Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: Eigenvalue-eigenfunction problem.
Two-point Boundary Value Problem.

Definition
A two-point BVP is the following: Given functions $p$, $q$, $g$, and constants $x_1 < x_2$, $y_1, y_2$, $b_1, b_2$, $\tilde{b}_1, \tilde{b}_2$, find a function $y$ solution of the differential equation

$$y'' + p(x) y' + q(x) y = g(x),$$

together with the extra, boundary conditions,

$$b_1 y(x_1) + b_2 y'(x_1) = y_1,$$
$$\tilde{b}_1 y(x_2) + \tilde{b}_2 y'(x_2) = y_2.$$
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Remarks:
- Both $y$ and $y'$ might appear in the boundary condition, evaluated at the same point.
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together with the extra, \textit{boundary conditions},

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Remarks:

\begin{itemize}
  \item Both \( y \) and \( y' \) might appear in the boundary condition, evaluated at the same point.
  \item In this notes we only study the case of constant coefficients, \( y'' + a_1 y' + a_0 y = g(x) \).
\end{itemize}
Two-point Boundary Value Problem.

**Example**

Examples of BVP.
Two-point Boundary Value Problem.

Example

Examples of BVP. Assume $x_1 \neq x_2$.

(1) Find $y$ solution of

$$y'' + a_1 y' + a_0 y = g(x), \quad y(x_1) = y_1, \quad y(x_2) = y_2.$$
Two-point Boundary Value Problem.

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Two-point Boundary Value Problem.

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Example from physics.

**Problem:** The equilibrium (time independent) temperature of a bar of length $L$ with insulated horizontal sides and the bar vertical extremes kept at fixed temperatures $T_0$, $T_L$ is the solution of the BVP:

$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$
Example from physics.

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- Example from physics.
- **Comparison: IVP vs BVP.**
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Comparison: IVP vs BVP.

Review: IVP:
Find the function values $y(t)$ solutions of the differential equation

$$y'' + a_1 y' + a_0 y = g(t),$$

together with the initial conditions

$$y(t_0) = y_1, \quad y'(t_0) = y_2.$$
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Remark: In physics:
- $y(t)$: Position at time $t$. 
Comparison: IVP vs BVP.

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Find the function values $y(t)$ solutions of the differential equation

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Remark: In physics:

- $y(t)$: Position at time $t$.
- Initial conditions: Position and velocity at the initial time $t_0$. 
Comparison: IVP vs BVP.

Review: BVP:
Find the function values \( y(x) \) solutions of the differential equation
\[
y'' + a_1 y' + a_0 y = g(x),
\]
together with the initial conditions
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y(x_1) = y_1, \quad y(x_2) = y_2.
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**Remark:** In physics:
- \( y(x) \): A physical quantity (temperature) at a position \( x \).
Comparison: IVP vs BVP.

Review: BVP:
Find the function values $y(x)$ solutions of the differential equation

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Remark: In physics:

- $y(x)$: A physical quantity (temperature) at a position $x$.
- **Boundary conditions**: Conditions at the boundary of the object under study, where $x_1 \neq x_2$. 
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Existence, uniqueness of solutions to BVP.

Review: The initial value problem.

Theorem (IVP)
Consider the homogeneous initial value problem:

\[ y'' + a_1 y' + a_0 y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1, \]

and let \( r_{\pm} \) be the roots of the characteristic polynomial

\[ p(r) = r^2 + a_1 r + a_0. \]

If \( r_+ \neq r_- \), real or complex, then for every choice of \( y_0, y_1 \), there exists a unique solution \( y \) to the initial value problem above.
Existence, uniqueness of solutions to BVP.

**Review:** The initial value problem.

**Theorem (IVP)**

*Consider the homogeneous initial value problem:*

\[
y'' + a_1 y' + a_0 y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1,
\]

*and let \( r_\pm \) be the roots of the characteristic polynomial*

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p(r) = r^2 + a_1 r + a_0.
\]

*If \( r_+ \neq r_- \), real or complex, then for every choice of \( y_0, y_1 \), there exists a unique solution \( y \) to the initial value problem above.*

**Summary:** The IVP above always has a unique solution, no matter what \( y_0 \) and \( y_1 \) we choose.
Existence, uniqueness of solutions to BVP.

Theorem (BVP)
Consider the homogeneous boundary value problem:

\[ y'' + a_1 y' + a_0 y = 0, \quad y(0) = y_0, \quad y(L) = y_1, \]

and let \( r_\pm \) be the roots of the characteristic polynomial

\[ p(r) = r^2 + a_1 r + a_0. \]

(A) If \( r_+ \neq r_- \), real, then for every choice of \( L \neq 0 \) and \( y_0, y_1 \), there exists a unique solution \( y \) to the BVP above.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \beta \neq 0 \), and \( \alpha, \beta \in \mathbb{R} \), then the solutions to the BVP above belong to one of these possibilities:

1. There exists a unique solution.
2. There exists no solution.
3. There exist infinitely many solutions.
Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. 

The general solution is

$$y(t) = c_1 e^{r_- t} + c_2 e^{r_+ t},$$

where $c_1, c_2 \in \mathbb{R}$. The initial conditions determine $c_1$ and $c_2$ as follows:

$$y(0) = y(t_0) = c_1 e^{r_- t_0} + c_2 e^{r_+ t_0},$$

$$y'(0) = y'(t_0) = c_1 r_- e^{r_- t_0} + c_2 r_+ e^{r_+ t_0}.$$ 

Using matrix notation,

$$\begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}.$$ 

The linear system above has a unique solution $c_1$ and $c_2$ for every constants $y(0)$ and $y'(0)$ iff

$$\det(Z) \neq 0,$$

where

$$Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix}.$$
Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

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The initial conditions determine $c_1$ and $c_2$. 

Using matrix notation,

$$
\begin{bmatrix}
  e^{r_- t_0} & e^{r_+ t_0} \\
  r_- e^{r_- t_0} & r_+ e^{r_+ t_0}
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix}
= 
\begin{bmatrix}
  y_0 \\
  y_1
\end{bmatrix}.
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The linear system above has a unique solution $c_1$ and $c_2$ for every constants $y_0$ and $y_1$ if

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$$y_0 = y(t_0)$$
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\[
y_1 = y'(t_0)
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Existence, uniqueness of solutions to BVP.

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The linear system above has a unique solution $c_1$ and $c_2$ for every constants $y_0$ and $y_1$ iff

$$\det(Z) \neq 0,$$ 

where

$$Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix}.$$
Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

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Using matrix notation,

$$\begin{bmatrix} e^{r^- t_0} & e^{r^+ t_0} \\
 -r_- e^{r^- t_0} & r_+ e^{r^+ t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$ 

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$\Rightarrow$ $\text{det}(Z) \neq 0$. 

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Existence, uniqueness of solutions to BVP.

Proof of IVP:

Recall: \[ Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \implies Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]
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Recall: \[ Z = \begin{bmatrix} e^{r_+ t_0} & e^{r_- t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows

\[ \det(Z) = (r_+ - r_-) e^{(r_+ + r_-) t_0} \]
Existence, uniqueness of solutions to BVP.

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\[
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Since \( r_+ \neq r_- \), the matrix \( Z \) is invertible
Existence, uniqueness of solutions to BVP.

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Since \( r_+ \neq r_- \), the matrix \( Z \) is invertible and so

\[ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Z^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]
Existence, uniqueness of solutions to BVP.

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Recall: \[ Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

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Since \( r_+ \neq r_- \), the matrix \( Z \) is invertible and so
\[
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Z^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.
\]

We conclude that for every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the IVP above has a unique solution. \( \square \)
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

\[ y(x) = c_1 e^{r_+ x} + c_2 e^{r_- x}, \]
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}. \]
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Proof of BVP: The general solution is

\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}. \]

The boundary conditions determine \( c_1 \) and \( c_2 \)
Existence, uniqueness of solutions to BVP.

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The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:

\[ y_0 = y(0) \]
Existence, uniqueness of solutions to BVP.

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\[ y_0 = y(0) = c_1 + c_2. \]
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\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}. \]

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\( y_1 = y(L) \)
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

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The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:

\[ y_0 = y(0) = c_1 + c_2. \]

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Existence, uniqueness of solutions to BVP.

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The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:

\[ y_0 = y(0) = c_1 + c_2. \]

\[ y_1 = y(L) = c_1 e^{r_- L} + c_2 e^{r_+ L} \]

Using matrix notation,

\[
\begin{bmatrix}
1 & 1 \\
1 & e^{r_+ L}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= 
\begin{bmatrix}
y_0 \\
y_1
\end{bmatrix}.
\]
Existence, uniqueness of solutions to BVP.

**Proof of BVP:** The general solution is

\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}. \]

The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:

\[ y_0 = y(0) = c_1 + c_2. \]

\[ y_1 = y(L) = c_1 e^{r_- L} + c_2 e^{r_+ L} \]

Using matrix notation,

\[
\begin{bmatrix}
1 & 1 \\
e^{r_- L} & e^{r_+ L}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
y_0 \\
y_1
\end{bmatrix}.
\]

The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff

\[ \text{det}(Z) \neq 0, \]

where \( Z = \begin{bmatrix}
1 & 1 \\
e^{r_- L} & e^{r_+ L}
\end{bmatrix} \Rightarrow Z \begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} = \begin{bmatrix}
y_0 \\
y_1
\end{bmatrix}. \]
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}. \]

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\[ y_0 = y(0) = c_1 + c_2. \]

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Using matrix notation,

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\begin{bmatrix}
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\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
y_0 \\
y_1
\end{bmatrix}.
\]

The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff the \( \det(Z) \neq 0 \),
Existence, uniqueness of solutions to BVP.

**Proof of BVP:** The general solution is

\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}. \]

The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:

\[ y_0 = y(0) = c_1 + c_2. \]
\[ y_1 = y(L) = c_1 e^{r_- L} + c_2 e^{r_+ L} \]

Using matrix notation,

\[
\begin{bmatrix}
1 & 1 \\
e^{r_- L} & e^{r_+ L}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
y_0 \\
y_1
\end{bmatrix}.
\]

The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff the \( \det(Z) \neq 0 \), where

\[
Z = \begin{bmatrix}
1 & 1 \\
e^{r_- L} & e^{r_+ L}
\end{bmatrix}
\]
Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

\[ y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}. \]

The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:

\[
    \begin{align*}
    y_0 &= y(0) = c_1 + c_2. \\
    y_1 &= y(L) = c_1 e^{r_- L} + c_2 e^{r_+ L}
    \end{align*}
\]

Using matrix notation,

\[
    \begin{bmatrix}
    1 & 1 \\
    e^{r_- L} & e^{r_+ L}
    \end{bmatrix}
    \begin{bmatrix}
    c_1 \\
    c_2
    \end{bmatrix}
    =
    \begin{bmatrix}
    y_0 \\
    y_1
    \end{bmatrix}.
\]

The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff the \( \det(Z) \neq 0 \), where

\[
    Z = \begin{bmatrix}
    1 & 1 \\
    e^{r_- L} & e^{r_+ L}
    \end{bmatrix}
\Rightarrow
    Z \begin{bmatrix}
    c_1 \\
    c_2
    \end{bmatrix}
    =
    \begin{bmatrix}
    y_0 \\
    y_1
    \end{bmatrix}.
\]
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \[ Z = \begin{bmatrix} 1 & 1 \\ e^{r-L} & e^{r+L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall:  \[ Z = \begin{bmatrix} 1 & 1 \\ e^{r^-L} & e^{r^+L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows

\[ \det(Z) = e^{r^+L} - e^{r^-L} \]
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \[ Z = \begin{bmatrix} 1 & 1 \\ e^{r_-L} & e^{r_+L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows

\[
\det(Z) = e^{r_+L} - e^{r_-L} \neq 0 \iff e^{r_+L} \neq e^{r_-L}. 
\]
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \( Z = \begin{bmatrix} 1 & 1 \\ e^{-r-L} & e^{r+L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \).

A simple calculation shows
\[
\det(Z) = e^{r+L} - e^{r-L} \neq 0 \iff e^{r+L} \neq e^{r-L}.
\]

(A) If \( r_+ \neq r_- \) and real-valued,

(1) If \( \beta L \neq n\pi \), then BVP has a unique solution.

(2) If \( \beta L = n\pi \) then BVP either has no solutions or it has infinitely many solutions.
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \( Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \).

A simple calculation shows

\[
\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}.
\]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: 

\[ Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows

\[ \det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}. \]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \( Z = \begin{bmatrix} 1 & 1 \\ e^{-L} & e^{+L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \)

A simple calculation shows

\[
\det(Z) = e^{+L} - e^{-L} \neq 0 \iff e^{+L} \neq e^{-L}.
\]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0. \)

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \),
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \[ Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \] \[ \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows
\[ \det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}. \]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0. \)

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \), then
\[ \det(Z) = e^{\alpha L}(e^{i\beta L} - e^{-i\beta L}) \]
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \[ Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows
\[
\text{det}(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}.
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(A) If \( r_+ \neq r_- \) and real-valued, then \( \text{det}(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \), then
\[
\text{det}(Z) = e^{\alpha L}(e^{i\beta L} - e^{-i\beta L}) \Rightarrow \text{det}(Z) = 2i e^{\alpha L} \sin(\beta L).
\]
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \( Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \).

A simple calculation shows

\[
\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}.
\]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \), then

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\det(Z) = e^{\alpha L}(e^{i\beta L} - e^{-i\beta L}) \Rightarrow \det(Z) = 2i e^{\alpha L} \sin(\beta L).
\]

Since \( \det(Z) = 0 \) iff \( \beta L = n\pi \), with \( n \) integer,
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \( Z = \begin{bmatrix} 1 & 1 \\ e^{r_-} L & e^{r_+} L \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \).

A simple calculation shows
\[
\det(Z) = e^{r_+} L - e^{r_-} L \neq 0 \iff e^{r_+} L \neq e^{r_-} L.
\]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \), then
\[
\det(Z) = e^{\alpha L} (e^{i\beta L} - e^{-i\beta L}) \Rightarrow \det(Z) = 2i \ e^{\alpha L} \sin(\beta L).
\]

Since \( \det(Z) = 0 \) iff \( \beta L = n\pi \), with \( n \) integer,

(1) If \( \beta L \neq n\pi \), then BVP has a unique solution.
Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: \[ Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows
\[ \det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \iff e^{r_+ L} \neq e^{r_- L}. \]

(A) If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \), then
\[ \det(Z) = e^{\alpha L}(e^{i\beta L} - e^{-i\beta L}) \Rightarrow \det(Z) = 2i e^{\alpha L} \sin(\beta L). \]

Since \( \det(Z) = 0 \) iff \( \beta L = n\pi \), with \( n \) integer,
\( (1) \) If \( \beta L \neq n\pi \), then BVP has a unique solution.
\( (2) \) If \( \beta L = n\pi \) then BVP either has no solutions or it has infinitely many solutions.
Existence, uniqueness of solutions to BVP.

**Example**
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1$$
Existence, uniqueness of solutions to BVP.

Example
Find \( y \) solution of the BVP

\[
y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.
\]

Solution: The characteristic polynomial is

\[
p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.
\]
Existence, uniqueness of solutions to BVP.

Example

Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$  

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$  

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$
Existence, uniqueness of solutions to BVP.

Example
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$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_\pm = \pm i.$$ 

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1,$$
Existence, uniqueness of solutions to BVP.

Example
Find \( y \) solution of the BVP

\[
y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.
\]

Solution: The characteristic polynomial is

\[
p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.
\]

The general solution is

\[
y(x) = c_1 \cos(x) + c_2 \sin(x).
\]

The boundary conditions are

\[
1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1
\]
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP
$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$  

Solution: The characteristic polynomial is
$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$  

The general solution is
$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are
$$1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 1, \quad c_2 \text{ free.}$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$  

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_\pm = \pm i.$$  

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 1, \quad c_2 \text{ free.}$$

We conclude:  
$$y(x) = \cos(x) + c_2 \sin(x), \text{ with } c_2 \in \mathbb{R}.$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$  

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$  

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$  

The boundary conditions are

$$1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 1, \quad c_2 \text{ free.}$$  

We conclude: $y(x) = \cos(x) + c_2 \sin(x)$, with $c_2 \in \mathbb{R}$.  

The BVP has infinitely many solutions. \( \triangle \)
Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$
Existence, uniqueness of solutions to BVP.

Example
Find \( y \) solution of the BVP

\[
y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.
\]

Solution: The characteristic polynomial is

\[
p(r) = r^2 + 1
\]
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$  

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP
\[ y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0. \]

Solution: The characteristic polynomial is
\[ p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i. \]

The general solution is
\[ y(x) = c_1 \cos(x) + c_2 \sin(x). \]
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$ 

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$ 

The boundary conditions are

$$1 = y(0) = c_1,$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \implies r_{\pm} = \pm i.$$ 

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 0 = y(\pi) = -c_1$$
Existence, uniqueness of solutions to BVP.

Example
Find \( y \) solution of the BVP
\[
  y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.
\]

Solution: The characteristic polynomial is
\[
p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.
\]

The general solution is
\[
y(x) = c_1 \cos(x) + c_2 \sin(x).
\]

The boundary conditions are
\[
1 = y(0) = c_1, \quad 0 = y(\pi) = -c_1
\]

The BVP has no solution.
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1$$
Existence, uniqueness of solutions to BVP.

