (x - 2)^{n+1} + \sum_{n=0}^{\infty} 2a_n (x - 2)^n.$$

We relabel the first sum: $\sum_{n=0}^{\infty} a_n (x - 2)^{n+1} = \sum_{n=1}^{\infty} a_{(n-1)} (x - 2)^n$. 
Examples of the Power Series Method

Example
Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x y = 0.$$ 

Solution: We relabel the $y''$, 

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^n.$$ 

Introduce $y''$ and $xy$ in the differential equation

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^n - \sum_{n=0}^{\infty} 2a_n(x-2)^n - \sum_{n=1}^{\infty} a_{n-1}(x-2)^n = 0$$

$$(2)(1)a_2 - 2a_0 + \sum_{n=1}^{\infty} \left[ (n+2)(n+1)a_{n+2} - 2a_n - a_{n-1} \right] (x-2)^n = 0.$$ 

The recurrence relation for the coefficients $a_n$ is:

$$a_2 - a_0 = 0, \quad (n+2)(n+1)a_{n+2} - 2a_n - a_{n-1} = 0, \quad n \geq 1.$$ 

We solve this recurrence relation for the first four coefficients,

$$n = 0 \quad a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = a_0,$$

$$n = 1 \quad (3)(2)a_3 - 2a_1 - a_0 = 0 \quad \Rightarrow \quad a_3 = \frac{a_0}{6} + \frac{a_1}{3},$$

$$n = 2 \quad (4)(3)a_4 - 2a_2 - a_1 = 0 \quad \Rightarrow \quad a_4 = \frac{a_0}{6} + \frac{a_1}{12}.$$ 

$$y \simeq a_0 + a_1(x-2) + a_0(x-2)^2 + \left( \frac{a_0}{6} + \frac{a_1}{3} \right) (x-2)^3 + \left( \frac{a_0}{6} + \frac{a_1}{12} \right) (x-2)^4.$$ 

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Examples of the Power Series Method

Example
Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x y = 0.$$ 

Solution: The recurrence relation is:

$$a_2 - a_0 = 0, \quad (n+2)(n+1)a_{n+2} - 2a_n - a_{n-1} = 0, \quad n \geq 1.$$ 

We solve this recurrence relation for the first four coefficients,

$$n = 0 \quad a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = a_0,$$

$$n = 1 \quad (3)(2)a_3 - 2a_1 - a_0 = 0 \quad \Rightarrow \quad a_3 = \frac{a_0}{6} + \frac{a_1}{3},$$

$$n = 2 \quad (4)(3)a_4 - 2a_2 - a_1 = 0 \quad \Rightarrow \quad a_4 = \frac{a_0}{6} + \frac{a_1}{12}.$$ 

$$y \simeq a_0 + a_1(x-2) + a_0(x-2)^2 + \left( \frac{a_0}{6} + \frac{a_1}{3} \right) (x-2)^3 + \left( \frac{a_0}{6} + \frac{a_1}{12} \right) (x-2)^4.$$
Examples of the Power Series Method

Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - xy = 0.$$  

Solution: The first terms in the power series expression for $y$ are

$$y \approx a_0 + a_1(x - 2) + a_0(x - 2)^2 + \left( \frac{a_0}{6} + \frac{a_1}{3} \right)(x - 2)^3 + \left( \frac{a_0}{6} + \frac{a_1}{12} \right)(x - 2)^4.$$  

$$y = a_0 \left[ 1 + (x - 2)^2 + \frac{1}{6}(x - 2)^3 + \frac{1}{6}(x - 2)^4 + \cdots \right]$$

$$+ a_1 \left[ (x - 2) + \frac{1}{3}(x - 2)^3 + \frac{1}{12}(x - 2)^4 + \cdots \right]$$

So the first three terms on each fundamental solution are given by

$$y_1 \approx 1 + (x - 2)^2 + \frac{1}{6}(x - 2)^3, \quad y_2 \approx (x - 2) + \frac{1}{3}(x - 2)^3 + \frac{1}{12}(x - 2)^4.$$