Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: Eigenvalue-eigenfunction problem.

Two-point Boundary Value Problem.

Definition

A *two-point BVP* is the following: Given functions p, q, g, and constants

$$x_1 < x_2, \quad y_1, y_2, \quad b_1, b_2, \quad \tilde{b}_1, \tilde{b}_2,$$

find a function y solution of the differential equation

$$y'' + p(x)\, y' + q(x)\, y = g(x),$$

together with the extra, *boundary conditions*,

$$b_1\, y(x_1) + b_2\, y'(x_1) = y_1,$$
$$\tilde{b}_1\, y(x_2) + \tilde{b}_2\, y'(x_2) = y_2.$$

Two-point Boundary Value Problem.

Definition

A *two-point BVP* is the following: Given functions p, q, g, and constants

$$x_1 < x_2, \quad y_1, y_2, \quad b_1, b_2, \quad \tilde{b}_1, \tilde{b}_2,$$

find a function y solution of the differential equation

$$y'' + p(x)\, y' + q(x)\, y = g(x),$$

together with the extra, *boundary conditions*,

$$b_1\, y(x_1) + b_2\, y'(x_1) = y_1,$$
$$\tilde{b}_1\, y(x_2) + \tilde{b}_2\, y'(x_2) = y_2.$$

Remarks:

- Both y and y' might appear in the boundary condition, evaluated at the same point.

Two-point Boundary Value Problem.

Definition

A *two-point BVP* is the following: Given functions p, q, g, and constants

$$x_1 < x_2, \quad y_1, y_2, \quad b_1, b_2, \quad \tilde{b}_1, \tilde{b}_2,$$

find a function y solution of the differential equation

$$y'' + p(x)\, y' + q(x)\, y = g(x),$$

together with the extra, *boundary conditions*,

$$b_1\, y(x_1) + b_2\, y'(x_1) = y_1,$$
$$\tilde{b}_1\, y(x_2) + \tilde{b}_2\, y'(x_2) = y_2.$$

Remarks:

- Both y and y' might appear in the boundary condition, evaluated at the same point.
- In this notes we only study the case of constant coefficients,

$$y'' + a_1\, y' + a_0\, y = g(x).$$

Two-point Boundary Value Problem.

Example

Examples of BVP.

Two-point Boundary Value Problem.

Example

Examples of BVP. Assume $x_1 \neq x_2$.

(1) Find y solution of

$$y'' + a_1\, y' + a_0\, y = g(x), \quad y(x_1) = y_1, \quad y(x_2) = y_2.$$

Two-point Boundary Value Problem.

Example

Examples of BVP. Assume $x_1 \neq x_2$.

(1) Find y solution of

$$y'' + a_1\, y' + a_0\, y = g(x), \quad y(x_1) = y_1, \quad y(x_2) = y_2.$$

(2) Find y solution of

$$y'' + a_1\, y' + a_0\, y = g(x), \quad y'(x_1) = y_1, \quad y'(x_2) = y_2.$$

Two-point Boundary Value Problem.

Example

Examples of BVP. Assume $x_1 \neq x_2$.

(1) Find y solution of

$$y'' + a_1\, y' + a_0\, y = g(x), \quad y(x_1) = y_1, \quad y(x_2) = y_2.$$

(2) Find y solution of

$$y'' + a_1\, y' + a_0\, y = g(x), \quad y'(x_1) = y_1, \quad y'(x_2) = y_2.$$

(3) Find y solution of

$$y'' + a_1\, y' + a_0\, y = g(x), \quad y(x_1) = y_1, \quad y'(x_2) = y_2.$$

Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- **Example from physics.**
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: Eigenvalue-eigenfunction problem.

Example from physics.

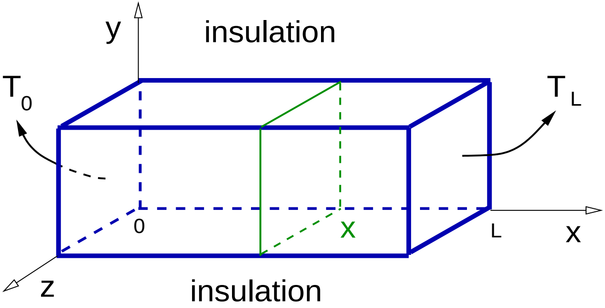
Problem: The equilibrium (time independent) temperature of a bar of length L with insulated horizontal sides and the bar vertical extremes kept at fixed temperatures T_0, T_L is the solution of the BVP:

$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$

Example from physics.

Problem: The equilibrium (time independent) temperature of a bar of length L with insulated horizontal sides and the bar vertical extremes kept at fixed temperatures T_0, T_L is the solution of the BVP:

$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$



Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- **Comparison: IVP vs BVP.**
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: Eigenvalue-eigenfunction problem.

Comparison: IVP vs BVP.

Review: IVP:

Find the function values $y(t)$ solutions of the differential equation

$$y'' + a_1\, y' + a_0\, y = g(t),$$

together with the initial conditions

$$y(t_0) = y_1, \quad y'(t_0) = y_2.$$

Comparison: IVP vs BVP.

Review: IVP:

Find the function values $y(t)$ solutions of the differential equation

$$y'' + a_1\, y' + a_0\, y = g(t),$$

together with the initial conditions

$$y(t_0) = y_1, \quad y'(t_0) = y_2.$$

Remark: In physics:

- $y(t)$: Position at time t.

Comparison: IVP vs BVP.

Review: IVP:
Find the function values $y(t)$ solutions of the differential equation

$$y'' + a_1\, y' + a_0\, y = g(t),$$

together with the initial conditions

$$y(t_0) = y_1, \quad y'(t_0) = y_2.$$

Remark: In physics:

- $y(t)$: Position at time t.
- Initial conditions: Position and velocity at the initial time t_0.

Comparison: IVP vs BVP.

Review: BVP:

Find the function values $y(x)$ solutions of the differential equation

$$y'' + a_1\, y' + a_0\, y = g(x),$$

together with the initial conditions

$$y(x_1) = y_1, \quad y(x_2) = y_2.$$

Comparison: IVP vs BVP.

Review: BVP:

Find the function values $y(x)$ solutions of the differential equation

$$y'' + a_1\, y' + a_0\, y = g(x),$$

together with the initial conditions

$$y(x_1) = y_1, \quad y(x_2) = y_2.$$

Remark: In physics:

- $y(x)$: A physical quantity (temperature) at a position x.

Comparison: IVP vs BVP.

Review: BVP:

Find the function values $y(x)$ solutions of the differential equation

$$y'' + a_1\, y' + a_0\, y = g(x),$$

together with the initial conditions

$$y(x_1) = y_1, \quad y(x_2) = y_2.$$

Remark: In physics:

- $y(x)$: A physical quantity (temperature) at a position x.
- Boundary conditions: Conditions at the boundary of the object under study, where $x_1 \neq x_2$.

Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- **Existence, uniqueness of solutions to BVP.**
- Particular case of BVP: Eigenvalue-eigenfunction problem.

Existence, uniqueness of solutions to BVP.

Review: The initial value problem.

Theorem (IVP)

Consider the homogeneous initial value problem:

$$y'' + a_1\, y' + a_0\, y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_1,$$

and let $r_\pm$ *be the roots of the characteristic polynomial*

$$p(r) = r^2 + a_1\, r + a_0.$$

If $r_+ \neq r_-$*, real or complex, then for every choice of* y_0*,* y_1*, there exists a unique solution* y *to the initial value problem above.*

Existence, uniqueness of solutions to BVP.

Review: The initial value problem.

Theorem (IVP)

Consider the homogeneous initial value problem:

$$y'' + a_1\, y' + a_0\, y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_1,$$

and let $r_\pm$ *be the roots of the characteristic polynomial*

$$p(r) = r^2 + a_1\, r + a_0.$$

If $r_+ \neq r_-$, *real or complex, then for every choice of* y_0, y_1, *there exists a unique solution* y *to the initial value problem above.*

Summary: The IVP above always has a unique solution, no matter what y_0 and y_1 we choose.

Existence, uniqueness of solutions to BVP.

Theorem (BVP)

Consider the homogeneous boundary value problem:

$$y'' + a_1\, y' + a_0\, y = 0, \qquad y(0) = y_0, \quad y(L) = y_1,$$

and let $r_\pm$ be the roots of the characteristic polynomial

$$p(r) = r^2 + a_1\, r + a_0.$$

(A) *If $r_+ \neq r_-$, real, then for every choice of $L \neq 0$ and y_0, y_1, there exists a unique solution y to the BVP above.*

(B) *If $r_\pm = \alpha \pm i\beta$, with $\beta \neq 0$, and $\alpha, \beta \in \mathbb{R}$, then the solutions to the BVP above belong to one of these possibilities:*

   (1) *There exists a unique solution.*

   (2) *There exists no solution.*

   (3) *There exist infinitely many solutions.*

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$.

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1 \, e^{r_- t} + c_2 \, e^{r_+ t},$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1\, e^{r_-\, t} + c_2\, e^{r_+\, t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1\, e^{r_- t} + c_2\, e^{r_+ t}, \qquad c_1, c_2 \in \mathbb{R}.$$

The initial conditions determine c_1 and c_2

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1 \, e^{r_- t} + c_2 \, e^{r_+ t}, \qquad c_1, c_2 \in \mathbb{R}.$$

The initial conditions determine c_1 and c_2 as follows:

$$y_0 = y(t_0)$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1\, e^{r_- t} + c_2\, e^{r_+ t}, \qquad c_1, c_2 \in \mathbb{R}.$$

The initial conditions determine c_1 and c_2 as follows:

$$y_0 = y(t_0) = c_1\, e^{r_- t_0} + c_2\, e^{r_+ t_0}$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1\, e^{r_- t} + c_2\, e^{r_+ t}, \qquad c_1, c_2 \in \mathbb{R}.$$

The initial conditions determine c_1 and c_2 as follows:

$$y_0 = y(t_0) = c_1\, e^{r_- t_0} + c_2\, e^{r_+ t_0}$$

$$y_1 = y'(t_0)$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1\, e^{r_-\, t} + c_2\, e^{r_+\, t}, \qquad c_1, c_2 \in \mathbb{R}.$$

The initial conditions determine c_1 and c_2 as follows:

$$y_0 = y(t_0) = c_1\, e^{r_-\, t_0} + c_2\, e^{r_+\, t_0}$$

$$y_1 = y'(t_0) = c_1 r_-\, e^{r_-\, t_0} + c_2 r_+\, e^{r_+\, t_0}$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1 \, e^{r_- t} + c_2 \, e^{r_+ t}, \qquad c_1, c_2 \in \mathbb{R}.$$

The initial conditions determine c_1 and c_2 as follows:

$$y_0 = y(t_0) = c_1 \, e^{r_- t_0} + c_2 \, e^{r_+ t_0}$$

$$y_1 = y'(t_0) = c_1 r_- \, e^{r_- t_0} + c_2 r_+ \, e^{r_+ t_0}$$

Using matrix notation,

$$\begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- \, e^{r_- t_0} & r_+ \, e^{r_+ t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1\, e^{r_- t} + c_2\, e^{r_+ t}, \qquad c_1, c_2 \in \mathbb{R}.$$

The initial conditions determine c_1 and c_2 as follows:

$$y_0 = y(t_0) = c_1\, e^{r_- t_0} + c_2\, e^{r_+ t_0}$$

$$y_1 = y'(t_0) = c_1 r_-\, e^{r_- t_0} + c_2 r_+\, e^{r_+ t_0}$$

Using matrix notation,

$$\begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_-\, e^{r_- t_0} & r_+\, e^{r_+ t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1\, e^{r_-\, t} + c_2\, e^{r_+\, t}, \qquad c_1, c_2 \in \mathbb{R}.$$

The initial conditions determine c_1 and c_2 as follows:

$$y_0 = y(t_0) = c_1\, e^{r_-\, t_0} + c_2\, e^{r_+\, t_0}$$

$$y_1 = y'(t_0) = c_1 r_-\, e^{r_-\, t_0} + c_2 r_+\, e^{r_+\, t_0}$$

Using matrix notation,

$$\begin{bmatrix} e^{r_-\, t_0} & e^{r_+\, t_0} \\ r_-\, e^{r_- t_0} & r_+\, e^{r_+\, t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the $\det(Z) \neq 0$,

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1\, e^{r_- t} + c_2\, e^{r_+ t}, \qquad c_1, c_2 \in \mathbb{R}.$$

The initial conditions determine c_1 and c_2 as follows:

$$y_0 = y(t_0) = c_1\, e^{r_- t_0} + c_2\, e^{r_+ t_0}$$

$$y_1 = y'(t_0) = c_1 r_-\, e^{r_- t_0} + c_2 r_+\, e^{r_+ t_0}$$

Using matrix notation,

$$\begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_-\, e^{r_- t_0} & r_+\, e^{r_+ t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the $\det(Z) \neq 0$, where

$$Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_-\, e^{r_- t_0} & r_+\, e^{r_+ t_0} \end{bmatrix}$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case $r_+ \neq r_-$. The general solution is

$$y(t) = c_1\, e^{r_- t} + c_2\, e^{r_+ t}, \qquad c_1, c_2 \in \mathbb{R}.$$

The initial conditions determine c_1 and c_2 as follows:

$$y_0 = y(t_0) = c_1\, e^{r_- t_0} + c_2\, e^{r_+ t_0}$$

$$y_1 = y'(t_0) = c_1 r_-\, e^{r_- t_0} + c_2 r_+\, e^{r_+ t_0}$$

Using matrix notation,

$$\begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_-\, e^{r_- t_0} & r_+\, e^{r_+ t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the $\det(Z) \neq 0$, where

$$Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_-\, e^{r_- t_0} & r_+\, e^{r_+ t_0} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

Existence, uniqueness of solutions to BVP.