**Example**
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$  

**Solution:** The characteristic polynomial is

$$p(r) = r^2 + 1 \implies r_{\pm} = \pm i.$$
Existence, uniqueness of solutions to BVP.

Example
Find \( y \) solution of the BVP

\[
y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.
\]

Solution: The characteristic polynomial is

\[
p(r) = r^2 + 1 \quad \Rightarrow \quad r_\pm = \pm i.
\]

The general solution is

\[
y(x) = c_1 \cos(x) + c_2 \sin(x).
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Existence, uniqueness of solutions to BVP.

Example
Find \( y \) solution of the BVP
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y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.
\]

Solution: The characteristic polynomial is
\[
p(r) = r^2 + 1 \implies r_{\pm} = \pm i.
\]

The general solution is
\[
y(x) = c_1 \cos(x) + c_2 \sin(x).
\]

The boundary conditions are
\[
1 = y(0) = c_1,
\]
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$ 

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$ 

The boundary conditions are

$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$  

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$  

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2 \quad \Rightarrow \quad c_1 = c_2 = 1.$$
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$  

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \implies r_{\pm} = \pm i.$$  

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2 \implies c_1 = c_2 = 1.$$  

We conclude: $y(x) = \cos(x) + \sin(x)$.
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$ 

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \Rightarrow r_{\pm} = \pm i.$$ 

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$ 

The boundary conditions are

$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2 \Rightarrow c_1 = c_2 = 1.$$ 

We conclude: $y(x) = \cos(x) + \sin(x)$.

The BVP has a unique solution.
Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: \textit{Eigenvalue-eigenfunction problem}.
Particular case of BVP: Eigenvalue-eigenfunction problem.

Problem:
Find a number $\lambda$ and a non-zero function $y$ solutions to the boundary value problem

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$
Particular case of BVP: Eigenvalue-eigenfunction problem.

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$$Av - \lambda v = 0.$$
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Differences:

$\rightarrow$ $A$ \quad \left\{ \begin{array}{l} \text{computing a second derivative and } \\ \text{applying the boundary conditions.} \end{array} \right.$
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Differences:
- $A$ $\longrightarrow$ \begin{cases} \text{computing a second derivative and} \\ \text{applying the boundary conditions.} \end{cases}
- $v$ $\longrightarrow$ \{a function $y$\}. 
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**Example**

Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP

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Remarks: We will show that:

1. If $\lambda \leq 0$, then the BVP has no solution.
2. If $\lambda > 0$, then there exist infinitely many eigenvalues $\lambda_n$ and eigenfunctions $y_n$, with $n$ any positive integer, given by

$$\lambda_n = \left(\frac{n \pi}{L}\right)^2,$$

$$y_n(x) = \sin\left(\frac{n \pi x}{L}\right),$$

3. Analogous results can be proven for the same equation but with different types of boundary conditions. For example, for $y(0) = 0$, $y'(L) = 0$; or for $y'(0) = 0$, $y'(L) = 0$. 
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The boundary conditions imply

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Since $y = 0$, there are NO non-zero solutions for $\lambda = 0$. 

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Solution: Case $\lambda < 0$. 
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Solution: Case $\lambda < 0$. Introduce the notation $\lambda = -\mu^2$. 
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$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$
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The boundary condition are

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Solution: Recall: $y(x) = c_1 e^{\mu x} + c_2 e^{\mu x}$ and

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We need to solve the linear system

$$
\begin{bmatrix}
1 & 1 \\
e^{\mu L} & e^{-\mu L}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
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$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

since $\det(Z) = e^{-\mu L} - e^{\mu L} \neq 0$ for $L \neq 0$, matrix $Z$ is invertible, so the linear system above has a unique solution $c_1 = 0$ and $c_2 = 0$. Since $y = 0$, there are NO non-zero solutions for $\lambda < 0$. 

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Solution: Case $\lambda > 0$. 

Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \Rightarrow r = \pm \mu i.$$ 

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$ 

The boundary conditions are

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y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.
\]

Solution: Case \( \lambda > 0 \). Introduce the notation \( \lambda = \mu^2 \). The characteristic equation is

\[
p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.
\]

The general solution is

\[
y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).
\]

The boundary condition are

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Particular case of BVP: Eigenvalue-eigenfunction problem.

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$\forall$
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$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$  \triangleq
Overview of Fourier Series (Sect. 6.2).

- Periodic functions.
- Orthogonality of Sines and Cosines.
- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.
Definition
A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* iff there exists $\tau > 0$ such that for all $x \in \mathbb{R}$ holds

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Notation:
A periodic function with period \( T \) is also called \( T \)-periodic.
Example
The following functions are periodic, with period $T$,

\[
\begin{align*}
f(x) &= \sin(x), & T &= 2\pi. \\
f(x) &= \cos(x), & T &= 2\pi. \\
f(x) &= \tan(x), & T &= \pi. \\
f(x) &= \sin(ax), & T &= \frac{2\pi}{a}.
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\[
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Show that the function below is periodic, and find its period,

\[ f(x) = e^x, \quad x \in [0, 2), \quad f(x - 2) = f(x). \]
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So the function is periodic with period \( T = 2 \).
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- **Orthogonality of Sines and Cosines.**
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Remark:
From now on we work on the following domain: \([-L, L]\).
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**Theorem (Orthogonality)**

*The following relations hold for all \( n, m \in \mathbb{N}, \)*

\[
\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) \, dx = \begin{cases} 
0 & n \neq m, \\
L & n = m \neq 0, \\
2L & n = m = 0,
\end{cases}
\]

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Remark:

The operation \( f \cdot g = \int_{-L}^{L} f(x) g(x) \, dx \) is an **inner product** in the vector space of functions. Like the dot product in \( \mathbb{R}^2 \).

Two functions \( f, g \) are orthogonal iff \( f \cdot g = 0 \).
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- Two functions \( f, g \), are orthogonal iff \( f \cdot g = 0 \).
Orthogonality of Sines and Cosines.

Recall:

\[
\cos(\theta) \cos(\phi) = \frac{1}{2} \left[ \cos(\theta + \phi) + \cos(\theta - \phi) \right];
\]

\[
\sin(\theta) \sin(\phi) = \frac{1}{2} \left[ \cos(\theta - \phi) - \cos(\theta + \phi) \right];
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Proof: First formula: If \( n = m = 0 \), it is simple to see that

\[
\int_{-L}^{L} \cos\left( \frac{n\pi x}{L} \right) \cos\left( \frac{m\pi x}{L} \right) \, dx = \int_{-L}^{L} \, dx = 2L.
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\]

In the case where one of \( n \) or \( m \) is non-zero, use the relation
\[
\int_{-L}^{L} \cos\left( \frac{n \pi x}{L} \right) \cos\left( \frac{m \pi x}{L} \right) \, dx = \frac{1}{2} \int_{-L}^{L} \cos\left[ \frac{(n + m) \pi x}{L} \right] \, dx \\
+ \frac{1}{2} \int_{-L}^{L} \cos\left[ \frac{(n - m) \pi x}{L} \right] \, dx.
\]
Orthogonality of Sines and Cosines.

**Proof:** Since one of $n$ or $m$ is non-zero,
Orthogonality of Sines and Cosines.

**Proof:** Since one of $n$ or $m$ is non-zero, holds

$$
\frac{1}{2} \int_{-L}^{L} \cos \left( \frac{(n + m)\pi x}{L} \right) \, dx = \frac{L}{2(n + m)\pi} \sin \left[ \frac{(n + m)\pi x}{L} \right] \bigg|_{-L}^{L} = 0.
$$

If $n = m \neq 0$, we have that

$$
\frac{1}{2} \int_{-L}^{L} \cos \left( \frac{(n - m)\pi x}{L} \right) \, dx = \frac{L}{2(n - m)\pi} \sin \left[ \frac{(n - m)\pi x}{L} \right] \bigg|_{-L}^{L} = 0.
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This establishes the first equation in the Theorem. The remaining equations are proven in a similar way.
Orthogonality of Sines and Cosines.

Proof: Since one of $n$ or $m$ is non-zero, holds

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We obtain that

$$\int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) \cos \left( \frac{m\pi x}{L} \right) \, dx = \frac{1}{2} \int_{-L}^{L} \cos \left[ \frac{(n - m)\pi x}{L} \right] \, dx.$$
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If we further restrict $n \neq m$, then

\[ \frac{1}{2} \int_{-L}^{L} \cos \left[ \frac{(n - m) \pi x}{L} \right] \, dx = \frac{L}{2(n - m)\pi} \sin \left[ \frac{(n - m) \pi x}{L} \right] \bigg|_{-L}^{L} = 0. \]
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\]

If \( n = m \neq 0 \), we have that

\[
\frac{1}{2} \int_{-L}^{L} \cos \left( \frac{(n - m)\pi x}{L} \right) \, dx = \frac{1}{2} \int_{-L}^{L} \, dx = L.
\]
Orthogonality of Sines and Cosines.

Proof: Since one of \( n \) or \( m \) is non-zero, holds

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We obtain that

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If we further restrict \( n \neq m \), then

\[
\frac{1}{2} \int_{-L}^{L} \cos\left[\frac{(n - m)\pi x}{L}\right] \, dx = \frac{L}{2(n - m)\pi} \sin\left[\frac{(n - m)\pi x}{L}\right] \bigg|_{-L}^{L} = 0.
\]

If \( n = m \neq 0 \), we have that

\[
\frac{1}{2} \int_{-L}^{L} \cos\left[\frac{(n - m)\pi x}{L}\right] \, dx = \frac{1}{2} \int_{-L}^{L} \, dx = L.
\]

This establishes the first equation in the Theorem. The remaining equations are proven in a similar way. \( \square \)
Overview of Fourier Series (Sect. 6.2).

- Periodic functions.
- Orthogonality of Sines and Cosines.
- **The Fourier Theorem: Continuous case.**
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.
The Fourier Theorem: Continuous case.

**Theorem (Fourier Series)**

*If the function* $f : [-L, L] \subset \mathbb{R} \rightarrow \mathbb{R}$ *is continuous, then* $f$ *can be expressed as an infinite series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \quad (1)$$

*with the constants* $a_n$ *and* $b_n$ *given by*

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 1.$$  

*Furthermore, the Fourier series in Eq. (1) provides a* $2L$-*periodic extension of function* $f$ *from the domain* $[-L, L] \subset \mathbb{R}$ *to* $\mathbb{R}$. 
The Fourier Theorem: Continuous case.

Sketch of the Proof:

Define the partial sum functions

\[ f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right] \]
The Fourier Theorem: Continuous case.

Sketch of the Proof:

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with \( a_n \) and \( b_n \) given by

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 0, \]

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- Express \(f_N\) as a convolution of Sine, Cosine, functions and the original function \(f\).
The Fourier Theorem: Continuous case.

Sketch of the Proof:

- Define the partial sum functions

\[ f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \]

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- Express \( f_N(x) \) as a convolution of Sine, Cosine, functions and the original function \( f \).

- Use the convolution properties to show that

\[ \lim_{N \to \infty} f_N(x) = f(x), \quad x \in [-L, L]. \]
Overview of Fourier Series (Sect. 6.2).

- Periodic functions.
- Orthogonality of Sines and Cosines.
- The Fourier Theorem: Continuous case.
  - Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
  - Example: Using the Fourier Theorem.
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: In this case \( L = 1 \).

The Fourier series expansion is

\[ f(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos\left( n\pi x \right) + b_n \sin\left( n\pi x \right) \right], \]

where the \( a_n \) and \( b_n \) are given in the Theorem.

We start with \( a_0 \),

\[ a_0 = \frac{1}{2} \left[ \int_{-1}^{0} (1 + x) \, dx + \int_{0}^{1} (1 - x) \, dx \right]. \]

We obtain:

\[ a_0 = 1. \]
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
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\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(n\pi x) + b_n \sin(n\pi x) \right], \]
Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

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Find the Fourier series expansion of the function

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\[ a_0 = \int_{-1}^{1} f(x) \, dx \]
Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

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\[ a_0 = \int_{-1}^{1} f(x) \, dx = \int_{-1}^{0} (1 + x) \, dx + \int_{0}^{1} (1 - x) \, dx. \]
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Example

Find the Fourier series expansion of the function

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f(x) = \begin{cases} 
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where the \( a_n, b_n \) are given in the Theorem. We start with \( a_0 \),

\[
a_0 = \int_{-1}^{1} f(x) \, dx = \int_{-1}^{0} (1 + x) \, dx + \int_{0}^{1} (1 - x) \, dx.
\]

\[
a_0 = \left( x + \frac{x^2}{2} \right) \bigg|_{-1}^{0} + \left( x - \frac{x^2}{2} \right) \bigg|_{0}^{1}
\]
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1].
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Solution: In this case \( L = 1 \). The Fourier series expansion is

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(n\pi x) + b_n \sin(n\pi x) \right], \]

where the \( a_n, b_n \) are given in the Theorem. We start with \( a_0 \),

\[ a_0 = \int_{-1}^{1} f(x) \, dx = \int_{-1}^{0} (1 + x) \, dx + \int_{0}^{1} (1 - x) \, dx. \]

\[ a_0 = \left( x + \frac{x^2}{2} \right) \bigg|_{-1}^{0} + \left( x - \frac{x^2}{2} \right) \bigg|_{0}^{1} = \left( 1 - \frac{1}{2} \right) + \left( 1 - \frac{1}{2} \right) \]
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
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We obtain: \( a_0 = 1 \).
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: Recall: \( a_0 = 1 \).
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: Recall: \( a_0 = 1 \). Similarly, the rest of the \( a_n \) are given by,

\[ a_n = \int_{-1}^{1} f(x) \cos(n\pi x) \, dx \]
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

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f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
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Solution: Recall: \( a_0 = 1 \). Similarly, the rest of the \( a_n \) are given by,

\[
a_n = \int_{-1}^{1} f(x) \cos(n\pi x) \, dx
\]

\[
a_n = \int_{-1}^{0} (1 + x) \cos(n\pi x) \, dx + \int_{0}^{1} (1 - x) \cos(n\pi x) \, dx.
\]
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
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\[ a_n = \int_{-1}^{0} (1 + x) \cos(n\pi x) \, dx + \int_{0}^{1} (1 - x) \cos(n\pi x) \, dx. \]

Recall the integrals \( \int \cos(n\pi x) \, dx = \frac{1}{n\pi} \sin(n\pi x), \)
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1].
\end{cases} \]

Solution: Recall: \( a_0 = 1 \). Similarly, the rest of the \( a_n \) are given by,

\[ a_n = \int_{-1}^{1} f(x) \cos(n \pi x) \, dx \]

\[ a_n = \int_{-1}^{0} (1 + x) \cos(n \pi x) \, dx + \int_{0}^{1} (1 - x) \cos(n \pi x) \, dx. \]

Recall the integrals \( \int \cos(n \pi x) \, dx = \frac{1}{n \pi} \sin(n \pi x) \), and

\[ \int x \cos(n \pi x) \, dx = \frac{x}{n \pi} \sin(n \pi x) + \frac{1}{n^2 \pi^2} \cos(n \pi x). \]
Example: Using the Fourier Theorem.

Example
Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: It is not difficult to see that

\[ a_n = \frac{1}{n\pi} \sin(n\pi x) \bigg|_0^1 + \left[ \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x) \right] \bigg|_0^1 \]

\[ + \frac{1}{n\pi} \sin(n\pi x) \bigg|_0^1 - \left[ \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x) \right] \bigg|_0^1 \]
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

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f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
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\end{cases}
\]

Solution: It is not difficult to see that

\[
a_n = \frac{1}{n\pi} \sin(n\pi x) \bigg|_{-1}^{0} + \left[ \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x) \right] \bigg|_{-1}^{0} \\
+ \frac{1}{n\pi} \sin(n\pi x) \bigg|_{0}^{1} - \left[ \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x) \right] \bigg|_{0}^{1} \\
= \left[ \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} \cos(-n\pi) \right] - \left[ \frac{1}{n^2\pi^2} \cos(n\pi) - \frac{1}{n^2\pi^2} \right].
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Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

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\[ a_n = \frac{1}{n\pi} \sin(n\pi x) \bigg|_0^0 + \frac{x}{n\pi} \sin(n\pi x) \bigg|_1^0 + \frac{1}{n^2\pi^2} \cos(n\pi x) \bigg|_0^0 \]

\[ + \frac{1}{n\pi} \sin(n\pi x) \bigg|_0^1 - \frac{x}{n\pi} \sin(n\pi x) \bigg|_1^1 - \frac{1}{n^2\pi^2} \cos(n\pi x) \bigg|_0^1 \]

\[ a_n = \left[ \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} \cos(-n\pi) \right] - \left[ \frac{1}{n^2\pi^2} \cos(n\pi) - \frac{1}{n^2\pi^2} \right]. \]

We then conclude that \( a_n = \frac{2}{n^2\pi^2} [1 - \cos(n\pi)] \).
Example: Using the Fourier Theorem.

Example
Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases} \]

Solution: Recall: \( a_0 = 1 \), and \( a_n = \frac{2}{n^2\pi^2} [1 - \cos(n\pi)] \).
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
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Solution: Recall: \( a_0 = 1 \), and \( a_n = \frac{2}{n^2\pi^2} \left[ 1 - \cos(n\pi) \right] \).

Finally, we must find the coefficients \( b_n \).
Example: Using the Fourier Theorem.

Example
Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
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Solution: Recall: \( a_0 = 1 \), and \( a_n = \frac{2}{n^2 \pi^2} [1 - \cos(n\pi)] \).

Finally, we must find the coefficients \( b_n \).

A similar calculation shows that \( b_n = 0 \).
Example: Using the Fourier Theorem.

Example
Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
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Finally, we must find the coefficients \( b_n \).

A similar calculation shows that \( b_n = 0 \).

Then, the Fourier series of \( f \) is given by

\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} [1 - \cos(n\pi)] \cos(n\pi x). \]
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
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f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \left[1 - \cos(n\pi)\right] \cos(n\pi x).
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We can obtain a simpler expression for the Fourier coefficients \( a_n \).
Example: Using the Fourier Theorem.

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We can obtain a simpler expression for the Fourier coefficients \( a_n \).

Recall the relations \( \cos(n\pi) = (-1)^n \).
Example: Using the Fourier Theorem.

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Recall the relations \( \cos(n \pi) = (-1)^n \), then

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\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [1 - (-1)^n] \cos(n\pi x). \]

\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [1 + (-1)^{n+1}] \cos(n\pi x). \]
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

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If \( n = 2k \),
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

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1 + x & x \in [-1, 0), \\
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Solution: Recall:

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\]

If \( n = 2k \), so \( n \) is even,
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: Recall: 

\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \left[ 1 + (-1)^{n+1} \right] \cos(n \pi x). \]

If \( n = 2k \), so \( n \) is even, so \( n + 1 = 2k + 1 \) is odd,
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: Recall:

\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \left[ 1 + (-1)^{n+1} \right] \cos(n\pi x). \]

If \( n = 2k \), so \( n \) is even, so \( n + 1 = 2k + 1 \) is odd, then

\[ a_{2k} = \frac{2}{(2k)^2 \pi^2} (1 - 1) \]
Example: Using the Fourier Theorem.

Example
Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: Recall: \( f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \left[1 + (-1)^{n+1}\right] \cos(n\pi x). \)

If \( n = 2k \), so \( n \) is even, so \( n + 1 = 2k + 1 \) is odd, then

\[ a_{2k} = \frac{2}{(2k)^2 \pi^2} (1 - 1) \quad \Rightarrow \quad a_{2k} = 0. \]
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: Recall:

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\[ a_{2k-1} = \frac{2}{(2k - 1)^2\pi^2} (1 + 1) \implies a_{2k-1} = \frac{4}{(2k - 1)^2\pi^2}. \]
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We conclude: \( f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{(2k - 1)^2 \pi^2} \cos((2k - 1)\pi x). \)
Overview of Fourier Series (Sect. 6.2).

- Periodic functions.
- Orthogonality of Sines and Cosines.
- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
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Recall:

Definition
A function \( f : [a, b] \to \mathbb{R} \) is called \textit{piecewise continuous} \( \text{iff} \) holds,

(a) \([a, b]\) can be partitioned in a finite number of sub-intervals such that \( f \) is continuous on the interior of these sub-intervals.

(b) \( f \) has finite limits at the endpoints of all sub-intervals.
The Fourier Theorem: Piecewise continuous case.

Theorem (Fourier Series)

If \( f : [-L, L] \subset \mathbb{R} \to \mathbb{R} \) is piecewise continuous, then the function

\[
f_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right]
\]

where \( a_n \) and \( b_n \) given by

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx, \quad n \geq 0,
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx, \quad n \geq 1.
\]

satisfies that:

(a) \( f_f(x) = f(x) \) for all \( x \) where \( f \) is continuous;

(b) \( f_f(x_0) = \frac{1}{2} \left[ \lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x) \right] \) for all \( x_0 \) where \( f \) is discontinuous.
Overview of Fourier Series (Sect. 6.2).

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b_n &= \frac{1}{n\pi} [1 - \cos(-n\pi) - \cos(n\pi) + 1] = \frac{2}{n\pi} [1 - \cos(n\pi)],
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If \( n = 2k \), then 
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Solution: Recall: \( b_{2k} = 0 \), and \( b_{2k} = \frac{4}{(2k - 1)\pi} \).
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a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx,
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Solution: Recall: \( b_{2k} = 0 \), \( b_{2k} = \frac{4}{(2k - 1)\pi} \), and \( a_n = 0 \).
Therefore, we conclude that

\[
f_F(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)} \sin((2k - 1)\pi x).
\]
Solving the Heat Equation (Sect. 6.3).

- The Heat Equation.
- The Initial-Boundary Value Problem.
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where $k > 0$ is the heat conductivity, units: $[k] = (\text{distance})^2/(\text{time})$. 

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- The Heat Equation.
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The Initial-Boundary Value Problem.

Definition

The IBVP for the one-dimensional Heat Equation is the following:
Given a constant $k > 0$ and a function $f : [0, L] \rightarrow \mathbb{R}$ with $f(0) = f(L) = 0$, find $u : [0, \infty) \times [0, L] \rightarrow \mathbb{R}$ solution of

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$$u(0, x) = f(x)$$

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Solving the Heat Equation (Sect. 6.3).

- The Heat Equation.
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**Summary:** IBVP for the Heat Equation.

**Propose:**

\[ u(t, x) = \sum_{n=1}^{\infty} c_n v_n(t) w_n(x). \]

where
- \( v_n(t) \): Solution of an IVP.
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Here $c_n$ are constants, $n = 1, 2, \ldots$. 
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I.C.: \( u_n(0, x) = w_n(x) \),

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The separation of variables method.

**Step 4:** (Key step.)
Transform the IBVP for $u_n$ into:

\[
\partial_t u_n(t, x) = \partial_x^2 u_n(t, x)
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Notice:
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\frac{\partial}{\partial t} u_n(t, x) = \text{Depends only on } t \\
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Transform the IBVP for $u_n$ into: *(a)* IVP for $v_n$; *(b)* BVP for $w_n$.

Notice:

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\frac{\partial^2}{\partial x^2} u_n(t, x) = \frac{\partial^2}{\partial x^2} [v_n(t) w_n(x)]
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Therefore, the equation \( \partial_t u_n = k \partial_x^2 u_n \)
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\frac{1}{k v_n(t)} \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)} \frac{d^2 w_n}{dx^2}(x).
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Depends only on \( t \) \hspace{1cm} \text{Depends only on} \ x.

- The Heat Equation has the following property:
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- We conclude that for appropriate constants $\lambda_m$ holds
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  The left-hand side depends only on \( t \), while the right-hand side depends only on \( x \).

- When this happens in a PDE, one can use the separation of variables method on that PDE.

- We conclude that for appropriate constants \( \lambda_m \) holds
  \[
  \frac{1}{k v_n(t)} \frac{dv_n(t)}{dt} = -\lambda_n, \quad \frac{1}{w_n(x)} \frac{d^2w_n(x)}{dx^2} = -\lambda_n.
  \]

- We have transformed the original PDE into infinitely many ODEs parametrized by \( n \), positive integer.
The separation of variables method.

**Summary Step 4:** The original *IBVP* for the Heat Equation, PDE, can be transformed into:

(a) We choose to solve the following IVP for \( v_n \):

\[
\frac{dv_n}{dt}(t) = -\lambda_n,
\]

I.C.: \( v_n(0) = 1 \).

Remark: This choice of I.C. simplifies the problem.

(b) The BVP for \( w_n \):

\[
\frac{d^2w_n}{dx^2}(x) = -\lambda_n,
\]

B.C.: \( w_n(0) = 0 \), \( w_n(L) = 0 \).

**Step 5:**

(a) Solve the IVP for \( v_n \).

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(b) The BVP for $w_n$,

$$\frac{1}{w_n(x)} \frac{d^2 w_n}{dx^2}(x) = -\lambda_n,$$
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\[
\frac{1}{w_n(x)} \frac{d^2 w_n(x)}{dx^2} = -\lambda_n, \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0.
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(a) Solve the IVP for $v_n$.

(b) Solve the BVP for $w_n$. 
The separation of variables method.

Step 5(a): Solving the IVP for $v_n$.

\[ v_n'(t) + k\lambda_n v_n(t) = 0, \]
The separation of variables method.

Step 5(a): Solving the IVP for $v_n$.

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The separation of variables method.

Step 5(a): Solving the IVP for $v_n$.

$$v_n'(t) + k\lambda_n v_n(t) = 0, \quad \text{I.C.:} \quad v_n(0) = 1.$$ 

The integrating factor method implies that $\mu(t) = e^{k\lambda_n t}$.

$$e^{k\lambda_n t} v_n'(t) + k\lambda_n e^{k\lambda_n t} v_n(t) = 0$$
The separation of variables method.

Step 5(a): Solving the IVP for \( v_n \).

\[
v'_n(t) + k\lambda_n v_n(t) = 0, \quad \text{I.C.:} \quad v_n(0) = 1.
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\[
e^{k\lambda_n t} v'_n(t) + k\lambda_n e^{k\lambda_n t} v_n(t) = 0 \quad \Rightarrow \quad \left[ e^{k\lambda_n t} v_n(t) \right]' = 0.
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\]

\[
e^{k\lambda_n t} v_n(t) = c_n \quad \Rightarrow \quad v_n(t) = c_n e^{-k\lambda_n t}.
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$$1 = v_n(0) = c$$
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The separation of variables method.

Step 5(a): Recall:  \( v_n(t) = e^{-k\lambda_n t} \).
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Step 5(a): Recall: \( v_n(t) = e^{-k\lambda_n t} \).

Step 5(b): Eigenvalue-eigenvector problem for \( w_n \):

Find the eigenvalues \( \lambda_n \) and the non-zero eigenfunctions \( w_n \) solutions of the BVP

\[
w_n''(x) + \lambda_n w_n(x) = 0
\]

B.C.: \( w_n(0) = 0 \), \( w_n(L) = 0 \).

We know that this problem has solution only for \( \lambda_n > 0 \).

Denote: \( \lambda_n = \mu_n^2 \).

Proposing \( w_n(x) = e^{r_n x} \), we get that

\[
p(r_n) = r_n^2 + \mu_n^2 = 0 \Rightarrow r_n = \pm \mu_n i
\]

The real-valued general solution is

\[
w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)
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$$w''_n(x) + \lambda_n w_n(x) = 0 \quad \text{B.C.: } w_n(0) = 0, \quad w_n(L) = 0.$$
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\]
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$$\mu_n L = n\pi$$
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\]

\[
\mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L} \quad \Rightarrow \quad \lambda_n = \left( \frac{n\pi}{L} \right)^2.
\]
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\]

Choosing \( c_2 = 1 \), we get \( w_n(x) = \sin(\frac{n\pi x}{L}) \).
The separation of variables method.

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\]

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\]

Choosing \( c_2 = 1 \), we get \( w_n(x) = \sin\left(\frac{n\pi x}{L}\right) \).

We conclude that: \( u_n(t, x) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \).
The separation of variables method.

Step 6: Recall: \[ u_n(t, x) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right). \]
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Step 6: Recall: \( u_n(t, x) = e^{-k \left( \frac{n\pi}{L} \right)^2 t} \sin \left( \frac{n\pi x}{L} \right) \).

Compute the solution to the IBVP for the Heat Equation,

\[
    u(t, x) = \sum_{n=1}^{\infty} c_n u_n(t, x).
\]
The separation of variables method.

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Compute the solution to the IBVP for the Heat Equation,

\[
u(t, x) = \sum_{n=1}^{\infty} c_n u_n(t, x).
\]

By construction, this solution satisfies the boundary conditions,

\[
u(t, 0) = 0, \quad u(t, L) = 0.
\]
The separation of variables method.

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By construction, this solution satisfies the boundary conditions,

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  u(t, 0) = 0, \quad u(t, L) = 0.
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Given a function \( f \) with \( f(0) = f(L) = 0 \), the solution \( u \) above satisfies the initial condition \( f(x) = u(0, x) \) iff holds
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Compute the solution to the IBVP for the Heat Equation,

\[
\begin{align*}
    u(t, x) &= \sum_{n=1}^{\infty} c_n u_n(t, x).
\end{align*}
\]

\[
\begin{align*}
    u(t, x) &= \sum_{n=1}^{\infty} c_n e^{-k \left( \frac{n \pi}{L} \right)^2 t} \sin \left( \frac{n \pi x}{L} \right).
\end{align*}
\]

By construction, this solution satisfies the boundary conditions,

\[
\begin{align*}
    u(t, 0) &= 0, \quad u(t, L) = 0.
\end{align*}
\]

Given a function \( f \) with \( f(0) = f(L) = 0 \), the solution \( u \) above satisfies the initial condition \( f(x) = u(0, x) \) iff holds

\[
\begin{align*}
    f(x) &= \sum_{n=1}^{\infty} c_n \sin \left( \frac{n \pi x}{L} \right).
\end{align*}
\]
The separation of variables method.

Recall:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-k \left( \frac{n\pi}{L} \right)^2 t} \sin \left( \frac{n\pi x}{L} \right), \quad f(x) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n\pi x}{L} \right). \]
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Recall:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-k \left( \frac{n\pi}{L} \right)^2 t} \sin \left( \frac{n\pi x}{L} \right), \quad f(x) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n\pi x}{L} \right). \]

This is a Sine Series for \( f \). The coefficients \( c_n \) are computed in the usual way.
The separation of variables method.

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This is a Sine Series for \( f \). The coefficients \( c_n \) are computed in the usual way. Recall the orthogonality relation
\[
\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \, dx = \begin{cases} 
0, & m \neq n, \\
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\]
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Multiply the equation for \( u \) by \( \sin \left( \frac{m\pi x}{L} \right) \) and integrate,

\[ \sum_{n=1}^{\infty} c_n \int_0^L \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{m\pi x}{L} \right) \, dx = \int_0^L f(x) \sin \left( \frac{m\pi x}{L} \right) \, dx. \]
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\]

\[
c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx, \quad u(t, x) = \sum_{n=1}^{\infty} c_n e^{-k \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right).\]
The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x). \]

Remark: The separation of variables method does not work for every PDE.
The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x). \]

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Propose:

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- \( \nu_n \): Solution of an IVP.
- \( w_n \): Solution of a BVP, an eigenvalue-eigenfunction problem.

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The separation of variables method does not work for every PDE.
Solving the Heat Equation (Sect. 6.3).

- The Heat Equation.
- The Initial-Boundary Value Problem.
- The separation of variables method.
- An example of separation of variables.
An example of separation of variables.

Example
Find the solution to the IBVP $4\partial_t u = \partial_x^2 u$, $t > 0$, $x \in [0, 2]$,

$u(0, x) = 3 \sin(\pi x/2)$, $u(t, 0) = 0$, $u(t, 2) = 0$. 
An example of separation of variables.

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Find the solution to the IBVP

$$4 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad x \in [0, 2],$$

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Solution: Let \( u_n(t, x) = v_n(t) w_n(x) \). Then

\[ 4w_n(x) \frac{dv}{dt}(t) = v_n(t) \frac{d^2 w}{dx^2}(x) \]
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The equations for $v_n$ and $w_n$ are

$$v'_n(t) + \frac{\lambda_n}{4} v_n(t) = 0,$$
An example of separation of variables.

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Find the solution to the IBVP \( 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \)
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Example

Find the solution to the IBVP \(4 \frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u, \quad t > 0, \quad x \in [0, 2],\)
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We solve for \(v_n\) with the initial condition \(v_n(0) = 1.\)
\[e^{\frac{\lambda_n}{4} t} v_n'(t) + \frac{\lambda_n}{4} e^{\frac{\lambda_n}{4} t} v_n(t) = 0\]
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Find the solution to the IBVP \[4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.\]

Solution: Let \(u_n(t, x) = v_n(t)w_n(x)\). Then
\[4w_n(x) \frac{dv}{dt}(t) = v_n(t) \frac{d^2w}{dx^2}(x) \quad \Rightarrow \quad \frac{4v'_n(t)}{v_n(t)} = \frac{w''_n(x)}{w_n(x)} = -\lambda_n.\]

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\[e^{\frac{\lambda_n}{4} t} v'_n(t) + \frac{\lambda_n}{4} e^{\frac{\lambda_n}{4} t} v_n(t) = 0 \quad \Rightarrow \quad \left[e^{\frac{\lambda_n}{4} t} v_n(t)\right]' = 0.\]
An example of separation of variables.