Proof of IVP:

Recall: $Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

Existence, uniqueness of solutions to BVP.

Proof of IVP:

Recall: $Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = (r_+ - r_-)\, e^{(r_+ + r_-)\, t_0}$$

Existence, uniqueness of solutions to BVP.

Proof of IVP:

Recall: $Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = (r_+ - r_-)\, e^{(r_+ + r_-)\, t_0} \neq 0 \quad \Leftrightarrow \quad r_+ \neq r_-.$$

Existence, uniqueness of solutions to BVP.

Proof of IVP:

Recall: $Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- \, e^{r_- t_0} & r_+ \, e^{r_+ t_0} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = (r_+ - r_-) \, e^{(r_+ + r_-) \, t_0} \neq 0 \quad \Leftrightarrow \quad r_+ \neq r_-.$$

Since $r_+ \neq r_-$, the matrix Z is invertible

Existence, uniqueness of solutions to BVP.

Proof of IVP:

Recall: $Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = \big(r_+ - r_-\big)\, e^{(r_+ + r_-)\, t_0} \neq 0 \quad \Leftrightarrow \quad r_+ \neq r_-.$$

Since $r_+ \neq r_-$, the matrix Z is invertible and so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Z^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

Existence, uniqueness of solutions to BVP.

Proof of IVP:

Recall: $Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = (r_+ - r_-)\, e^{(r_+ + r_-)\, t_0} \neq 0 \quad \Leftrightarrow \quad r_+ \neq r_-.$$

Since $r_+ \neq r_-$, the matrix Z is invertible and so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Z^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

We conclude that for every choice of y_0 and y_1, there exist a unique value of c_1 and c_2, so the IVP above has a unique solution. $\square$

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1\, e^{r_- x} + c_2\, e^{r_+ x},$$

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1\, e^{r_- x} + c_2\, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1 \, e^{r_- x} + c_2 \, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine c_1 and c_2

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1 \, e^{r_- x} + c_2 \, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine c_1 and c_2 as follows:

$$y_0 = y(0)$$

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1\, e^{r_- x} + c_2\, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine c_1 and c_2 as follows:

$$y_0 = y(0) = c_1 + c_2.$$

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1\, e^{r_- x} + c_2\, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine c_1 and c_2 as follows:

$$y_0 = y(0) = c_1 + c_2.$$

$$y_1 = y(L)$$

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1 \, e^{r_- x} + c_2 \, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine c_1 and c_2 as follows:

$$y_0 = y(0) = c_1 + c_2.$$

$$y_1 = y(L) = c_1 \, e^{r_- L} + c_2 \, e^{r_+ L}$$

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1 \, e^{r_- x} + c_2 \, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine c_1 and c_2 as follows:

$$y_0 = y(0) = c_1 + c_2.$$

$$y_1 = y(L) = c_1 \, e^{r_- L} + c_2 \, e^{r_+ L}$$

Using matrix notation,

$$\begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1\, e^{r_- x} + c_2\, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine c_1 and c_2 as follows:

$$y_0 = y(0) = c_1 + c_2.$$

$$y_1 = y(L) = c_1\, e^{r_- L} + c_2\, e^{r_+ L}$$

Using matrix notation,

$$\begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1\, e^{r_- x} + c_2\, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine c_1 and c_2 as follows:

$$y_0 = y(0) = c_1 + c_2.$$

$$y_1 = y(L) = c_1\, e^{r_- L} + c_2\, e^{r_+ L}$$

Using matrix notation,

$$\begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the $\det(Z) \neq 0$,

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1\, e^{r_- x} + c_2\, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine c_1 and c_2 as follows:

$$y_0 = y(0) = c_1 + c_2.$$

$$y_1 = y(L) = c_1\, e^{r_- L} + c_2\, e^{r_+ L}$$

Using matrix notation,

$$\begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the $\det(Z) \neq 0$, where

$$Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix}$$

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1\, e^{r_- x} + c_2\, e^{r_+ x}, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine c_1 and c_2 as follows:

$$y_0 = y(0) = c_1 + c_2.$$

$$y_1 = y(L) = c_1\, e^{r_- L} + c_2\, e^{r_+ L}$$

Using matrix notation,

$$\begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the $\det(Z) \neq 0$, where

$$Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L}$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \quad \Leftrightarrow \quad e^{r_+ L} \neq e^{r_- L}.$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \quad \Leftrightarrow \quad e^{r_+ L} \neq e^{r_- L}.$$

(A) If $r_+ \neq r_-$ and real-valued,

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \quad \Leftrightarrow \quad e^{r_+ L} \neq e^{r_- L}.$$

(A) If $r_+ \neq r_-$ and real-valued, then $\det(Z) \neq 0$.

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \quad \Leftrightarrow \quad e^{r_+ L} \neq e^{r_- L}.$$

(A) If $r_+ \neq r_-$ and real-valued, then $\det(Z) \neq 0$.

We conclude: For every choice of y_0 and y_1, there exist a unique value of c_1 and c_2, so the BVP in (A) above has a unique solution.

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \quad \Leftrightarrow \quad e^{r_+ L} \neq e^{r_- L}.$$

(A) If $r_+ \neq r_-$ and real-valued, then $\det(Z) \neq 0$.

We conclude: For every choice of y_0 and y_1, there exist a unique value of c_1 and c_2, so the BVP in (A) above has a unique solution.

(B) If $r_\pm = \alpha \pm i\beta$, with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$,

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \quad \Leftrightarrow \quad e^{r_+ L} \neq e^{r_- L}.$$

(A) If $r_+ \neq r_-$ and real-valued, then $\det(Z) \neq 0$.

We conclude: For every choice of y_0 and y_1, there exist a unique value of c_1 and c_2, so the BVP in (A) above has a unique solution.

(B) If $r_\pm = \alpha \pm i\beta$, with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$, then

$$\det(Z) = e^{\alpha L}\big(e^{i\beta L} - e^{-i\beta L}\big)$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \quad \Leftrightarrow \quad e^{r_+ L} \neq e^{r_- L}.$$

(A) If $r_+ \neq r_-$ and real-valued, then $\det(Z) \neq 0$.

We conclude: For every choice of y_0 and y_1, there exist a unique value of c_1 and c_2, so the BVP in (A) above has a unique solution.

(B) If $r_\pm = \alpha \pm i\beta$, with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$, then

$$\det(Z) = e^{\alpha L}\left(e^{i\beta L} - e^{-i\beta L}\right) \;\Rightarrow\; \det(Z) = 2i\, e^{\alpha L} \sin(\beta L).$$

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \quad \Leftrightarrow \quad e^{r_+ L} \neq e^{r_- L}.$$

(A) If $r_+ \neq r_-$ and real-valued, then $\det(Z) \neq 0$.