Example
Find the solution to the IBVP
\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\pi x / 2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall: \[ e^{\lambda_n^4 t} \nu_n(t) \]' = 0.
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\[ 4 \frac{\partial t}{\partial t} u = \frac{\partial^2}{\partial x^2} u, \quad t > 0, \quad x \in [0, 2], \]
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Next the BVP: \[ w_n''(x) + \lambda_n w_n(x) = 0, \text{ with } w_n(0) = w_n(L) = 0. \]
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Find the solution to the IBVP $4 \frac{\partial_t u}{t} = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

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Next the BVP: \[ w''_n(x) + \lambda_n w_n(x) = 0, \text{ with } w_n(0) = w_n(L) = 0. \]
Since \( \lambda_n > 0 \), introduce \( \lambda_n = \mu_n^2 \). The characteristic polynomial is

\[ p(r) = r^2 + \mu_n^2. \]
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Solution: Recall: \[\left[ e^{\frac{\lambda_n}{4} t} \nu_n(t) \right]' = 0.\] Therefore,
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Find the solution to the IBVP \(4 \frac{\partial t}{\partial t} u = \frac{\partial^2 x}{\partial x^2} u, \quad t > 0, \quad x \in [0, 2],\)
\[u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.\]

Solution: Recall: \(\left[ e^{\frac{\lambda_n}{4} t} v_n(t) \right]' = 0.\) Therefore,
\[v_n(t) = c e^{-\frac{\lambda_n}{4} t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4} t}.\]

Next the BVP: \(w_n''(x) + \lambda_n w_n(x) = 0,\) with \(w_n(0) = w_n(L) = 0.\)

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\[p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm \mu_n i.\]

The general solution, \(w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x).\)
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Example
Find the solution to the IBVP $4 \partial_t u = \partial_x^2 u$, $t > 0$, $x \in [0, 2]$, $u(0, x) = 3 \sin(\pi x/2)$, $u(t, 0) = 0$, $u(t, 2) = 0$.

Solution: Recall: $[e^{\frac{\lambda_n}{4} t} v_n(t)]' = 0$. Therefore,

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The boundary conditions imply

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Find the solution to the IBVP \( 4\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u, \quad t > 0, \quad x \in [0, 2], \)
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Solution: Recall: \( [e^{\frac{\lambda_n}{4} t} \nu_n(t)]' = 0. \) Therefore,
\[ \nu_n(t) = c e^{-\frac{\lambda_n}{4} t}, \quad 1 = \nu_n(0) = c \quad \Rightarrow \quad \nu_n(t) = e^{-\frac{\lambda n}{4} t}. \]

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Solution: Recall: \( v_n(t) = e^{-\frac{\lambda_n}{4} t}, \) and \( w_n(x) = c_2 \sin(\mu_n x). \)
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Find the solution to the IBVP $4 \partial_t u = \partial_x^2 u$, $t > 0$, $x \in [0, 2]$, $u(0, x) = 3 \sin(\pi x/2)$, $u(t, 0) = 0$, $u(t, 2) = 0$.

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\[ 0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0. \]

Then, \( \mu_n 2 = n\pi \).
An example of separation of variables.

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Find the solution to the IBVP

\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin\left(\frac{\pi x}{2}\right), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall: \( v_n(t) = e^{-\frac{\lambda_n}{4} t} \), and \( w_n(x) = c_2 \sin(\mu_n x) \).

\[ 0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0. \]

Then, \( \mu_n 2 = n\pi \), that is, \( \mu_n = \frac{n\pi}{2} \).
An example of separation of variables.

Example
Find the solution to the IBVP

$$4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$  

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4} t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$  

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \frac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n\pi}{2}\right)^2, \quad w_n(x) = \sin\left(\frac{n\pi x}{2}\right).$$
An example of separation of variables.

Example
Find the solution to the IBVP
\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall: \( v_n(t) = e^{-\lambda_n t/4}, \) and \( w_n(x) = c_2 \sin(\mu_n x). \)

\[ 0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0. \]

Then, \( \mu_n 2 = n\pi, \) that is, \( \mu_n = \frac{n\pi}{2}. \) Choosing \( c_2 = 1, \) we conclude,

\[ \lambda_n = \left( \frac{n\pi}{2} \right)^2, \quad w_n(x) = \sin\left( \frac{n\pi x}{2} \right). \]

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left( \frac{n\pi}{4} \right)^2 t} \sin\left( \frac{n\pi x}{2} \right). \]
An example of separation of variables.

Example
Find the solution to the IBVP
\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall:
\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right). \]
An example of separation of variables.

Example
Find the solution to the IBVP

\[ 4 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad x \in [0, 2], \]

\[ u(0, x) = 3 \sin\left(\frac{\pi x}{2}\right), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right). \]

The initial condition is

\[ 3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right). \]
An example of separation of variables.

Example

Find the solution to the IBVP $4 \partial_t u = \partial_x^2 u$, $t > 0$, $x \in [0, 2]$,  
$u(0, x) = 3 \sin(\pi x / 2)$,  
$u(t, 0) = 0$,  
$u(t, 2) = 0$.

Solution: Recall:  
$u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right)$.

The initial condition is  
$3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right)$.

The orthogonality of the sine functions implies  
$3 \int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx$. 
An example of separation of variables.

Example
Find the solution to the IBVP $4 \partial_t u = \partial_x^2 u$, $t > 0$, $x \in [0, 2]$, $u(0, x) = 3 \sin(\pi x/2)$, $u(t, 0) = 0$, $u(t, 2) = 0$.

Solution: Recall: $u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right)$.

The initial condition is $3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right)$.

The orthogonality of the sine functions implies

$$3 \int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx.$$ 

If $m \neq 1$, then $0 = c_m \frac{2}{2}$,
An example of separation of variables.

Example
Find the solution to the IBVP

\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]

\[ u(0, x) = 3 \sin(\pi x / 2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right). \]

The initial condition is

\[ 3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right). \]

The orthogonality of the sine functions implies

\[ 3 \int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx. \]

If \( m \neq 1 \), then \( 0 = c_m \frac{2}{2} \), that is, \( c_m = 0 \) for \( m \neq 1 \).
An example of separation of variables.

Example

Find the solution to the IBVP

\[ 4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]
\[ u(0, x) = 3 \sin(\frac{\pi x}{2}), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right). \]

The initial condition is

\[ 3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right). \]

The orthogonality of the sine functions implies

\[ 3 \int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) \, dx. \]

If \( m \neq 1 \), then \( 0 = c_m \frac{2}{2} \), that is, \( c_m = 0 \) for \( m \neq 1 \). Therefore,

\[ 3 \sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right) \]
An example of separation of variables.

Example

Find the solution to the IBVP

\[ 4 \frac{\partial t u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad x \in [0, 2], \]

\[ u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0. \]

Solution: Recall:

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n \pi}{4}\right)^2 t} \sin\left(\frac{n \pi x}{2}\right). \]

The initial condition is

\[ 3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n \pi x}{2}\right). \]

The orthogonality of the sine functions implies

\[ 3 \int_{0}^{2} \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m \pi x}{2}\right) dx = \sum_{n=1}^{\infty} \int_{0}^{2} \sin\left(\frac{n \pi x}{2}\right) \sin\left(\frac{m \pi x}{2}\right) dx. \]

If \( m \neq 1 \), then \( 0 = c_m \frac{2}{2} \), that is, \( c_m = 0 \) for \( m \neq 1 \). Therefore,

\[ 3 \sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right) \implies c_1 = 3. \]
An example of separation of variables.

Example
Find the solution to the IBVP \( 4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \)
\[
  u(0, x) = 3 \sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.
\]

Solution: We conclude that
\[
  u(t, x) = 3 e^{-\left(\frac{\pi}{4}\right)^2 t} \sin\left(\frac{\pi x}{2}\right).
\]
Review for Final Exam.

- Exam is cumulative.
- Heat equation and Fourier Series not included.
- 10-12 problems.
- Two hours.
- Integration and Laplace Transform tables included.

- **Not in the exam:** Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Power Series Methods (Chptr. 3).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).
Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution:

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right].$$

Since $f$ is odd and periodic, the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{2}{L} \int_{L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx = \frac{1}{L} \int_{-1}^{0} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx.$$

$$b_n = 2 \left[ \int_{0}^{1} \sin \left( \frac{n\pi x}{L} \right) \, dx \right] = \frac{1}{n\pi} \left[ \cos \left( \frac{n\pi - 1}{L} \right) - \cos \left( \frac{n\pi - 1}{L} \right) \right] = \frac{1}{n\pi}. $$

$$b_n = \frac{2}{n\pi} \left[ \cos \left( \frac{n\pi x}{L} \right) - 1 \right] \bigg|_{L}^{L} = \frac{2}{n\pi} \left( 1 - \cos \left( \frac{n\pi}{L} \right) \right).$$
Fourier Series: Even/Odd-periodic extensions.

Example
Graph the odd-periodic extension of \( f(x) = 1 \) for \( x \in (-1, 0) \), and then find the Fourier Series of this extension.

Solution: The Fourier series is

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f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right].
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b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left( \frac{n \pi x}{L} \right) \, dx
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Fourier Series: Even/Odd-periodic extensions.

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Since $f$ is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx.$$
Fourier Series: Even/Odd-periodic extensions.

Example
Graph the odd-periodic extension of \( f(x) = 1 \) for \( x \in (-1, 0) \), and then find the Fourier Series of this extension.

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\]

\[
b_n = 2 \int_{0}^{1} (-1) \sin(n\pi x) \, dx
\]
Fourier Series: Even/Odd-periodic extensions.

Example
Graph the odd-periodic extension of \( f(x) = 1 \) for \( x \in (-1, 0) \), and then find the Fourier Series of this extension.

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f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right].
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\]

\[
b_n = 2 \int_{0}^{1} (-1) \sin(n\pi x) \, dx = (-2) \left. \frac{(-1)}{n\pi} \cos(n\pi x) \right|_{0}^{1},
\]
Fourier Series: Even/Odd-periodic extensions.

Example
Graph the odd-periodic extension of \( f(x) = 1 \) for \( x \in (-1, 0) \), and then find the Fourier Series of this extension.

Solution: The Fourier series is

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 f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right].
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\]

\[
b_n = 2 \int_{0}^{1} (-1) \sin(n\pi x) \, dx = (-2) \frac{(-1)^n}{n\pi} \cos(n\pi x) \bigg|_{0}^{1},
\]

\[
b_n = \frac{2}{n\pi} \left[ \cos(n\pi) - 1 \right]
\]
Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right].$$

Since $f$ is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

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$$b_n = 2 \int_{0}^{1} (-1) \sin(n\pi x) \, dx = (-2) \left[ \frac{(-1)}{n\pi} \cos(n\pi x) \right]_{0}^{1},$$  

$$b_n = \frac{2}{n\pi} \left[ \cos(n\pi) - 1 \right] \Rightarrow b_n = \frac{2}{n\pi} \left[ (-1)^n - 1 \right].$$
Example

Graph the odd-periodic extension of \( f(x) = 1 \) for \( x \in (-1, 0) \), and then find the Fourier Series of this extension.

Solution: Recall: \( b_n = \frac{2}{n\pi} [(-1)^n - 1] \).
Example
Graph the odd-periodic extension of \( f(x) = 1 \) for \( x \in (-1, 0) \), and then find the Fourier Series of this extension.

Solution: Recall: \( b_n = \frac{2}{n\pi} \left[ (-1)^n - 1 \right] \).

If \( n = 2k \),
Example

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Solution: Recall: \( b_n = \frac{2}{n\pi} \left[ (-1)^n - 1 \right] \).

If \( n = 2k \), then \( b_{2k} = \frac{2}{2k\pi} \left[ (-1)^{2k} - 1 \right] \).
Example

Graph the odd-periodic extension of \( f(x) = 1 \) for \( x \in (-1, 0) \), and then find the Fourier Series of this extension.

Solution: Recall: \( b_n = \frac{2}{n\pi} \left[ (-1)^n - 1 \right] \).

If \( n = 2k \), then \( b_{2k} = \frac{2}{2k\pi} \left[ (-1)^{2k} - 1 \right] = 0. \)
Fourier Series: Even/Odd-periodic extensions.

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Graph the odd-periodic extension of \( f(x) = 1 \) for \( x \in (-1, 0) \), and then find the Fourier Series of this extension.

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Fourier Series: Even/Odd-periodic extensions.

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If \( n = 2k \), then \( b_{2k} = \frac{2}{2k\pi} [(-1)^{2k} - 1] = 0 \).

If \( n = 2k - 1 \),
\[
b_{(2k-1)} = \frac{2}{(2k-1)\pi} [(-1)^{2k-1} - 1]
\]
Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \frac{2}{n\pi} \left[(-1)^n - 1\right]$.

If $n = 2k$, then $b_{2k} = \frac{2}{2k\pi} \left[(-1)^{2k} - 1\right] = 0$.

If $n = 2k - 1$, $b_{(2k-1)} = \frac{2}{(2k-1)\pi} \left[(-1)^{2k-1} - 1\right] = -\frac{4}{(2k-1)\pi}$.
Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of \( f(x) = 1 \) for \( x \in (-1, 0) \), and then find the Fourier Series of this extension.

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If \( n = 2k \), then \( b_{2k} = \frac{2}{2k\pi} \left[ (-1)^{2k} - 1 \right] = 0 \).

If \( n = 2k - 1 \),
\[
b_{(2k-1)} = \frac{2}{(2k-1)\pi} \left[ (-1)^{2k-1} - 1 \right] = -\frac{4}{(2k-1)\pi}.
\]

We conclude: \( f(x) = -\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin[(2k-1)\pi x] \). \( \triangle \)
Fourier Series: Even/Odd-periodic extensions.

Example
Graph the odd-periodic extension of \( f(x) = 2 - x \) for \( x \in (0, 2) \), and then find the Fourier Series of this extension.
Fourier Series: Even/Odd-periodic extensions.

Example
Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right].$$
Fourier Series: Even/Odd-periodic extensions.

Example
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Fourier Series: Even/Odd-periodic extensions.

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Fourier Series: Even/Odd-periodic extensions.

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Graph the odd-periodic extension of \( f(x) = 2 - x \) for \( x \in (0, 2) \), and then find the Fourier Series of this extension.

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f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right].
\]

Since \( f \) is odd and periodic, then the Fourier Series is a Sine Series, that is, \( a_n = 0 \).
Example
Graph the odd-periodic extension of \( f(x) = 2 - x \) for \( x \in (0, 2) \), and then find the Fourier Series of this extension.

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b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx
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Fourier Series: Even/Odd-periodic extensions.

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b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx,
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Fourier Series: Even/Odd-periodic extensions.

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Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

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Example
Graph the odd-periodic extension of \( f(x) = 2 - x \) for \( x \in (0, 2) \), and then find the Fourier Series of this extension.

Solution: \( b_n = 2 \int_0^2 \sin \left( \frac{n\pi x}{2} \right) \, dx - \int_0^2 x \sin \left( \frac{n\pi x}{2} \right) \, dx \).
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\[
\int \sin \left( \frac{n\pi x}{2} \right) \, dx = \frac{-2}{n\pi} \cos \left( \frac{n\pi x}{2} \right),
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I = \int x \sin \left( \frac{n\pi x}{2} \right) \, dx, \quad \left\{ \begin{array}{c}
u = x, \quad v' = \sin \left( \frac{n\pi x}{2} \right)
\end{array} \right.
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 u' = 1, \quad v = -\frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)
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$I = -\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)$.
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\]

So, we get
\[
b_n = 2 \left[\frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]_0^2 + \left[\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]_0^2 - \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)_0^2.
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\]

\[
b_n = \frac{-4}{n\pi} [\cos(n\pi) - 1] + \frac{4}{n\pi} \cos(n\pi) - 0
\]
Example

Graph the odd-periodic extension of \( f(x) = 2 - x \) for \( x \in (0, 2) \), and then find the Fourier Series of this extension.

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\[
b_n = -\frac{4}{n\pi} [\cos(n\pi) - 1] + \left[\frac{4}{n\pi} \cos(n\pi) - 0\right] \Rightarrow b_n = \frac{4}{n\pi}.
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Fourier Series: Even/Odd-periodic extensions.

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Graph the odd-periodic extension of \( f(x) = 2 - x \) for \( x \in (0, 2) \), and then find the Fourier Series of this extension.

Solution: \( I = \frac{-2x}{n\pi} \cos \left( \frac{n\pi x}{2} \right) - \int \left( \frac{-2}{n\pi} \right) \cos \left( \frac{n\pi x}{2} \right) \, dx \).

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We conclude: \( f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{n\pi x}{2} \right) \). \( \triangle \)
Fourier Series: Even/Odd-periodic extensions.

Example
Graph the even-periodic extension of \( f(x) = 2 - x \) for \( x \in [0, 2] \), and then find the Fourier Series of this extension.

Solution:
The Fourier series is
\[
f(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right].
\]
Since \( f \) is even and periodic, the Fourier Series is a Cosine Series, that is, \( b_n = 0 \).
\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx = \int_{0}^{2} (2 - x) \, dx = \text{base x height} = 2.
\]

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Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

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Graph the even-periodic extension of \( f(x) = 2 - x \) for \( x \in [0, 2] \), and then find the Fourier Series of this extension.

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Example

Graph the even-periodic extension of \( f(x) = 2 - x \) for \( x \in [0, 2] \), and then find the Fourier Series of this extension.

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Graph the even-periodic extension of \( f(x) = 2 - x \) for \( x \in [0, 2] \), and then find the Fourier Series of this extension.

Solution: \( a_n = 2 \int_0^2 \cos\left(\frac{n\pi x}{2}\right) \, dx - \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) \, dx \).

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\end{array} \right.
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$I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right)$. 
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Graph the even-periodic extension of \( f(x) = 2 - x \) for \( x \in [0, 2] \), and then find the Fourier Series of this extension.