We conclude: For every choice of y_0 and y_1, there exist a unique value of c_1 and c_2, so the BVP in (A) above has a unique solution.

(B) If $r_\pm = \alpha \pm i\beta$, with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$, then

$$\det(Z) = e^{\alpha L}\big(e^{i\beta L} - e^{-i\beta L}\big) \;\Rightarrow\; \det(Z) = 2i\, e^{\alpha L} \sin(\beta L).$$

Since $\det(Z) = 0$ iff $\beta L = n\pi$, with n integer,

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \quad \Leftrightarrow \quad e^{r_+ L} \neq e^{r_- L}.$$

(A) If $r_+ \neq r_-$ and real-valued, then $\det(Z) \neq 0$.

We conclude: For every choice of y_0 and y_1, there exist a unique value of c_1 and c_2, so the BVP in (A) above has a unique solution.

(B) If $r_\pm = \alpha \pm i\beta$, with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$, then

$$\det(Z) = e^{\alpha L}\big(e^{i\beta L} - e^{-i\beta L}\big) \;\Rightarrow\; \det(Z) = 2i\, e^{\alpha L} \sin(\beta L).$$

Since $\det(Z) = 0$ iff $\beta L = n\pi$, with n integer,

(1) If $\beta L \neq n\pi$, then BVP has a unique solution.

Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall: $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \quad \Rightarrow \quad Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \quad \Leftrightarrow \quad e^{r_+ L} \neq e^{r_- L}.$$

(A) If $r_+ \neq r_-$ and real-valued, then $\det(Z) \neq 0$.

We conclude: For every choice of y_0 and y_1, there exist a unique value of c_1 and c_2, so the BVP in (A) above has a unique solution.

(B) If $r_\pm = \alpha \pm i\beta$, with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$, then

$$\det(Z) = e^{\alpha L}\left(e^{i\beta L} - e^{-i\beta L}\right) \;\Rightarrow\; \det(Z) = 2i\, e^{\alpha L} \sin(\beta L).$$

Since $\det(Z) = 0$ iff $\beta L = n\pi$, with n integer,

(1) If $\beta L \neq n\pi$, then BVP has a unique solution.

(2) If $\beta L = n\pi$ then BVP either has no solutions or it has infinitely many solutions. □

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1,$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 1, \quad c_2 \text{ free}.$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 1, \quad c_2 \text{ free.}$$

We conclude: $y(x) = \cos(x) + c_2 \sin(x)$, with $c_2 \in \mathbb{R}$.

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 1, \quad c_2 \text{ free}.$$

We conclude: $y(x) = \cos(x) + c_2 \sin(x)$, with $c_2 \in \mathbb{R}$.

The BVP has infinitely many solutions. $\lhd$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1,$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 0 = y(\pi) = -c_1$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 0 = y(\pi) = -c_1$$

The BVP has no solution. $\triangleleft$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1,$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2 \quad \Rightarrow \quad c_1 = c_2 = 1.$$

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2 \quad \Rightarrow \quad c_1 = c_2 = 1.$$

We conclude: $y(x) = \cos(x) + \sin(x)$.

Existence, uniqueness of solutions to BVP.

Example

Find y solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$

Solution: The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2 \quad \Rightarrow \quad c_1 = c_2 = 1.$$

We conclude: $y(x) = \cos(x) + \sin(x)$.

The BVP has a unique solution. $\triangleleft$

Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: **Eigenvalue-eigenfunction problem.**

Particular case of BVP: Eigenvalue-eigenfunction problem.

Problem:

Find a number λ and a non-zero function y solutions to the boundary value problem

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Problem:

Find a number λ and a non-zero function y solutions to the boundary value problem

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remark: This problem is similar to the eigenvalue-eigenvector problem in Linear Algebra:

Particular case of BVP: Eigenvalue-eigenfunction problem.

Problem:
Find a number λ and a non-zero function y solutions to the boundary value problem

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remark: This problem is similar to the eigenvalue-eigenvector problem in Linear Algebra: Given an $n \times n$ matrix A, find λ and a non-zero n-vector $\mathbf{v}$ solutions of

$$A\mathbf{v} - \lambda\,\mathbf{v} = \mathbf{0}.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Problem:

Find a number λ and a non-zero function y solutions to the boundary value problem

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remark: This problem is similar to the eigenvalue-eigenvector problem in Linear Algebra: Given an $n \times n$ matrix A, find λ and a non-zero n-vector $\mathbf{v}$ solutions of

$$A\mathbf{v} - \lambda\,\mathbf{v} = \mathbf{0}.$$

Differences:

- $A \quad \longrightarrow \quad \left\{ \begin{matrix} \text{computing a second derivative and} \\ \text{applying the boundary conditions.} \end{matrix} \right\}$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Problem:
Find a number λ and a non-zero function y solutions to the boundary value problem

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remark: This problem is similar to the eigenvalue-eigenvector problem in Linear Algebra: Given an $n \times n$ matrix A, find λ and a non-zero n-vector $\mathbf{v}$ solutions of

$$A\mathbf{v} - \lambda\,\mathbf{v} = \mathbf{0}.$$

Differences:

- $A \quad \longrightarrow \quad \left\{ \begin{array}{l} \text{computing a second derivative and} \\ \text{applying the boundary conditions.} \end{array} \right\}$
- $\mathbf{v} \quad \longrightarrow \quad \{\text{a function } y\}.$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remarks: We will show that:

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remarks: We will show that:

(1) If $\lambda \leqslant 0$, then the BVP has no solution.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remarks: We will show that:

(1) If $\lambda \leqslant 0$, then the BVP has no solution.

(2) If $\lambda > 0$, then there exist infinitely many eigenvalues λ_n and eigenfunctions y_n, with n any positive integer,

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remarks: We will show that:

(1) If $\lambda \leqslant 0$, then the BVP has no solution.

(2) If $\lambda > 0$, then there exist infinitely many eigenvalues λ_n and eigenfunctions y_n, with n any positive integer, given by

$$\lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \qquad y_n(x) = \sin\Big(\frac{n\pi x}{L}\Big),$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remarks: We will show that:

(1) If $\lambda \leqslant 0$, then the BVP has no solution.

(2) If $\lambda > 0$, then there exist infinitely many eigenvalues λ_n and eigenfunctions y_n, with n any positive integer, given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right),$$

(3) Analogous results can be proven for the same equation but with different types of boundary conditions.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remarks: We will show that:

(1) If $\lambda \leqslant 0$, then the BVP has no solution.

(2) If $\lambda > 0$, then there exist infinitely many eigenvalues λ_n and eigenfunctions y_n, with n any positive integer, given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right),$$

(3) Analogous results can be proven for the same equation but with different types of boundary conditions. For example, for $y(0) = 0,\; y'(L) = 0$;

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Remarks: We will show that:

(1) If $\lambda \leqslant 0$, then the BVP has no solution.

(2) If $\lambda > 0$, then there exist infinitely many eigenvalues λ_n and eigenfunctions y_n, with n any positive integer, given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right),$$

(3) Analogous results can be proven for the same equation but with different types of boundary conditions. For example, for $y(0) = 0,\ y'(L) = 0$; or for $y'(0) = 0,\ y'(L) = 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda = 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda = 0$. The equation is

$$y'' = 0$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda = 0$. The equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda = 0$. The equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x.$$

The boundary conditions imply

$$0 = y(0)$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda = 0$. The equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x.$$

The boundary conditions imply

$$0 = y(0) = c_1,$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda = 0$. The equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x.$$

The boundary conditions imply

$$0 = y(0) = c_1, \quad 0 = c_1 + c_2 L$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda = 0$. The equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x.$$

The boundary conditions imply

$$0 = y(0) = c_1, \quad 0 = c_1 + c_2 L \quad \Rightarrow \quad c_1 = c_2 = 0.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda = 0$. The equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x.$$

The boundary conditions imply

$$0 = y(0) = c_1, \quad 0 = c_1 + c_2 L \quad \Rightarrow \quad c_1 = c_2 = 0.$$

Since $y = 0$, there are NO non-zero solutions for $\lambda = 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda < 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda < 0$. Introduce the notation $\lambda = -\mu^2$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda < 0$. Introduce the notation $\lambda = -\mu^2$. The characteristic equation is

$$p(r) = r^2 - \mu^2 = 0$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda < 0$. Introduce the notation $\lambda = -\mu^2$. The characteristic equation is

$$p(r) = r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda < 0$. Introduce the notation $\lambda = -\mu^2$. The characteristic equation is

$$p(r) = r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu.$$

The general solution is

$$y(x) = c_1\, e^{\mu x} + c_2\, e^{-\mu x}.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda < 0$. Introduce the notation $\lambda = -\mu^2$. The characteristic equation is

$$p(r) = r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu.$$

The general solution is

$$y(x) = c_1\, e^{\mu x} + c_2\, e^{-\mu x}.$$

The boundary condition are

$$0 = y(0)$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda < 0$. Introduce the notation $\lambda = -\mu^2$. The characteristic equation is

$$p(r) = r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu.$$

The general solution is

$$y(x) = c_1\, e^{\mu x} + c_2\, e^{-\mu x}.$$

The boundary condition are

$$0 = y(0) = c_1 + c_2,$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda < 0$. Introduce the notation $\lambda = -\mu^2$. The characteristic equation is

$$p(r) = r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu.$$

The general solution is

$$y(x) = c_1\, e^{\mu x} + c_2\, e^{-\mu x}.$$

The boundary condition are

$$0 = y(0) = c_1 + c_2,$$

$$0 = y(L) = c_1\, e^{\mu L} + c_2\, e^{-\mu L}.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $y(x) = c_1\, e^{\mu x} + c_2\, e^{\mu x}$ and

$$c_1 + c_2 = 0, \qquad c_1\, e^{\mu L} + c_2\, e^{-\mu L} = 0.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $y(x) = c_1\, e^{\mu x} + c_2\, e^{\mu x}$ and

$$c_1 + c_2 = 0, \qquad c_1\, e^{\mu L} + c_2\, e^{-\mu L} = 0.$$

We need to solve the linear system

$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $y(x) = c_1\, e^{\mu x} + c_2\, e^{\mu x}$ and

$$c_1 + c_2 = 0, \qquad c_1\, e^{\mu L} + c_2\, e^{-\mu L} = 0.$$

We need to solve the linear system

$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;\Leftrightarrow\; Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $y(x) = c_1\, e^{\mu x} + c_2\, e^{\mu x}$ and

$$c_1 + c_2 = 0, \qquad c_1\, e^{\mu L} + c_2\, e^{-\mu L} = 0.$$

We need to solve the linear system

$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;\Leftrightarrow\; Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix}$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $y(x) = c_1\, e^{\mu x} + c_2\, e^{\mu x}$ and

$$c_1 + c_2 = 0, \qquad c_1\, e^{\mu L} + c_2\, e^{-\mu L} = 0.$$

We need to solve the linear system

$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;\Leftrightarrow\; Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix}$$

Since $\det(Z) = e^{-\mu L} - e^{\mu L} \neq 0$ for $L \neq 0$, matrix Z is invertible, so the linear system above has a unique solution $c_1 = 0$ and $c_2 = 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $y(x) = c_1\, e^{\mu x} + c_2\, e^{\mu x}$ and

$$c_1 + c_2 = 0, \qquad c_1\, e^{\mu L} + c_2\, e^{-\mu L} = 0.$$

We need to solve the linear system

$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;\Leftrightarrow\; Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix}$$

Since $\det(Z) = e^{-\mu L} - e^{\mu L} \neq 0$ for $L \neq 0$, matrix Z is invertible, so the linear system above has a unique solution $c_1 = 0$ and $c_2 = 0$.

Since $y = 0$, there are NO non-zero solutions for $\lambda < 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda > 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm \mu i.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

The boundary condition are

$$0 = y(0)$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

The boundary condition are

$$0 = y(0) = c_1,$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

The boundary condition are

$$0 = y(0) = c_1, \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

The boundary condition are

$$0 = y(0) = c_1, \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(L) = c_2 \sin(\mu L),$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

The boundary condition are

$$0 = y(0) = c_1, \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(L) = c_2 \sin(\mu L), \quad c_2 \neq 0$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

The boundary condition are

$$0 = y(0) = c_1, \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(L) = c_2 \sin(\mu L), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu L) = 0.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

The non-zero solution condition is the reason for $c_2 \neq 0$.

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

The non-zero solution condition is the reason for $c_2 \neq 0$. Hence

$$\sin(\mu L) = 0$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

The non-zero solution condition is the reason for $c_2 \neq 0$. Hence

$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

The non-zero solution condition is the reason for $c_2 \neq 0$. Hence

$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L}.$$

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

The non-zero solution condition is the reason for $c_2 \neq 0$. Hence

$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L}.$$

Recalling that $\lambda_n = \mu_n^2$, and choosing $c_2 = 1$,

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example

Find every $\lambda \in \mathbb{R}$ and non-zero functions y solutions of the BVP

$$y''(x) + \lambda\, y(x) = 0, \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

Solution: Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

The non-zero solution condition is the reason for $c_2 \neq 0$. Hence

$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L}.$$

Recalling that $\lambda_n = \mu_n^2$, and choosing $c_2 = 1$, we conclude

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right). \qquad \triangleleft$$

Overview of Fourier Series (Sect. 6.2).

- Periodic functions.
- Orthogonality of Sines and Cosines.
- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.

Periodic functions.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* iff there exists $\tau > 0$ such that for all $x \in \mathbb{R}$ holds

$$f(x + \tau) = f(x).$$

Periodic functions.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* iff there exists $\tau > 0$ such that for all $x \in \mathbb{R}$ holds

$$f(x + \tau) = f(x).$$

Remark: f is invariant under translations by τ.