Solution: Recall: \( L = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \int \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \, dx \).

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\[ a_n = 2\left[\frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]_0^2 - \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]_0^2 - \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right)_0^2 \]
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\]

We conclude: \( f(x) = 1 + \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos\left(\frac{(2k-1)\pi x}{2}\right) \). \( \triangle \)
Review for Final Exam.

- Fourier Series expansions (Chptr. 6).
- **Eigenvalue-Eigenfunction BVP (Chptr. 6).**
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Power Series Methods (Chptr. 3).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).
Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(8) = 0.$$
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Solution: Since \( \lambda > 0 \), introduce \( \lambda = \mu^2 \),

\[ y_n(x) = \sin \left( \frac{n\pi}{8} x \right), \quad n = 1, 2, \ldots \]
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$\mu = \frac{n \pi}{8}$,  

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The general solution is \( y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x) \).
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Find the solution of the BVP

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A Boundary Value Problem.

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Then, \( y'(x) = -c_1 \sin(x) + c_2 \cos(x) \). The B.C. imply:

\[ 1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x). \]

\[ 0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \quad \Rightarrow \quad c_1 = -\frac{\sin(\pi/3)}{\cos(\pi/3)}. \]

\[ c_1 = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3} \quad \Rightarrow \quad y(x) = -\sqrt{3} \cos(x) + \sin(x). \]
Review for Final Exam.

- Fourier Series expansions (Chptr. 6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- **Systems of linear Equations (Chptr. 5).**
- Laplace transforms (Chptr. 4).
- Power Series Methods (Chptr. 3).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).
Systems of linear Equations.

**Summary:** Find solutions of $x' = A x$, with $A$ a $2 \times 2$ matrix.
Systems of linear Equations.

**Summary:** Find solutions of $\mathbf{x}' = A \mathbf{x}$, with $A$ a $2 \times 2$ matrix.
First find the eigenvalues $\lambda_i$ and the eigenvectors $\mathbf{v}^{(i)}$ of $A$. 
Summary: Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix. First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(a) If $\lambda_1 \neq \lambda_2$, real,
Systems of linear Equations.

Summary: Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix. First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{v^{(1)}, v^{(2)}\}$ are linearly independent,
Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A \mathbf{x}$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $\mathbf{v}^{(i)}$ of $A$.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{v}^{(2)} e^{\lambda_2 t}$.
Systems of linear Equations.

**Summary:** Find solutions of $\mathbf{x}' = A \mathbf{x}$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $\mathbf{v}^{(i)}$ of $A$.

(a) If $\lambda_1 \neq \lambda_2$, real, then \{$\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$\} are linearly independent, and the general solution is $\mathbf{x}(x) = c_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{v}^{(2)} e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex,
Systems of linear Equations.

Summary: Find solutions of \( x' = A x \), with \( A \) a \( 2 \times 2 \) matrix.

First find the eigenvalues \( \lambda_i \) and the eigenvectors \( v^{(i)} \) of \( A \).

(a) If \( \lambda_1 \neq \lambda_2 \), real, then \( \{v^{(1)}, v^{(2)}\} \) are linearly independent, and the general solution is \( x(x) = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t} \).

(b) If \( \lambda_1 \neq \lambda_2 \), complex, then denoting \( \lambda_\pm = \alpha \pm \beta i \) and \( v^{(\pm)} = a \pm bi \),
Summary: Find solutions of $\mathbf{x}' = A \mathbf{x}$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $\mathbf{v}^{(i)}$ of $A$.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{v}^{(2)} e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = a \pm b i$, the complex-valued fundamental solutions $\mathbf{x}^{(\pm)} = (a \pm b i) e^{(\alpha \pm \beta i) t}$.
Systems of linear Equations.

Summary: Find solutions of \( \mathbf{x}' = A \mathbf{x} \), with \( A \) a \( 2 \times 2 \) matrix.

First find the eigenvalues \( \lambda_i \) and the eigenvectors \( \mathbf{v}^{(i)} \) of \( A \).

(a) If \( \lambda_1 \neq \lambda_2 \), real, then \( \{ \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \} \) are linearly independent, and the general solution is 
\[
\mathbf{x}(x) = c_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{v}^{(2)} e^{\lambda_2 t}.
\]

(b) If \( \lambda_1 \neq \lambda_2 \), complex, then denoting \( \lambda_{\pm} = \alpha \pm \beta i \) and \( \mathbf{v}^{(\pm)} = a \pm b i \), the complex-valued fundamental solutions 
\[
\mathbf{x}^{(\pm)} = (a \pm b i) e^{(\alpha \pm \beta i) t}
\]
\[
\mathbf{x}^{(\pm)} = e^{\alpha t} (a \pm b i) [\cos(\beta t) + i \sin(\beta t)].
\]
Systems of linear Equations.

**Summary:** Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{v^{(1)}, v^{(2)}\}$ are linearly independent, and the general solution is

$$x(x) = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}.$$  

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_{\pm} = \alpha \pm \beta i$ and $v^{(\pm)} = a \pm bi$, the complex-valued fundamental solutions

$$x^{(\pm)} = (a \pm bi) e^{(\alpha \pm \beta i) t}$$

$$x^{(\pm)} = e^{\alpha t} (a \pm bi) [\cos(\beta t) + i \sin(\beta t)].$$  

$$x^{(\pm)} = e^{\alpha t} [a \cos(\beta t) - b \sin(\beta t)] \pm ie^{\alpha t} [a \sin(\beta t) + b \cos(\beta t)].$$
Systems of linear Equations.

Summary: Find solutions of \( \mathbf{x}' = A \mathbf{x} \), with \( A \) a \( 2 \times 2 \) matrix.

First find the eigenvalues \( \lambda_i \) and the eigenvectors \( \mathbf{v}^{(i)} \) of \( A \).

(a) If \( \lambda_1 \neq \lambda_2 \), real, then \( \{ \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \} \) are linearly independent, and the general solution is

\[
\mathbf{x}(t) = c_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{v}^{(2)} e^{\lambda_2 t}.
\]

(b) If \( \lambda_1 \neq \lambda_2 \), complex, then denoting \( \lambda_{\pm} = \alpha \pm \beta i \) and \( \mathbf{v}^{(\pm)} = a \pm bi \), the complex-valued fundamental solutions

\[
\mathbf{x}^{(\pm)} = (a \pm bi) e^{(\alpha \pm \beta i) t}
\]

\[
\mathbf{x}^{(\pm)} = e^{\alpha t} (a \pm bi) \left[ \cos(\beta t) + i \sin(\beta t) \right].
\]

\[
\mathbf{x}^{(\pm)} = e^{\alpha t} \left[ a \cos(\beta t) - b \sin(\beta t) \right] \pm ie^{\alpha t} \left[ a \sin(\beta t) + b \cos(\beta t) \right].
\]

Real-valued fundamental solutions are
Systems of linear Equations.

Summary: Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{v^{(1)}, v^{(2)}\}$ are linearly independent, and the general solution is $x(x) = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $v^{(\pm)} = a \pm bi$, the complex-valued fundamental solutions are $x^{(\pm)} = e^{\alpha t} (a \pm bi) [\cos(\beta t) + i \sin(\beta t)]$.

Real-valued fundamental solutions are $x^{(1)} = e^{\alpha t} [a \cos(\beta t) - b \sin(\beta t)]$, $x^{(2)} = e^{\alpha t} [a \sin(\beta t) + b \cos(\beta t)]$. 
Systems of linear Equations.

**Summary:** Find solutions of \( x' = Ax \), with \( A \) a \( 2 \times 2 \) matrix.

First find the eigenvalues \( \lambda_i \) and the eigenvectors \( v^{(i)} \) of \( A \).

(a) If \( \lambda_1 \neq \lambda_2 \), real, then \( \{v^{(1)}, v^{(2)}\} \) are linearly independent, and the general solution is \( x(x) = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t} \).

(b) If \( \lambda_1 \neq \lambda_2 \), complex, then denoting \( \lambda_{\pm} = \alpha \pm \beta i \) and \( v^{(\pm)} = a \pm b i \), the complex-valued fundamental solutions

\[
x^{(\pm)} = (a \pm bi) e^{(\alpha \pm \beta i) t}.
\]

\[
x^{(\pm)} = e^{\alpha t} (a \pm bi) \left[ \cos(\beta t) + i \sin(\beta t) \right].
\]

Real-valued fundamental solutions are

\[
x^{(1)} = e^{\alpha t} \left[ a \cos(\beta t) - b \sin(\beta t) \right],
\]

\[
x^{(2)} = e^{\alpha t} \left[ a \sin(\beta t) + b \cos(\beta t) \right].
\]
Systems of linear Equations.

**Summary:** Find solutions of $\mathbf{x}' = A\mathbf{x}$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $\mathbf{v}^{(i)}$ of $A$. 
Systems of linear Equations.

**Summary:** Find solutions of \( x' = Ax \), with \( A \) a \( 2 \times 2 \) matrix. First find the eigenvalues \( \lambda_i \) and the eigenvectors \( v^{(i)} \) of \( A \).

(c) If \( \lambda_1 = \lambda_2 = \lambda \), real,
Systems of linear Equations.

**Summary:** Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{v^{(1)}, v^{(2)}\}$ are linearly independent,
Systems of linear Equations.

Summary: Find solutions of \( \mathbf{x}' = A \mathbf{x} \), with \( A \) a \( 2 \times 2 \) matrix.

First find the eigenvalues \( \lambda_i \) and the eigenvectors \( \mathbf{v}^{(i)} \) of \( A \).

(c) If \( \lambda_1 = \lambda_2 = \lambda \), real, and their eigenvectors \( \{ \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \} \) are linearly independent, then the general solution is

\[
\mathbf{x}(t) = c_1 \mathbf{v}^{(1)} e^{\lambda t} + c_2 \mathbf{v}^{(2)} e^{\lambda t}.
\]
Summary: Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{v^{(1)}, v^{(2)}\}$ are linearly independent, then the general solution is

$$x(t) = c_1 v^{(1)} e^{\lambda t} + c_2 v^{(2)} e^{\lambda t}.$$ 

(d) If $\lambda_1 = \lambda_2 = \lambda$, real,
Summary: Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{v^{(1)}, v^{(2)}\}$ are linearly independent, then the general solution is

$$x(x) = c_1 v^{(1)} e^{\lambda t} + c_2 v^{(2)} e^{\lambda t}.$$ 

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $v$, then find $w$ solution of $(A - \lambda I)w = v$. Then fundamental solutions to the differential equation are given by

$$x^{(1)} = v e^{\lambda t},$$

$$x^{(2)} = (v t + w) e^{\lambda t}.$$ 

Then, the general solution is

$$x = c_1 v e^{\lambda t} + c_2 (v t + w) e^{\lambda t}.$$
Summary: Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{v^{(1)}, v^{(2)}\}$ are linearly independent, then the general solution is

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Systems of linear Equations.

**Summary:** Find solutions of \( \mathbf{x}' = A \mathbf{x} \), with \( A \) a \( 2 \times 2 \) matrix.

First find the eigenvalues \( \lambda_i \) and the eigenvectors \( \mathbf{v}^{(i)} \) of \( A \).

(c) If \( \lambda_1 = \lambda_2 = \lambda \), real, and their eigenvectors \( \{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\} \) are linearly independent, then the general solution is

\[
\mathbf{x}(t) = c_1 \mathbf{v}^{(1)} e^{\lambda t} + c_2 \mathbf{v}^{(2)} e^{\lambda t}.
\]

(d) If \( \lambda_1 = \lambda_2 = \lambda \), real, and there is only one eigendirection \( \mathbf{v} \), then find \( \mathbf{w} \) solution of \( (A - \lambda I)\mathbf{w} = \mathbf{v} \). Then fundamental solutions to the differential equation are given by

\[
\mathbf{x}^{(1)} = \mathbf{v} e^{\lambda t},
\]
Systems of linear Equations.

Summary: Find solutions of \( x' = Ax \), with \( A \) a \( 2 \times 2 \) matrix.

First find the eigenvalues \( \lambda_i \) and the eigenvectors \( v^{(i)} \) of \( A \).

(c) If \( \lambda_1 = \lambda_2 = \lambda \), real, and their eigenvectors \( \{v^{(1)}, v^{(2)}\} \) are linearly independent, then the general solution is

\[
x(x) = c_1 v^{(1)} e^{\lambda t} + c_2 v^{(2)} e^{\lambda t}.
\]

(d) If \( \lambda_1 = \lambda_2 = \lambda \), real, and there is only one eigendirection \( v \), then find \( w \) solution of \( (A - \lambda I)w = v \). Then fundamental solutions to the differential equation are given by

\[
x^{(1)} = v e^{\lambda t}, \quad x^{(2)} = (v t + w) e^{\lambda t}.
\]
Systems of linear Equations.

**Summary:** Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{v^{(1)}, v^{(2)}\}$ are linearly independent, then the general solution is

$$x(x) = c_1 v^{(1)} e^{\lambda t} + c_2 v^{(2)} e^{\lambda t}.$$  

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$$x^{(1)} = v e^{\lambda t}, \quad x^{(2)} = (v t + w) e^{\lambda t}.$$  

Then, the general solution is

$$x = c_1 v e^{\lambda t} + c_2 (v t + w) e^{\lambda t}.$$
Systems of linear Equations.

Example

Find the solution to: \( \mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)
Systems of linear Equations.

Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix}
\]
Systems of linear Equations.

Example

Find the solution to: \( \mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8
\]
Systems of linear Equations.

Example

Find the solution to: \( \mathbf{x}' = A \mathbf{x} \), \( \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \), \( A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \).

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]

\[
\begin{align*}
\text{Case } \lambda + 1 &= 3, \\
A - 3I &= \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \\
&\Rightarrow v_1 = 2v_2 \\
\text{Case } \lambda - 1 &= -3, \\
A + 3I &= \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\
&\Rightarrow v_1 = -v_2
\end{align*}
\]
Systems of linear Equations.

Example
Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:
\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]
\[
p(\lambda) = \lambda^2 - 9 = 0
\]
Systems of linear Equations.

Example

Find the solution to: \( \mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]

\[
p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.
\]
Systems of linear Equations.

Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]

\[
p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_\pm = \pm 3.
\]

Case \( \lambda_+ = 3, \)
Systems of linear Equations.

Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]

\[
p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.
\]

Case \( \lambda_+ = 3, \)

\( A - 3I \)
Systems of linear Equations.

Example

Find the solution to: \( \mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]

\[
p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.
\]

Case \( \lambda_+ = 3 \),

\[
A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix}
\]
Systems of linear Equations.

Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]

\[
p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.
\]

Case \( \lambda_+ = 3 \),

\[
A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}
\]
Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]

\[
p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.
\]

Case \( \lambda_+ = 3 \),

\[
A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2
\]
Systems of linear Equations.

Example

Find the solution to: \( \mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]

\[
p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.
\]

Case \( \lambda_+ = 3 \),

\[
A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = 2\mathbf{v}_2 \quad \Rightarrow \quad \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]
Systems of linear Equations.

Example
Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:
\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]
\[
p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.
\]

Case \( \lambda_{+} = 3 \),
\[
A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = 2v_2 \quad \Rightarrow \quad v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

Case \( \lambda_{-} = -3 \),
Systems of linear Equations.

Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \).

Solution:

\[
p(\lambda) = \begin{vmatrix} 1 - \lambda & 4 \\ 2 & -1 - \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]

\[
p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.
\]

Case \( \lambda_+ = 3 \),

\[
A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

Case \( \lambda_- = -3 \),

\[
A + 3I
\]
Systems of linear Equations.

Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \).

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]

\[
p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.
\]

Case \( \lambda_+ = 3 \),

\[
A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

Case \( \lambda_- = -3 \),

\[
A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}
\]
Systems of linear Equations.

Example

Find the solution to: \( \mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
\]

\[
p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_\pm = \pm 3.
\]

Case \( \lambda_+ = 3, \)

\[
A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_1 = 2\mathbf{v}_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

Case \( \lambda_- = -3, \)

\[
A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\]
Systems of linear Equations.

Example

Find the solution to: \( \mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
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\[
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\]
Systems of linear Equations.

Example

Find the solution to:  \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
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Case \( \lambda_+ = 3, \)

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A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow v^{(+) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}
\]

Case \( \lambda_- = -3, \)

\[
A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -v_2 \Rightarrow v^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
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Systems of linear Equations.

Example

Find the solution to: \( x' = A x \), \( x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \), \( A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \).

Solution: Recall: \( \lambda_{\pm} = \pm 3 \), \( \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \), \( \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \).
Example

Find the solution to: $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$.

Solution: Recall: $\lambda_{\pm} = \pm 3$, $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$. 
Systems of linear Equations.

Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution: Recall: \( \lambda = \pm 3, \quad v^+(+) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad v^-(−) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \)

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The initial condition implies,

\[
\begin{bmatrix} 3 \\ 2 \end{bmatrix} = x(0)
\]
Systems of linear Equations.

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Find the solution to:  \( \mathbf{x}' = A \mathbf{x} \),  \( \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \),  \( A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \).

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\]
Systems of linear Equations.

Example

Find the solution to: \( x' = Ax \), \( x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution: Recall: \( \lambda_{\pm} = \pm 3, \quad v^{(+) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad v^{-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \)

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\]

\[
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2 + 1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\]
Systems of linear Equations.

Example

Find the solution to:  \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

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Systems of linear Equations.

Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

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\]

We conclude: \( x(t) = \frac{5}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}. \) △
Review for Final Exam.

- Fourier Series expansions (Chptr. 6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- **Laplace transforms (Chptr. 4).**
- Power Series Methods (Chptr. 3).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).
Laplace transforms.

Summary:

- Main Properties:
Laplace transforms.

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\[ \mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0); \quad (18) \]
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- **Main Properties:**
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  \mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0); 
  \]
  \[
  e^{-cs} \mathcal{L}[f(t)] = \mathcal{L}[u_c(t) f(t - c)]; 
  \]
  \[
  \mathcal{L}[f(t)] \bigg|_{s=c} = \mathcal{L}[e^{ct} f(t)]; 
  \]

- **Convolutions:**
  \[
  \mathcal{L}[f \ast g(t)] = \mathcal{L}[f(t)] \mathcal{L}[g(t)]; 
  \]

- **Partial fraction decompositions, completing the squares.**
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Example

Use L.T. to find the solution to the IVP

\[ y'' + 9y = u_5(t), \quad y(0) = 3, \quad y'(0) = 2. \]
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Solution: Compute \( \mathcal{L}[y''] + 9 \mathcal{L}[y] = \mathcal{L}[u_5(t)] \)
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\[ (s^2 + 9) \mathcal{L}[y] - 3s - 2 = \frac{e^{-5s}}{s} \]

\[ \mathcal{L}[y] = \frac{(3s + 2)}{(s^2 + 9)} + e^{-5s} \frac{1}{s(s^2 + 9)}. \]
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Use L.T. to find the solution to the IVP
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Solution: Compute \( \mathcal{L}[y''] + 9 \mathcal{L}[y] = \mathcal{L}[u_5(t)] = \frac{e^{-5s}}{s} \), and recall,

\[ \mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy(0) - y'(0) \quad \Rightarrow \quad \mathcal{L}[y''] = s^2 \mathcal{L}[y] - 3s - 2. \]

\[ (s^2 + 9) \mathcal{L}[y] - 3s - 2 = \frac{e^{-5s}}{s}. \]

\[ \mathcal{L}[y] = \frac{(3s + 2)}{(s^2 + 9)} + e^{-5s} \frac{1}{s(s^2 + 9)}. \]

\[ \mathcal{L}[y] = 3 \frac{s}{(s^2 + 9)} + \frac{2}{3} \frac{3}{(s^2 + 9)} + e^{-5s} \frac{1}{s(s^2 + 9)}. \]
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Use L.T. to find the solution to the IVP

\[ y'' + 9y = u_5(t), \quad y(0) = 3, \quad y'(0) = 2. \]

Solution: Recall \( L[y] = 3 \frac{s}{(s^2 + 9)} + \frac{2}{3} \frac{3}{(s^2 + 9)} + e^{-5s} \frac{1}{s(s^2 + 9)}. \)
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\[ \mathcal{L}[y] = 3 \mathcal{L}[\cos(3t)] + \frac{2}{3} \mathcal{L}[\sin(3t)] + e^{-5s} \frac{1}{s(s^2 + 9)}. \]
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Partial fractions on

\[ H(s) = \frac{1}{s(s^2 + 9)} \]
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\[ H(s) = \frac{1}{s(s^2 + 9)} = \frac{a}{s} + \frac{(bs + c)}{(s^2 + 9)} = \frac{a(s^2 + 9) + (bs + c)s}{s(s^2 + 9)}, \]
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\[ 1 = as^2 + 9a + bs^2 + cs \]
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Use L.T. to find the solution to the IVP

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Solution: Recall

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\[ 1 = as^2 + 9a + bs^2 + cs = (a + b)s^2 + cs + 9a \]

\[ a = \frac{1}{9}, \]
Laplace transforms.

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Use L.T. to find the solution to the IVP

\[ y'' + 9y = u_5(t), \quad y(0) = 3, \quad y'(0) = 2. \]

Solution: Recall \( \mathcal{L}[y] = 3 \frac{s}{s^2 + 9} + 2 \frac{3}{3(s^2 + 9)} + e^{-5s} \frac{1}{s(s^2 + 9)}. \)

\[ \mathcal{L}[y] = 3 \mathcal{L}[\cos(3t)] + \frac{2}{3} \mathcal{L}[\sin(3t)] + e^{-5s} \frac{1}{s(s^2 + 9)}. \]

Partial fractions on \( H(s) = \frac{1}{s(s^2 + 9)} = \frac{a}{s} + \frac{(bs + c)}{(s^2 + 9)} = \frac{a(s^2 + 9) + (bs + c)s}{s(s^2 + 9)}, \)

\[ 1 = as^2 + 9a + bs^2 + cs = (a + b)s^2 + cs + 9a \]

\[ a = \frac{1}{9}, \quad c = 0, \]
Laplace transforms.

Example

Use L.T. to find the solution to the IVP

\[ y'' + 9y = u_5(t), \quad y(0) = 3, \quad y'(0) = 2. \]

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\[ \mathcal{L}[y] = 3 \frac{s}{s^2 + 9} + \frac{2}{3} \frac{3}{s^2 + 9} + e^{-5s} \frac{1}{s(s^2 + 9)}. \]

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Example

Use L.T. to find the solution to the IVP

\[ y'' + 9y = u_5(t), \quad y(0) = 3, \quad y'(0) = 2. \]

Solution: So, \( L[y] = 3 L[\cos(3t)] + \frac{2}{3} L[\sin(3t)] + e^{-5s} H(s), \) and

\[
H(s) = \frac{1}{s(s^2 + 9)} = \frac{1}{9} \left[ \frac{1}{s} - \frac{s}{s^2 + 9} \right] = \frac{1}{9} \left( L[u(t)] - L[\cos(3t)]\right)
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\]

Therefore, we conclude that,

\[
y(t) = 3 \cos(3t) + \frac{2}{3} \sin(3t) + \frac{u_5(t)}{9} \left[ 1 - \cos(3(t - 5)) \right].
\]
Review for Final Exam.

- Fourier Series expansions (Chptr. 6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- **Power Series Methods (Chptr. 3).**
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).
Power series solutions (Chptr. 3).

Summary: Solve: \( a(x) y'' + b(x) y' + c(x) y = 0 \) near \( x_0 \).
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Solutions: If \( y(x) = |x - x_0|^r \), then \( r \) is solution of the indicial equation
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Power series solutions (Chptr. 3).

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$$y(x) = (c_1 + c_2 \ln |x - x_0|) |x - x_0|^r.$$
Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$. 
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\begin{align*}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad xy = \sum_{n=0}^{\infty} a_n x^{(n+1)}. \\
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y''(x) &= \sum_{n=0}^{\infty} n(n-1) a_n x^{(n-2)}.
\end{align*}
\]

\[
\sum_{n=0}^{\infty} n(n-1) a_n x^{(n-2)} + \sum_{n=0}^{\infty} (-3n) a_n x^{(n-1)} + \sum_{n=0}^{\infty} a_n x^{(n+1)} = 0.
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Power series solutions (Chptr. 3).

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\[
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$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} (-3)(n+1)a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$
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Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} (-3)(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$
Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$
\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} (-3)(n+1)a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0.
$$

$$(2)(1)a_2 + (-3)(1)a_1 + \sum_{n=1}^{\infty} \left[ (n + 2)(n + 1)a_{n+2} - 3(n + 1)a_{n+1} + a_{n-1} \right] x^n = 0$$
Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$
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$$

$$
(2)(1)a_2 + (-3)(1)a_1 + \sum_{n=1}^{\infty} [(n + 2)(n + 1)a_{n+2} - 3(n + 1)a_{n+1} + a_{n-1}] x^n = 0
$$

We conclude: $2a_2 - 3a_1 = 0$, and

$$
(n + 2)(n + 1)a_{n+2} - 3(n + 1)a_{n+1} + a_{n-1} = 0, \quad n \geq 1.
$$
Power series solutions (Chptr. 3).

Example
Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$. 

Solution:
Recall: $a_2^2 - 3a_1 = 0$, and $(n + 2)(n + 1)a_n + 2 - 3(n + 1)a_{n + 1} + a_n - 1 = 0$, $n \geq 1$.
Therefore, $a_2 = 3a_1$, and $n = 1$ in the other equation implies $(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \Rightarrow a_3 = a_2 - a_0$.

Using the equation for $a_2$, we obtain $a_3 = 3a_1 - a_0$.

$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ 

$y(x) = a_0 + a_1 x + 3a_1 x^2 + (3a_1 - a_0) x^3 + \cdots$
Power series solutions (Chptr. 3).

Example
Find the first two terms on the power series expansion around \( x_0 = 0 \) of each fundamental solution of \( y'' - 3y' + xy = 0 \).

Solution: Recall: \( 2a_2 - 3a_1 = 0 \), and

\[
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Example

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\[
(n + 2)(n + 1)a_{n+2} - 3(n + 1)a_{n+1} + a_{n-1} = 0, \quad n \geq 1.
\]

Therefore, \( a_2 = \frac{3}{2} a_1 \).
Power series solutions (Chptr. 3).

Example
Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n + 2)(n + 1)a_{n+2} - 3(n + 1)a_{n+1} + a_{n-1} = 0, \quad n \geq 1.$$ 

Therefore, $a_2 = \frac{3}{2}a_1$, and $n = 1$ in the other equation implies
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Solution: Recall: $2a_2 - 3a_1 = 0$, and

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Therefore, $a_2 = \frac{3}{2} a_1$, and $n = 1$ in the other equation implies

$$ (3)(2)a_3 - 3(2)a_2 + a_0 = 0 $$
Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

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Therefore, $a_2 = \frac{3}{2} a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$
Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

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Using the equation for $a_2$ we obtain
Power series solutions (Chptr. 3).

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Solution: Recall: $2a_2 - 3a_1 = 0$, and

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Power series solutions (Chptr. 3).

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Find the first two terms on the power series expansion around \( x_0 = 0 \) of each fundamental solution of \( y'' - 3y' + xy = 0 \).

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\[
(n + 2)(n + 1)a_{n+2} - 3(n + 1)a_{n+1} + a_{n-1} = 0, \quad n \geq 1.
\]
Therefore, \( a_2 = \frac{3}{2} a_1 \), and \( n = 1 \) in the other equation implies
\[
(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.
\]
Using the equation for \( a_2 \) we obtain \( a_3 = \frac{3}{2} a_1 - \frac{a_0}{6} \).

\[ y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \]
Example
Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n + 2)(n + 1)a_{n+2} - 3(n + 1)a_{n+1} + a_{n-1} = 0, \quad n \geq 1.$$ 

Therefore, $a_2 = \frac{3}{2} a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$ 

Using the equation for $a_2$ we obtain $a_3 = \frac{3}{2} a_1 - \frac{a_0}{6}$.

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y(x) = a_0 + a_1 x + \frac{3}{2} a_1 x^2 + \left(\frac{3}{2} a_1 - \frac{a_0}{6}\right) x^3 + \cdots$$
Example
Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $y(x) = a_0 + a_1 x + \frac{3}{2} a_1 x^2 + \left(\frac{3}{2} a_1 - \frac{a_0}{6}\right) x^3 + \cdots$. 
Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $y(x) = a_0 + a_1 x + \frac{3}{2} a_1 x^2 + \left(\frac{3}{2} a_1 - \frac{a_0}{6}\right) x^3 + \cdots$.

$$y(x) = a_0 \left(1 - \frac{1}{6} x^3 + \cdots\right) + a_1 \left(x + \frac{3}{2} x^2 + \frac{3}{2} x^3 + \cdots\right),$$
Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around \( x_0 = 0 \) of each fundamental solution of \( y'' - 3y' + xy = 0 \).

Solution: Recall: 
\[
y(x) = a_0 + a_1 x + \frac{3}{2} a_1 x^2 + \left( \frac{3}{2} a_1 - \frac{a_0}{6} \right) x^3 + \cdots.
\]

\[
y(x) = a_0 \left( 1 - \frac{1}{6} x^3 + \cdots \right) + a_1 \left( x + \frac{3}{2} x^2 + \frac{3}{2} x^3 + \cdots \right),
\]

We conclude that:

\[
y_1(x) = 1 - \frac{1}{6} x^3 + \cdots,
\]

\[
y_2(x) = x + \frac{3}{2} x^2 + \frac{3}{2} x^3 + \cdots.
\]

\( \triangle \)
Review for Final Exam.

- Fourier Series expansions (Chptr. 6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Power Series Methods (Chptr. 3).
- **Second order linear equations (Chptr. 2).**
- First order differential equations (Chptr. 1).
Second order linear equations.

Summary: Solve $y'' + a_1 y' + a_0 y = g(t)$. 
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \),
Second order linear equations.

Summary: Solve $y'' + a_1 y' + a_0 y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where $r$ is a root of $p(r) = r^2 + a_1 r + a_0$. 
Second order linear equations.

Summary: Solve $y'' + a_1 y' + a_0 y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where $r$ is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real,
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 r + a_0 \).

(a) If \( r_1 \neq r_2 \), real, then the general solution is

\[
y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.
\]
Second order linear equations.

**Summary:** Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 r + a_0 \).

(a) If \( r_1 \neq r_2 \), real, then the general solution is
\[ y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \]

(b) If \( r_1 \neq r_2 \), complex,
Second order linear equations.

**Summary:** Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 r + a_0 \).

(a) If \( r_1 \neq r_2 \), real, then the general solution is

\[
y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.
\]

(b) If \( r_1 \neq r_2 \), complex, then denoting \( r_\pm = \alpha \pm \beta i \),
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 r + a_0 \).

(a) If \( r_1 \neq r_2 \), real, then the general solution is

\[
  y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.
\]

(b) If \( r_1 \neq r_2 \), complex, then denoting \( r_\pm = \alpha \pm \beta i \), complex-valued fundamental solutions are

\[
  y_\pm(t) = e^{(\alpha \pm \beta i)t}
\]
Second order linear equations.

Summary: Solve $y'' + a_1 y' + a_0 y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where $r$ is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$ 

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t} \left[ \cos(\beta t) \pm i \sin(\beta t) \right],$$

If $r_1 = r_2 = r$, real, then the general solution is

$$y(t) = (c_1 + c_2 t) e^{rt}.$$
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 + a_0 \).

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\]

(b) If \( r_1 \neq r_2 \), complex, then denoting \( r_\pm = \alpha \pm \beta i \), complex-valued fundamental solutions are
\[
y_\pm(t) = e^{(\alpha \pm \beta i)t} \iff y_\pm(t) = e^{\alpha t} \left[ \cos(\beta t) \pm i \sin(\beta t) \right],
\]
and real-valued fundamental solutions are
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 r + a_0 \).

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\]

(b) If \( r_1 \neq r_2 \), complex, then denoting \( r_\pm = \alpha \pm \beta i \), complex-valued fundamental solutions are

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y_\pm(t) = e^{(\alpha \pm \beta i)t} \iff y_\pm(t) = e^{\alpha t} \left[ \cos(\beta t) \pm i \sin(\beta t) \right],
\]

and real-valued fundamental solutions are

\[
y_1(t) = e^{\alpha t} \cos(\beta t),
\]

\[
y_2(t) = e^{\alpha t} \sin(\beta t).
\]
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).

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y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.
\]

(b) If \( r_1 \neq r_2 \), complex, then denoting \( r_{\pm} = \alpha \pm \beta i \), complex-valued fundamental solutions are
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y_{\pm}(t) = e^{(\alpha \pm \beta i)t} \iff y_{\pm}(t) = e^{\alpha t} \left[ \cos(\beta t) \pm i \sin(\beta t) \right],
\]
and real-valued fundamental solutions are
\[
y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).
\]
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 r + a_0 \).

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\]

(b) If \( r_1 \neq r_2 \), complex, then denoting \( r_\pm = \alpha \pm \beta i \), complex-valued fundamental solutions are
\[
y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t} \left[ \cos(\beta t) \pm i \sin(\beta t) \right],
\]
and real-valued fundamental solutions are
\[
y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).
\]

If \( r_1 = r_2 = r \), real,
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 r + a_0 \).

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y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.
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(b) If \( r_1 \neq r_2 \), complex, then denoting \( r_{\pm} = \alpha \pm \beta i \), complex-valued fundamental solutions are
\[
y_{\pm}(t) = e^{(\alpha \pm \beta i)t} \iff y_{\pm}(t) = e^{\alpha t} \left[ \cos(\beta t) \pm i \sin(\beta t) \right],
\]
and real-valued fundamental solutions are
\[
y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).
\]
If \( r_1 = r_2 = r \), real, then the general solution is
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y(t) = (c_1 + c_2 t) e^{rt}.
\]
Second order linear equations.

**Remark:** Case (c) is solved using the *reduction of order method*.
Second order linear equations.

Remark: Case (c) is solved using the \textit{reduction of order method}. See page 170 in the textbook.
Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.
Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$. 
Second order linear equations.

Remark: Case (c) is solved using the reduction of order method. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients:
Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution $y_p$
Second order linear equations.