Periodic functions.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* iff there exists $\tau > 0$ such that for all $x \in \mathbb{R}$ holds

$$f(x + \tau) = f(x).$$

Remark: f is invariant under translations by τ.

Definition

A *period* T of a periodic function f is the smallest value of τ such that $f(x + \tau) = f(x)$ holds.

Periodic functions.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* iff there exists $\tau > 0$ such that for all $x \in \mathbb{R}$ holds

$$f(x + \tau) = f(x).$$

Remark: f is invariant under translations by τ.

Definition

A *period* T of a periodic function f is the smallest value of τ such that $f(x + \tau) = f(x)$ holds.

Notation:

A periodic function with period T is also called T-periodic.

Periodic functions.

Example

The following functions are periodic, with period T,

$$f(x) = \sin(x), \qquad T = 2\pi.$$

$$f(x) = \cos(x), \qquad T = 2\pi.$$

$$f(x) = \tan(x), \qquad T = \pi.$$

$$f(x) = \sin(ax), \qquad T = \frac{2\pi}{a}.$$

Periodic functions.

Example

The following functions are periodic, with period T,

$$f(x) = \sin(x), \qquad T = 2\pi.$$

$$f(x) = \cos(x), \qquad T = 2\pi.$$

$$f(x) = \tan(x), \qquad T = \pi.$$

$$f(x) = \sin(ax), \qquad T = \frac{2\pi}{a}.$$

The proof of the latter statement is the following:

Periodic functions.

Example

The following functions are periodic, with period T,

$$f(x) = \sin(x), \qquad T = 2\pi.$$

$$f(x) = \cos(x), \qquad T = 2\pi.$$

$$f(x) = \tan(x), \qquad T = \pi.$$

$$f(x) = \sin(ax), \qquad T = \frac{2\pi}{a}.$$

The proof of the latter statement is the following:

$$f\left(x + \frac{2\pi}{a}\right)$$

Periodic functions.

Example

The following functions are periodic, with period T,

$$f(x) = \sin(x), \qquad T = 2\pi.$$

$$f(x) = \cos(x), \qquad T = 2\pi.$$

$$f(x) = \tan(x), \qquad T = \pi.$$

$$f(x) = \sin(ax), \qquad T = \frac{2\pi}{a}.$$

The proof of the latter statement is the following:

$$f\left(x + \frac{2\pi}{a}\right) = \sin\left(ax + a\frac{2\pi}{a}\right)$$

Periodic functions.

Example

The following functions are periodic, with period T,

$$f(x) = \sin(x), \qquad T = 2\pi.$$

$$f(x) = \cos(x), \qquad T = 2\pi.$$

$$f(x) = \tan(x), \qquad T = \pi.$$

$$f(x) = \sin(ax), \qquad T = \frac{2\pi}{a}.$$

The proof of the latter statement is the following:

$$f\left(x + \frac{2\pi}{a}\right) = \sin\left(ax + a\frac{2\pi}{a}\right) = \sin(ax + 2\pi)$$

Periodic functions.

Example

The following functions are periodic, with period T,

$$f(x) = \sin(x), \qquad T = 2\pi.$$

$$f(x) = \cos(x), \qquad T = 2\pi.$$

$$f(x) = \tan(x), \qquad T = \pi.$$

$$f(x) = \sin(ax), \qquad T = \frac{2\pi}{a}.$$

The proof of the latter statement is the following:

$$f\left(x + \frac{2\pi}{a}\right) = \sin\left(ax + a\frac{2\pi}{a}\right) = \sin(ax + 2\pi) = \sin(ax)$$

Periodic functions.

Example

The following functions are periodic, with period T,

$$f(x) = \sin(x), \qquad T = 2\pi.$$

$$f(x) = \cos(x), \qquad T = 2\pi.$$

$$f(x) = \tan(x), \qquad T = \pi.$$

$$f(x) = \sin(ax), \qquad T = \frac{2\pi}{a}.$$

The proof of the latter statement is the following:

$$f\left(x + \frac{2\pi}{a}\right) = \sin\left(ax + a\frac{2\pi}{a}\right) = \sin(ax + 2\pi) = \sin(ax) = f(x).$$

$\triangleleft$

Periodic functions.

Example

Show that the function below is periodic, and find its period,

$$f(x) = e^x, \qquad x \in [0, 2), \qquad f(x-2) = f(x).$$

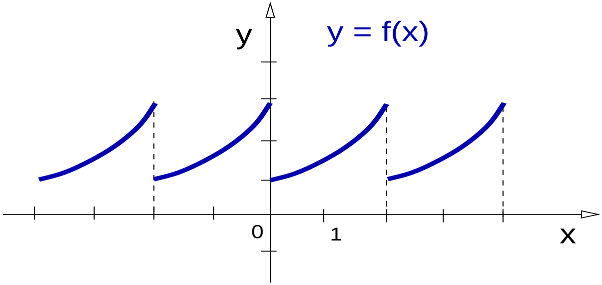
Periodic functions.

Example

Show that the function below is periodic, and find its period,

$$f(x) = e^x, \qquad x \in [0, 2), \qquad f(x-2) = f(x).$$

Solution: We just graph the function,



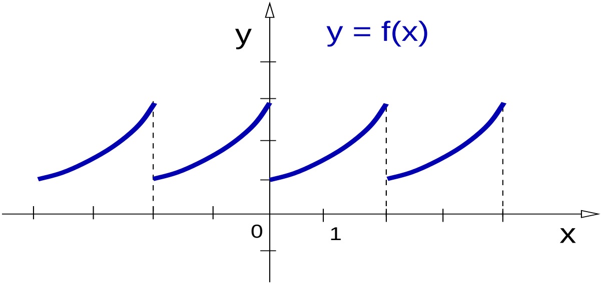
Periodic functions.

Example

Show that the function below is periodic, and find its period,

$$f(x) = e^x, \qquad x \in [0, 2), \qquad f(x-2) = f(x).$$

Solution: We just graph the function,



So the function is periodic with period $T = 2$. $\triangleleft$

Overview of Fourier Series (Sect. 6.2).

- Periodic functions.
- **Orthogonality of Sines and Cosines.**
- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.

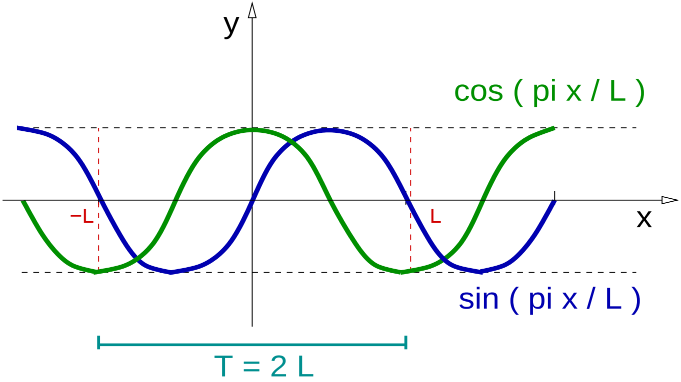
Orthogonality of Sines and Cosines.

Remark:
From now on we work on the following domain: $[-L, L]$.

Orthogonality of Sines and Cosines.

Remark:

From now on we work on the following domain: $[-L, L]$.



Orthogonality of Sines and Cosines.

Theorem (Orthogonality)

The following relations hold for all n, $m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\, \cos\Big(\frac{m\pi x}{L}\Big)\, dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big)\, \sin\Big(\frac{m\pi x}{L}\Big)\, dx = \begin{cases} 0 & n \neq m, \\ L & n = m, \end{cases}$$

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\, \sin\Big(\frac{m\pi x}{L}\Big)\, dx = 0.$$

Orthogonality of Sines and Cosines.

Theorem (Orthogonality)

The following relations hold for all $n, m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\, \cos\Big(\frac{m\pi x}{L}\Big)\, dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big)\, \sin\Big(\frac{m\pi x}{L}\Big)\, dx = \begin{cases} 0 & n \neq m, \\ L & n = m, \end{cases}$$

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\, \sin\Big(\frac{m\pi x}{L}\Big)\, dx = 0.$$

Remark:

- The operation $f \cdot g = \displaystyle\int_{-L}^{L} f(x)\, g(x)\, dx$ is an inner product in the vector space of functions.

Orthogonality of Sines and Cosines.

Theorem (Orthogonality)

The following relations hold for all $n,\, m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big) \, \cos\Big(\frac{m\pi x}{L}\Big) \, dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big) \, \sin\Big(\frac{m\pi x}{L}\Big) \, dx = \begin{cases} 0 & n \neq m, \\ L & n = m, \end{cases}$$

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big) \, \sin\Big(\frac{m\pi x}{L}\Big) \, dx = 0.$$

Remark:

- The operation $f \cdot g = \displaystyle\int_{-L}^{L} f(x)\, g(x)\, dx$ is an inner product in the vector space of functions. Like the dot product is in $\mathbb{R}^2$.

Orthogonality of Sines and Cosines.

Theorem (Orthogonality)

The following relations hold for all $n, m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big)\,\sin\Big(\frac{m\pi x}{L}\Big)\,dx = \begin{cases} 0 & n \neq m, \\ L & n = m, \end{cases}$$

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\sin\Big(\frac{m\pi x}{L}\Big)\,dx = 0.$$

Remark:

- The operation $f \cdot g = \displaystyle\int_{-L}^{L} f(x)\,g(x)\,dx$ is an inner product in the vector space of functions. Like the dot product is in $\mathbb{R}^2$.
- Two functions f, g, are orthogonal iff $f \cdot g = 0$.

Orthogonality of Sines and Cosines.

Recall:

$$\cos(\theta)\,\cos(\phi) = \frac{1}{2}\left[\cos(\theta+\phi) + \cos(\theta-\phi)\right];$$

$$\sin(\theta)\,\sin(\phi) = \frac{1}{2}\left[\cos(\theta-\phi) - \cos(\theta+\phi)\right];$$

$$\sin(\theta)\,\cos(\phi) = \frac{1}{2}\left[\sin(\theta+\phi) + \sin(\theta-\phi)\right].$$

Orthogonality of Sines and Cosines.

Recall:

$$\cos(\theta)\,\cos(\phi) = \frac{1}{2}\big[\cos(\theta+\phi) + \cos(\theta-\phi)\big];$$

$$\sin(\theta)\,\sin(\phi) = \frac{1}{2}\big[\cos(\theta-\phi) - \cos(\theta+\phi)\big];$$

$$\sin(\theta)\,\cos(\phi) = \frac{1}{2}\big[\sin(\theta+\phi) + \sin(\theta-\phi)\big].$$

Proof: First formula:

Orthogonality of Sines and Cosines.

Recall:
$$\cos(\theta)\,\cos(\phi) = \frac{1}{2}\big[\cos(\theta+\phi) + \cos(\theta-\phi)\big];$$
$$\sin(\theta)\,\sin(\phi) = \frac{1}{2}\big[\cos(\theta-\phi) - \cos(\theta+\phi)\big];$$
$$\sin(\theta)\,\cos(\phi) = \frac{1}{2}\big[\sin(\theta+\phi) + \sin(\theta-\phi)\big].$$

Proof: First formula: If $n = m = 0$, it is simple to see that

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \int_{-L}^{L} dx = 2L.$$

Orthogonality of Sines and Cosines.

Recall:
$$\cos(\theta)\,\cos(\phi) = \frac{1}{2}\big[\cos(\theta+\phi) + \cos(\theta-\phi)\big];$$
$$\sin(\theta)\,\sin(\phi) = \frac{1}{2}\big[\cos(\theta-\phi) - \cos(\theta+\phi)\big];$$
$$\sin(\theta)\,\cos(\phi) = \frac{1}{2}\big[\sin(\theta+\phi) + \sin(\theta-\phi)\big].$$

Proof: First formula: If $n = m = 0$, it is simple to see that

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \int_{-L}^{L} dx = 2L.$$

In the case where one of n or m is non-zero, use the relation

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx$$
$$+\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

Orthogonality of Sines and Cosines.

Proof: Since one of n or m is non-zero,

Orthogonality of Sines and Cosines.

Proof: Since one of n or m is non-zero, holds

$$\frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx = \frac{L}{2(n+m)\pi}\,\sin\Big[\frac{(n+m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

Orthogonality of Sines and Cosines.

Proof: Since one of n or m is non-zero, holds

$$\frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx = \frac{L}{2(n+m)\pi}\,\sin\Big[\frac{(n+m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

We obtain that

$$\int_{-L}^{L}\cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

Orthogonality of Sines and Cosines.

Proof: Since one of n or m is non-zero, holds

$$\frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx = \frac{L}{2(n+m)\pi}\,\sin\Big[\frac{(n+m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

We obtain that

$$\int_{-L}^{L}\cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

If we further restrict $n \neq m$, then

$$\frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx = \frac{L}{2(n-m)\pi}\,\sin\Big[\frac{(n-m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

Orthogonality of Sines and Cosines.

Proof: Since one of n or m is non-zero, holds

$$\frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx = \frac{L}{2(n+m)\pi}\,\sin\Big[\frac{(n+m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

We obtain that

$$\int_{-L}^{L}\cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

If we further restrict $n \neq m$, then

$$\frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx = \frac{L}{2(n-m)\pi}\,\sin\Big[\frac{(n-m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

If $n = m \neq 0$, we have that

$$\frac{1}{2}\int_{-L}^{L}\cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx = \frac{1}{2}\int_{-L}^{L}dx = L.$$

Orthogonality of Sines and Cosines.