Remark: Case (c) is solved using the reduction of order method. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution $y_p$ using the guessing table, $g \rightarrow y_p$. 
Second order linear equations.

Remark: Case (c) is solved using the reduction of order method. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution $y_p$ using the guessing table, $g \rightarrow y_p$.

(ii) Variation of parameters:
Second order linear equations.

Remark: Case (c) is solved using the reduction of order method. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: \( g \neq 0 \).

(i) Undetermined coefficients: Guess the particular solution \( y_p \) using the guessing table, \( g \rightarrow y_p \).

(ii) Variation of parameters: If \( y_1 \) and \( y_2 \) are fundamental solutions to the homogeneous equation,
Second order linear equations.

Remark: Case (c) is solved using the reduction of order method. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution $y_p$ using the guessing table, $g \rightarrow y_p$.

(ii) Variation of parameters: If $y_1$ and $y_2$ are fundamental solutions to the homogeneous equation, and $W$ is their Wronskian,
Second order linear equations.

Remark: Case (c) is solved using the reduction of order method. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: \( g \neq 0 \).

(i) Undetermined coefficients: Guess the particular solution \( y_p \) using the guessing table, \( g \rightarrow y_p \).

(ii) Variation of parameters: If \( y_1 \) and \( y_2 \) are fundamental solutions to the homogeneous equation, and \( W \) is their Wronskian, then \( y_p = u_1y_1 + u_2y_2 \).
Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution $y_p$ using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If $y_1$ and $y_2$ are fundamental solutions to the homogeneous equation, and $W$ is their Wronskian, then $y_p = u_1y_1 + u_2y_2$, where

$$u'_1 = -\frac{y_2g}{W},$$
Second order linear equations.

Remark: Case (c) is solved using the reduction of order method. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: \( g \neq 0 \).

(i) Undetermined coefficients: Guess the particular solution \( y_p \) using the guessing table, \( g \rightarrow y_p \).

(ii) Variation of parameters: If \( y_1 \) and \( y_2 \) are fundamental solutions to the homogeneous equation, and \( W \) is their Wronskian, then \( y_p = u_1 y_1 + u_2 y_2 \), where

\[
\begin{align*}
u_1' &= -\frac{y_2 g}{W}, \\
u_2' &= \frac{y_1 g}{W}.
\end{align*}
\]
Second order linear equations.

Example
Knowing that $y_1(x) = x^2$ solves $x^2 y'' - 4x y' + 6y = 0$, with $x > 0$, find a second solution $y_2$ not proportional to $y_1$. 

Solution:
Use the reduction of order method.
We verify that $y_1(x) = x^2$ solves the equation,

$x^2 (2) - 4x (2) + 6x^2 = 0$.

Look for a solution $y_2(x) = v(x) y_1(x)$,

and find an equation for $v$.

$y_2 = x^2 v$,  
$y_2' = x^2 v' + 2xv$, 
$y_2'' = x^2 v'' + 4xv' + 2v$.

$x^2 (x^2 v'' + 4xv' + 2v) - 4x (x^2 v' + 2xv) + 6 (x^2 v) = 0$.

$x^4 v'' + (4x^3 - 4x^3) v' + (2x^2 - 8x^2 + 6x^2) v = 0$.

$v'' = 0$  
$⇒ v = c_1 + c_2 x$  
$⇒ y_2 = c_1 y_1 + c_2 x y_1$.

Choose $c_1 = 0$, $c_2 = 1$.

Hence $y_2(x) = x^3$, and $y_1(x) = x^2$. 

◁
Second order linear equations.

Example
Knowing that $y_1(x) = x^2$ solves $x^2 y'' - 4x y' + 6y = 0$, with $x > 0$, find a second solution $y_2$ not proportional to $y_1$.

Solution: Use the reduction of order method.
Second order linear equations.

Example
Knowing that $y_1(x) = x^2$ solves $x^2 y'' - 4x y' + 6y = 0$, with $x > 0$, find a second solution $y_2$ not proportional to $y_1$.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

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Look for a solution $y_2(x) = v(x) y_1(x)$,
Second order linear equations.

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Second order linear equations.

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Second order linear equations.

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Knowing that \( y_1(x) = x^2 \) solves \( x^2 y'' - 4x y' + 6y = 0 \), with \( x > 0 \), find a second solution \( y_2 \) not proportional to \( y_1 \).

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\[ y_2 = x^2 v, \quad y'_2 = x^2 v' + 2xv, \quad y''_2 = x^2 v'' + 4xv' + 2v. \]
Second order linear equations.

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\]

\[
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$$x^4 v'' + (4x^3 - 4x^3) v' + (2x^2 - 8x^2 + 6x^2) v = 0.$$ 

Choose $c_1 = 0$, $c_2 = 1$. Hence $y_2(x) = x^3$, and $y_1(x) = x^2$. \[\blacksquare\]
Second order linear equations.

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x^4 v'' + (4x^3 - 4x^3) v' + (2x^2 - 8x^2 + 6x^2) v = 0.
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y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4x v' + 2v.
\]

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\]

\[
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\]

\[
v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x
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x^2(x^2 v'' + 4xv' + 2v) - 4x(x^2 v' + 2xv) + 6(x^2 v) = 0.
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x^4 v'' + (4x^3 - 4x^3) v' + (2x^2 - 8x^2 + 6x^2) v = 0.
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v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2x y_1.
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Choose $c_1 = 0$, $c_2 = 1$. Hence $y_2(x) = x^3$, and $y_1(x) = x^2$. ◄
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$
Second order linear equations.

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Second order linear equations.

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$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}. $$

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$$y(t) = e^{rt},$$
Second order linear equations.

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Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3$$
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$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$ 

$$r_{\pm} = \frac{1}{2} [2 \pm \sqrt{4 + 12}]$$
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Example
Find the solution \( y \) to the initial value problem

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y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.
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Solution: (1) Solve the homogeneous equation.

\[
y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.
\]

\[
r_{\pm} = \frac{1}{2} [2 \pm \sqrt{4 + 12}] = \frac{1}{2} [2 \pm \sqrt{16}] = 1 \pm 2
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Find the solution \( y \) to the initial value problem

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Fundamental solutions: \( y_1(t) = e^{3t} \) and \( y_2(t) = e^{-t}. \)
Second order linear equations.

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(2) Guess $y_p$.  

Second order linear equations.

Example

Find the solution \( y \) to the initial value problem

\[
y'' - 2y' - 3y = 3 \, e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.
\]

Solution: 

(1) Solve the homogeneous equation.

\[
y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.
\]

\[
r_{\pm} = \frac{1}{2} \left[ 2 \pm \sqrt{4 + 12} \right] = \frac{1}{2} \left[ 2 \pm \sqrt{16} \right] = 1 \pm 2 \quad \Rightarrow \quad \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}
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(2) Guess \( y_p \). Since \( g(t) = 3 \, e^{-t} \)
Second order linear equations.

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y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.
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Solution: (1) Solve the homogeneous equation.

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Fundamental solutions: \( y_1(t) = e^{3t} \) and \( y_2(t) = e^{-t} \).

(2) Guess \( y_p \). Since \( g(t) = 3e^{-t} \) \( \Rightarrow \) \( y_p(t) = ke^{-t} \).
Second order linear equations.

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Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

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Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess $y_p$. Since $g(t) = 3 e^{-t} \Rightarrow y_p(t) = k e^{-t}$.

But this $y_p = k e^{-t}$
Second order linear equations.

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Find the solution $y$ to the initial value problem

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Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess $y_p$. Since $g(t) = 3 e^{-t}$ $\Rightarrow$ $y_p(t) = k e^{-t}$.

But this $y_p = k e^{-t}$ is solution of the homogeneous equation.
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

\[y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.
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Solution: (1) Solve the homogeneous equation.

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Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess $y_p$. Since $g(t) = 3e^{-t} \Rightarrow y_p(t) = ke^{-t}$.

But this $y_p = ke^{-t}$ is solution of the homogeneous equation.
Then propose $y_p(t) = k't e^{-t}$. 

Second order linear equations.

**Example**

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

**Solution:** Recall: $y_p(t) = kt e^{-t}$. 

This is correct, since $te^{-t}$ is not a solution of the homogeneous equation.

Find the undetermined coefficient $k$.

$y'_p = ke^{-t} - kt e^{-t}$,

$y''_p = -2ke^{-t} + kt e^{-t}$.

$(-2ke^{-t} + kt e^{-t}) - 2(ke^{-t} - kt e^{-t}) - 3(kt e^{-t}) = 3e^{-t}$

$(-2 + t - 2 + 2t - 3t)ke^{-t} = 3e^{-t}$

$\Rightarrow -4k = 3 \Rightarrow k = -\frac{3}{4}$.

We obtain:

$y_p(t) = -\frac{3}{4}t e^{-t}$. 

Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}. $$

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not solution of the homogeneous equation.
Example
Find the solution \( y \) to the initial value problem

\[
y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.
\]

Solution: Recall: \( y_p(t) = k t e^{-t} \). This is correct, since \( t e^{-t} \) is not solution of the homogeneous equation.

(3) Find the undetermined coefficient \( k \).
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not solution of the homogeneous equation.

(3) Find the undetermined coefficient $k$.

$$y_p' = k e^{-t} - kt e^{-t},$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not solution of the homogeneous equation.

(3) Find the undetermined coefficient $k$.

$$y'_p = k e^{-t} - kt e^{-t}, \quad y''_p = -2k e^{-t} + kt e^{-t}.$$
Second order linear equations.

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Find the solution $y$ to the initial value problem

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$$y'_p = k e^{-t} - kt e^{-t}, \quad y''_p = -2k e^{-t} + kt e^{-t}.$$ 

$$(-2k e^{-t} + kt e^{-t}) - 2(k e^{-t} - kt e^{-t}) - 3(kt e^{-t}) = 3 e^{-t}$$
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Example
Find the solution $y$ to the initial value problem

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Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not solution of the homogeneous equation.

(3) Find the undetermined coefficient $k$.

$$y_p' = k e^{-t} - kt e^{-t}, \quad y_p'' = -2k e^{-t} + kt e^{-t}.$$ 

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$$(-2 + t - 2 + 2t - 3t) k e^{-t} = 3 e^{-t} \Rightarrow -4k = 3$$
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Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}. $$

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$$(-2k e^{-t} + kt e^{-t}) - 2(k e^{-t} - kt e^{-t}) - 3(kt e^{-t}) = 3 e^{-t} $$

$$(−2 + t − 2 + 2t − 3t) k e^{-t} = 3 e^{-t} \Rightarrow −4k = 3 \Rightarrow k = −\frac{3}{4}. $$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}. $$

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not solution of the homogeneous equation.

(3) Find the undetermined coefficient $k$.

$$y_p' = k e^{-t} - kt e^{-t}, \quad y_p'' = -2k e^{-t} + kt e^{-t}. $$

$$(-2k e^{-t} + kt e^{-t}) - 2(k e^{-t} - kt e^{-t}) - 3(kt e^{-t}) = 3 e^{-t}$$

$$(-2 + t - 2 + 2t - 3t) k e^{-t} = 3 e^{-t} \Rightarrow -4k = 3 \Rightarrow k = -\frac{3}{4}. $$

We obtain: $y_p(t) = -\frac{3}{4} t e^{-t}$. 
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

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Second order linear equations.

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(4) Find the general solution:
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}. $$

Solution: Recall: $y_p(t) = -\frac{3}{4} t e^{-t}$.

(4) Find the general solution: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t}$. 
Second order linear equations.

Example
Find the solution \( y \) to the initial value problem

\[
y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.
\]

Solution: Recall: \( y_p(t) = -\frac{3}{4} t e^{-t} \).

(4) Find the general solution: \( y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t} \).

(5) Impose the initial conditions.
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

Solution: Recall:  $y_p(t) = -\frac{3}{4}te^{-t}$.

(4) Find the general solution:  $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4}te^{-t}$.

(5) Impose the initial conditions. The derivative function is
Second order linear equations.

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Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y_p(t) = -\frac{3}{4}t e^{-t}$.

(4) Find the general solution: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4}t e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1 e^{3t} - c_2 e^{-t} - \frac{3}{4}(e^{-t} - t e^{-t}).$$
Second order linear equations.

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Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

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$$y'(t) = 3c_1 e^{3t} - c_2 e^{-t} - \frac{3}{4} (e^{-t} - t e^{-t}).$$

$$1 = y(0)$$
Second order linear equations.

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Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

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$$1 = y(0) = c_1 + c_2,$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

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$$\frac{1}{4} = y'(0)$$
Second order linear equations.

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\[
1 = y(0) = c_1 + c_2, \quad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}.
\]
Second order linear equations.

**Example**

Find the solution \( y \) to the initial value problem

\[ y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}. \]

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\[ 1 = y(0) = c_1 + c_2, \quad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}. \]

\[ \begin{cases} c_1 + c_2 = 1, \\ 3c_1 - c_2 = 1 \end{cases} \]
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

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$$1 = y(0) = c_1 + c_2, \quad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}.$$ 

$$\begin{align*}
    c_1 + c_2 &= 1, \\
    3c_1 - c_2 &= 1
\end{align*} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
Second order linear equations.

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Find the solution $y$ to the initial value problem

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Second order linear equations.

Example
Find the solution \( y \) to the initial value problem

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y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.
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Solution: Recall: \( y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t} \), and

\[
\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.
\]
Second order linear equations.

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Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}. $$

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Since $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$. 

Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$ 

Since $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$, we obtain,

$$y(t) = \frac{1}{2} \left( e^{3t} + e^{-t} \right) - \frac{3}{4} t e^{-t}.$$ 

$\triangle$
Review for Final Exam.

- Fourier Series expansions (Chptr. 6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Power Series Methods (Chptr. 3).
- Second order linear equations (Chptr. 2).
- **First order differential equations (Chptr. 1).**
First order differential equations.

Summary:

▶ Linear, first order equations: \( y' + p(t) y = q(t) \).

▶ Separable, non-linear equations: \( h(y) y' = g(t) \).

Integrate with the substitution: \( u = y(t) \), \( du = y(t) \, dt \), that is, \( \int h(u) \, du = \int g(t) \, dt + c \).

The solution can be found in implicit or explicit form.

▶ Homogeneous equations can be converted into separable equations.

▶ Applications: Modeling problems from Sect. 2.3.
First order differential equations.

Summary:

- **Linear, first order equations:** \( y' + p(t) y = q(t) \).
  
  Use the integrating factor method: \( \mu(t) = e^{\int p(t) \, dt} \).
First order differential equations.

Summary:

- **Linear**, first order equations: \( y' + p(t) y = q(t) \).
  
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  Integrate with the substitution: \( u = y(t), \ du = y'(t) \, dt, \)
First order differential equations.

Summary:

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Applications: Modeling problems from Sect. 2.3.
First order differential equations.

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- **Homogeneous equations** can be converted into separable equations.

- **Applications**: Modeling problems from Sect. 2.3.
First order differential equations.

Summary:

- Bernoulli equations: \( y' + p(t) \, y = q(t) \, y^n \), with \( n \in \mathbb{R} \).
First order differential equations.

Summary:
- Bernoulli equations: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).

Read page 77 in the textbook,
First order differential equations.

Summary:
- Bernoulli equations: \( y' + p(t)y = q(t)y^n \), with \( n \in \mathbb{R} \).

Read page 77 in the textbook, page 11 in the Lecture Notes.
First order differential equations.

Summary:

- Bernoulli equations: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).

Read page 77 in the textbook, page 11 in the Lecture Notes.

A Bernoulli equation for \( y \) can be converted into a linear equation for
First order differential equations.

Summary:

- Bernoulli equations: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).
  
  Read page 77 in the textbook, page 11 in the Lecture Notes.
  A Bernoulli equation for \( y \) can be converted into a linear equation for \( v = \frac{1}{y^{n-1}} \).
First order differential equations.

Summary:

- **Bernoulli equations**: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).

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  A Bernoulli equation for \( y \) can be converted into a linear equation for \( v = \frac{1}{y^{n-1}} \).

- **Exact equations** and integrating factors.
First order differential equations.

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- Bernoulli equations: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).

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- Exact equations and integrating factors.

  \[ N(x, y) y' + M(x, y) = 0. \]
First order differential equations.

Summary:

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- **Exact equations** and integrating factors.

  \[ N(x, y) y' + M(x, y) = 0. \]

  The equation is exact iff \( \partial_x N = \partial_y M \).
First order differential equations.

Summary:

- **Bernoulli equations**: \( y' + p(t)y = q(t)y^n \), with \( n \in \mathbb{R} \).
  
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  A Bernoulli equation for \( y \) can be converted into a linear equation for \( v = \frac{1}{y^{n-1}} \).

- **Exact equations** and integrating factors.
  
  \[ N(x, y)y' + M(x, y) = 0. \]
  
  The equation is exact iff \( \partial_x N = \partial_y M \).
  
  If the equation is exact, then there is a potential function \( \psi \),

\[ \psi(x, y(x)) = c. \]
First order differential equations.

Summary:

▶ **Bernoulli equations:** \[ y' + p(t) y = q(t) y^n, \text{ with } n \in \mathbb{R}. \]

Read page 77 in the textbook, page 11 in the Lecture Notes.

A Bernoulli equation for \( y \) can be converted into a linear equation for \( v = \frac{1}{y^{n-1}}. \)

▶ **Exact equations** and integrating factors.

\[
N(x, y) y' + M(x, y) = 0.
\]

The equation is exact iff \( \partial_x N = \partial_y M. \)

If the equation is exact, then there is a potential function \( \psi, \) such that \( N = \partial_y \psi. \)
First order differential equations.

Summary:
- Bernoulli equations: \( y' + p(t)y = q(t)y^n \), with \( n \in \mathbb{R} \).

Read page 77 in the textbook, page 11 in the Lecture Notes.

A Bernoulli equation for \( y \) can be converted into a linear equation for \( v = \frac{1}{y^{n-1}} \).

- Exact equations and integrating factors.

\[ N(x, y)y' + M(x, y) = 0. \]

The equation is exact iff \( \partial_x N = \partial_y M \).

If the equation is exact, then there is a potential function \( \psi \), such that \( N = \partial_y \psi \) and \( M = \partial_x \psi \).
First order differential equations.

Summary:

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- **Exact equations** and integrating factors.

  \[
  N(x, y) y' + M(x, y) = 0.
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  The equation is exact iff \( \partial_x N = \partial_y M \).

  If the equation is exact, then there is a potential function \( \psi \), such that \( N = \partial_y \psi \) and \( M = \partial_x \psi \).

  The solution of the differential equation is
  
  \[
  \psi(x, y(x)) = c.
  \]
First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations. (Just by looking at it: \( y' + a(t)y = b(t) \).)
2. Bernoulli equations. (Just by looking at it: \( y' + a(t)y = b(t)y^n \).)
3. Separable equations. (Few manipulations: \( h(y)y' = g(t) \).)
4. Homogeneous equations. (Several manipulations: \( y' = F(y/t) \).)
5. Exact equations. (Check one equation: \( Ny' + M = 0 \), and \( \partial_t N = \partial_y M \).)
6. Exact equation with integrating factor. (Very complicated to check.)
First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: \( y' + a(t) y = b(t) \).)
First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
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2. Bernoulli equations.
   (Just by looking at it: \( y' + a(t) y = b(t) y^n \).)
First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: \( y' + a(t) y = b(t) \).)

2. Bernoulli equations.
   (Just by looking at it: \( y' + a(t) y = b(t) y^n \).)

   (Few manipulations: \( h(y) y' = g(t) \).)
First order differential equations.

**Advice:** In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. **Linear equations.**
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   (Just by looking at it: \( y' + a(t) y = b(t) y^n \).)

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   (Few manipulations: \( h(y) y' = g(t) \).)

4. **Homogeneous equations.**
   (Several manipulations: \( y' = F(y/t) \).)

5. **Exact equations.**
   (Check one equation: \( N y' + M = 0 \), and \( \partial_t N = \partial_y M \).)

6. **Exact equation with integrating factor.**
   (Very complicated to check.)
First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: \( y' + a(t) y = b(t) \).)

2. Bernoulli equations.
   (Just by looking at it: \( y' + a(t) y = b(t) y^n \).)

   (Few manipulations: \( h(y) y' = g(t) \).)

   (Several manipulations: \( y' = F(y/t) \).)

5. Exact equations.
   (Check one equation: \( N y' + M = 0 \), and \( \partial_t N = \partial_y M \).)
First order differential equations.

**Advice:** In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. **Linear equations.**
   (Just by looking at it: $y' + a(t)y = b(t)$.)
2. **Bernoulli equations.**
   (Just by looking at it: $y' + a(t)y = b(t)y^n$.)
3. **Separable equations.**
   (Few manipulations: $h(y)y' = g(t)$.)
4. **Homogeneous equations.**
   (Several manipulations: $y' = F(y/t)$.)
5. **Exact equations.**
   (Check one equation: $Ny' + M = 0$, and $\partial_t N = \partial_y M$.)
6. **Exact equation with integrating factor.**
   (Very complicated to check.)
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: The sum of the powers in \( x \) and \( y \) on every term is the same number,
First order differential equations.

Example
Find all solutions of $y' = \frac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in $x$ and $y$ on every term is the same number, two in this example.
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: The sum of the powers in \( x \) and \( y \) on every term is the same number, two in this example. The equation is homogeneous.
First order differential equations.

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$v(x) = \frac{y}{x}$
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v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}. 
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\[
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Example
Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

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First order differential equations.

Example
Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: Recall: \( v' = \frac{1 + v}{v} \). This is a separable equation.
First order differential equations.

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Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: Recall: \( v' = \frac{1 + v}{v} \). This is a separable equation.

\[
\frac{v(x)}{1 + v(x)} \, v'(x) = \frac{1}{x}
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v = \frac{y}{x} \quad \Rightarrow \quad 1 + \frac{y(x)}{x} - \ln\left|1 + \frac{y(x)}{x}\right| = \ln|x| + c. \quad \triangleq
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First order differential equations.

Example

Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$
First order differential equations.

Example

Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$  

Solution: This is a Bernoulli equation,
First order differential equations.

Example
Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$ 

Solution: This is a Bernoulli equation, 

$$y' + y = -e^{2x} y^n,$$
First order differential equations.

Example
Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$  

Solution: This is a Bernoulli equation, $y' + y = -e^{2x} y^n, \quad n = 3.$
First order differential equations.

Example
Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$ 

Solution: This is a Bernoulli equation, $y' + y = -e^{2x} y^n, \quad n = 3$.

Divide by $y^3$. 

First order differential equations.

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Divide by $y^3$. That is, $\frac{y'}{y^3} + \frac{1}{y^2} = -e^{2x}$. 

First order differential equations.

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Let \( v = \frac{1}{y^2} \).
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Let $v = \frac{1}{y^2}$. Since $v' = -2 \frac{y'}{y^3}$, we obtain
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Divide by $y^3$. That is, \[\frac{y'}{y^3} + \frac{1}{y^2} = -e^{2x}.\]

Let $v = \frac{1}{y^2}$. Since $v' = -2 \frac{y'}{y^3}$, we obtain $-\frac{1}{2} v' + v = -e^{2x}$. 

First order differential equations.

Example
Find the solution $y$ to the initial value problem
\[ y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}. \]

Solution: This is a Bernoulli equation, $y' + y = -e^{2x} y^n$, $n = 3$.

Divide by $y^3$. That is, \[ \frac{y'}{y^3} + \frac{1}{y^2} = -e^{2x}. \]

Let $v = \frac{1}{y^2}$. Since $v' = -2 \frac{y'}{y^3}$, we obtain \[-\frac{1}{2} v' + v = -e^{2x}. \]

We obtain the linear equation \[ v' - 2v = 2e^{2x}. \]
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Divide by \( y^3 \). That is, \( \frac{y'}{y^3} + \frac{1}{y^2} = -e^{2x} \).

Let \( v = \frac{1}{y^2} \). Since \( v' = -2 \frac{y'}{y^3} \), we obtain \( -\frac{1}{2} v' + v = -e^{2x} \).

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Use the integrating factor method.
First order differential equations.

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Use the integrating factor method. $\mu(x) = e^{-2x}$. 
First order differential equations.

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$$e^{-2x} v' - 2 e^{-2x} v = 2$$
First order differential equations.

Example
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Use the integrating factor method. $\mu(x) = e^{-2x}$.

$$e^{-2x} v' - 2e^{-2x} v = 2 \quad \Rightarrow \quad (e^{-2x} v)' = 2.$$
First order differential equations.

Example
Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$ 

Solution: Recall: $v = \frac{1}{y^2}$ and $(e^{-2x} v)' = 2$. 

The initial condition $y(0) = \frac{1}{3}$ implies: Choose $y +$. 

$$1 = y(0) = \sqrt{2x + c} \Rightarrow c = 9 \Rightarrow y(x) = e^{-x} \sqrt{2x + 9}.$$
First order differential equations.

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Find the solution $y$ to the initial value problem

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$$e^{-2x} v = 2x + c$$
First order differential equations.

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Find the solution $y$ to the initial value problem

\[ y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}. \]

Solution: Recall: $v = \frac{1}{y^2}$ and $(e^{-2x} v)' = 2$.

\[
e^{-2x} v = 2x + c \implies v(x) = (2x + c) e^{2x}
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$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$ 

Solution: Recall: $v = \frac{1}{y^2}$ and $(e^{-2x} v)' = 2$.

$$e^{-2x} v = 2x + c \Rightarrow v(x) = (2x + c) e^{2x} \Rightarrow \frac{1}{y^2} = (2x + c) e^{2x}.$$
First order differential equations.

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$$e^{-2x} v = 2x + c \Rightarrow v(x) = (2x + c) e^{2x} \Rightarrow \frac{1}{y^2} = (2x + c) e^{2x}.$$

$$y^2 = \frac{1}{e^{2x} (2x + c)}$$
First order differential equations.

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$$e^{-2x} v = 2x + c \Rightarrow v(x) = (2x + c) e^{2x} \Rightarrow \frac{1}{y^2} = (2x + c) e^{2x}.$$ 

$$y^2 = \frac{1}{e^{2x} (2x + c)} \quad \Rightarrow \quad y_\pm(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$
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$$e^{-2x} v = 2x + c \Rightarrow v(x) = (2x + c)e^{2x} \Rightarrow \frac{1}{y^2} = (2x + c)e^{2x}.$$

$$y^2 = \frac{1}{e^{2x}(2x + c)} \Rightarrow y_\pm(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3$
First order differential equations.

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Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.\]$$

Solution: Recall: $v = \frac{1}{y^2}$ and $(e^{-2x} v)' = 2.$

$$e^{-2x} v = 2x + c \implies v(x) = (2x + c) e^{2x} \implies \frac{1}{y^2} = (2x + c) e^{2x}.$$  

$$y^2 = \frac{1}{e^{2x} (2x + c)} \implies y_{\pm}(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$  

The initial condition $y(0) = 1/3 > 0$
First order differential equations.

Example
Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \frac{1}{y^2}$ and $(e^{-2x} v)' = 2.$

$$e^{-2x} v = 2x + c \implies v(x) = (2x + c) e^{2x} \implies \frac{1}{y^2} = (2x + c) e^{2x}.$$

$$y^2 = \frac{1}{e^{2x} (2x + c)} \implies y_{\pm}(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies:
First order differential equations.

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\]

\[
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First order differential equations.

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\frac{1}{3} = y_+(0) = \frac{1}{\sqrt{c}}
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First order differential equations.

Example
Find the solution $y$ to the initial value problem

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$$e^{-2x} v = 2x + c \Rightarrow v(x) = (2x + c) e^{2x} \Rightarrow \frac{1}{y^2} = (2x + c) e^{2x}. $$

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The initial condition $y(0) = 1/3 > 0$ implies: Choose $y_+$.

$$\frac{1}{3} = y_+(0) = \frac{1}{\sqrt{c}} \Rightarrow c = 9 \Rightarrow y(x) = \frac{e^{-x}}{\sqrt{2x + 9}}. $$
First order differential equations.

Example
Find all solutions of \(2xy^2 + 2y + 2x^2yy' + 2xy' = 0\).
First order differential equations.

Example
Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0$.

Solution: Re-write the equation is a more organized way,
First order differential equations.

Example
Find all solutions of \( 2xy^2 + 2y + 2x^2y^2 + 2x \cdot y' = 0 \).

Solution: Re-write the equation is a more organized way,

\[
[2x^2y + 2x]y' + [2xy^2 + 2y] = 0.
\]
First order differential equations.

Example
Find all solutions of \( 2xy^2 + 2y + 2x^2y\ y' + 2x\ y' = 0. \)

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\[ N = [2x^2y + 2x] \]
Example
Find all solutions of \( 2xy^2 + 2y + 2x^2y \frac{dy}{dx} + 2x \frac{dy}{dx} = 0. \)

Solution: Re-write the equation is a more organized way,

\[
[2x^2y + 2x] \frac{dy}{dx} + [2xy^2 + 2y] = 0.
\]

\[ N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2. \]
First order differential equations.

Example
Find all solutions of $2xy^2 + 2y + 2x^2y\ y' + 2x\ y' = 0$.

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x] y' + [2xy^2 + 2y] = 0.$$  \( \hspace{1cm} \)

$$N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.$$  \( \hspace{1cm} \)

$$M = [2xy^2 + 2y]$$
First order differential equations.

Example
Find all solutions of \(2xy^2 + 2y + 2x^2 y \, y' + 2x \, y' = 0\).

Solution: Re-write the equation is a more organized way,
\[
[2x^2y + 2x] \, y' + [2xy^2 + 2y] = 0.
\]

\[N = [2x^2y + 2x] \Rightarrow \partial_x N = 4xy + 2.\]

\[M = [2xy^2 + 2y] \Rightarrow \partial_y M = 4xy + 2.\]
First order differential equations.

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\[
\begin{aligned}
\Rightarrow & \quad \partial_x N = \partial_y M.
\end{aligned}
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First order differential equations.

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\Rightarrow \quad \partial_x N = \partial_y M.
\]
The equation is exact.
First order differential equations.

Example
Find all solutions of \( 2xy^2 + 2y + 2x^2 y' + 2x y' = 0. \)

Solution: Re-write the equation is a more organized way,

\[
[2x^2 y + 2x] y' + [2xy^2 + 2y] = 0.
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\[
N = [2x^2 y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.
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M = [2xy^2 + 2y] \quad \Rightarrow \quad \partial_y M = 4xy + 2.
\]

\[
\Rightarrow \quad \partial_x N = \partial_y M.
\]

The equation is exact. There exists a potential function \( \psi \) with
First order differential equations.

Example
Find all solutions of \[2xy^2 + 2y + 2x^2y \, y' + 2x \, y' = 0.\]

Solution: Re-write the equation is a more organized way, \[[2x^2y + 2x] \, y' + [2xy^2 + 2y] = 0.\]

\[
\begin{align*}
N &= [2x^2y + 2x] &\implies \partial_x N = 4xy + 2. \\
M &= [2xy^2 + 2y] &\implies \partial_y M = 4xy + 2.
\end{align*}
\]

\[\implies \partial_x N = \partial_y M.\]

The equation is exact. There exists a potential function \(\psi\) with
\[
\partial_y \psi = N,
\]
First order differential equations.

**Example**
Find all solutions of \(2xy^2 + 2y + 2x^2y \, y' + 2x \, y' = 0\).

**Solution:** Re-write the equation in a more organized way,

\[
[2x^2y + 2x] \, y' + [2xy^2 + 2y] = 0.
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N &= [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2, \\
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\]
First order differential equations.

Example
Find all solutions of $2xy^2 + 2y + 2x^2y \, y' + 2x \, y' = 0$.

Solution: Re-write the equation is a more organized way,

$[2x^2y + 2x] \, y' + [2xy^2 + 2y] = 0$.

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N &= [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2. \\
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\]

\[
\begin{align*}
\Rightarrow \quad \partial_x N &= \partial_y M.
\end{align*}
\]

The equation is exact. There exists a potential function $\psi$ with

\[
\begin{align*}
\partial_y \psi &= N, \quad \partial_x \psi = M.
\end{align*}
\]

$\partial_y \psi = 2x^2y + 2x$
First order differential equations.

Example
Find all solutions of \( 2xy^2 + 2y + 2x^2 y \, y' + 2x \, y' = 0 \).

Solution: Re-write the equation is a more organized way,
\[
[2x^2 y + 2x] \, y' + [2xy^2 + 2y] = 0.
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N = [2x^2 y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.
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The equation is exact. There exists a potential function \( \psi \) with
\[
\partial_y \psi = N, \quad \partial_x \psi = M.
\]
\[
\partial_y \psi = 2x^2 y + 2x \quad \Rightarrow \quad \psi(x, y) = x^2 y^2 + 2xy + g(x).
\]
First order differential equations.

Example
Find all solutions of \( 2xy^2 + 2y + 2x^2y \, y' + 2x \, y' = 0 \).

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\partial_x N &= \partial_y M.
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\[
2xy^2 + 2y + g'(x) = \partial_x \psi
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Find all solutions of \(2xy^2 + 2y + 2x^2y \ y' + 2x \ y' = 0\).

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First order differential equations.

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Find all solutions of \( 2xy^2 + 2y + 2x^2 y \, y' + 2x \, y' = 0 \).

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Find all solutions of $2xy^2 + 2y + 2x^2y \, y' + 2x \, y' = 0$.

Solution: Re-write the equation in a more organized way,

$$[2x^2y + 2x] \, y' + [2xy^2 + 2y] = 0.$$  

Given:

$N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.$  
$M = [2xy^2 + 2y] \quad \Rightarrow \quad \partial_y M = 4xy + 2.$

\begin{align*}
\Rightarrow \quad \partial_x N &= \partial_y M.
\end{align*}

The equation is exact. There exists a potential function $\psi$ with

$\partial_y \psi = N, \quad \partial_x \psi = M.$

$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x, y) = x^2y^2 + 2xy + g(x).$

$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y \quad \Rightarrow \quad g'(x) = 0.$

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$$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y \quad \Rightarrow \quad g'(x) = 0.$$

$$\psi(x, y) = x^2y^2 + 2xy + c, \quad x^2 \, y^2(x) + 2x \, y(x) + c = 0.$$