Proof: Since one of n or m is non-zero, holds

$$\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx = \frac{L}{2(n+m)\pi}\,\sin\Big[\frac{(n+m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

We obtain that

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

If we further restrict $n \neq m$, then

$$\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx = \frac{L}{2(n-m)\pi}\,\sin\Big[\frac{(n-m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

If $n = m \neq 0$, we have that

$$\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx = \frac{1}{2}\int_{-L}^{L} dx = L.$$

This establishes the first equation in the Theorem. The remaining equations are proven in a similar way. □

Overview of Fourier Series (Sect. 6.2).

- Periodic functions.
- Orthogonality of Sines and Cosines.
- **The Fourier Theorem: Continuous case.**
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.

The Fourier Theorem: Continuous case.

Theorem (Fourier Series)

If the function $f : [-L, L] \subset \mathbb{R} \to \mathbb{R}$ *is continuous, then* f *can be expressed as an infinite series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big] \qquad (1)$$

with the constants a_n *and* b_n *given by*

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

Furthermore, the Fourier series in Eq. (1) provides a $2L$*-periodic extension of function* f *from the domain* $[-L, L] \subset \mathbb{R}$ *to* $\mathbb{R}$.

The Fourier Theorem: Continuous case.

Sketch of the Proof:

- Define the partial sum functions

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

The Fourier Theorem: Continuous case.

Sketch of the Proof:

- Define the partial sum functions

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big]$$

with a_n and b_n given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

The Fourier Theorem: Continuous case.

Sketch of the Proof:

- Define the partial sum functions

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big]$$

  with a_n and b_n given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

- Express f_N as a convolution of Sine, Cosine, functions and the original function f.

The Fourier Theorem: Continuous case.

Sketch of the Proof:

- Define the partial sum functions

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

  with a_n and b_n given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \qquad n \geqslant 1.$$

- Express f_N as a convolution of Sine, Cosine, functions and the original function f.
- Use the convolution properties to show that

$$\lim_{N\to\infty} f_N(x) = f(x), \qquad x \in [-L, L].$$

□

Overview of Fourier Series (Sect. 6.2).

- Periodic functions.
- Orthogonality of Sines and Cosines.
- The Fourier Theorem: Continuous case.
- **Example: Using the Fourier Theorem.**
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: In this case $L = 1$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: In this case $L = 1$. The Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x) \big],$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: In this case $L = 1$. The Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x) \big],$$

where the a_n, b_n are given in the Theorem.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: In this case $L = 1$. The Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x) \big],$$

where the a_n, b_n are given in the Theorem. We start with a_0,

$$a_0 = \int_{-1}^{1} f(x)\, dx$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: In this case $L = 1$. The Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x)\big],$$

where the a_n, b_n are given in the Theorem. We start with a_0,

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (1 + x)\,dx + \int_{0}^{1} (1 - x)\,dx.$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: In this case $L = 1$. The Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x) \big],$$

where the a_n, b_n are given in the Theorem. We start with a_0,

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (1 + x)\,dx + \int_{0}^{1} (1 - x)\,dx.$$

$$a_0 = \Big(x + \frac{x^2}{2} \Big)\Big|_{-1}^{0} + \Big(x - \frac{x^2}{2} \Big)\Big|_{0}^{1}$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: In this case $L = 1$. The Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x) \big],$$

where the a_n, b_n are given in the Theorem. We start with a_0,

$$a_0 = \int_{-1}^{1} f(x)\, dx = \int_{-1}^{0} (1 + x)\, dx + \int_{0}^{1} (1 - x)\, dx.$$

$$a_0 = \Big(x + \frac{x^2}{2} \Big)\Big|_{-1}^{0} + \Big(x - \frac{x^2}{2} \Big)\Big|_{0}^{1} = \Big(1 - \frac{1}{2} \Big) + \Big(1 - \frac{1}{2} \Big)$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: In this case $L = 1$. The Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x) \big],$$

where the a_n, b_n are given in the Theorem. We start with a_0,

$$a_0 = \int_{-1}^{1} f(x)\, dx = \int_{-1}^{0} (1 + x)\, dx + \int_{0}^{1} (1 - x)\, dx.$$

$$a_0 = \Big(x + \frac{x^2}{2} \Big)\Big|_{-1}^{0} + \Big(x - \frac{x^2}{2} \Big)\Big|_{0}^{1} = \Big(1 - \frac{1}{2} \Big) + \Big(1 - \frac{1}{2} \Big)$$

We obtain: $a_0 = 1$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$. Similarly, the rest of the a_n are given by,

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$. Similarly, the rest of the a_n are given by,

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx$$

$$a_n = \int_{-1}^{0} (1 + x) \cos(n\pi x)\, dx + \int_{0}^{1} (1 - x) \cos(n\pi x)\, dx.$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$. Similarly, the rest of the a_n are given by,

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx$$

$$a_n = \int_{-1}^{0} (1 + x) \cos(n\pi x)\, dx + \int_{0}^{1} (1 - x) \cos(n\pi x)\, dx.$$

Recall the integrals $\displaystyle\int \cos(n\pi x)\, dx = \frac{1}{n\pi} \sin(n\pi x),$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$. Similarly, the rest of the a_n are given by,

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx$$

$$a_n = \int_{-1}^{0} (1 + x) \cos(n\pi x)\, dx + \int_{0}^{1} (1 - x) \cos(n\pi x)\, dx.$$

Recall the integrals $\displaystyle\int \cos(n\pi x)\, dx = \frac{1}{n\pi} \sin(n\pi x)$, and

$$\int x \cos(n\pi x)\, dx = \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x).$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: It is not difficult to see that

$$a_n = \frac{1}{n\pi} \sin(n\pi x)\Big|_{-1}^{0} + \left[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x)\right]\Big|_{-1}^{0}$$
$$+ \frac{1}{n\pi} \sin(n\pi x)\Big|_{0}^{1} - \left[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x)\right]\Big|_{0}^{1}$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: It is not difficult to see that

$$a_n = \frac{1}{n\pi} \sin(n\pi x)\Big|_{-1}^{0} + \Big[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x)\Big]\Big|_{-1}^{0}$$
$$+ \frac{1}{n\pi} \sin(n\pi x)\Big|_{0}^{1} - \Big[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x)\Big]\Big|_{0}^{1}$$

$$a_n = \Big[\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} \cos(-n\pi)\Big] - \Big[\frac{1}{n^2\pi^2} \cos(n\pi) - \frac{1}{n^2\pi^2}\Big].$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: It is not difficult to see that

$$a_n = \frac{1}{n\pi} \sin(n\pi x)\Big|_{-1}^{0} + \Big[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x)\Big]\Big|_{-1}^{0}$$
$$+ \frac{1}{n\pi} \sin(n\pi x)\Big|_{0}^{1} - \Big[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x)\Big]\Big|_{0}^{1}$$

$$a_n = \Big[\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} \cos(-n\pi)\Big] - \Big[\frac{1}{n^2\pi^2} \cos(n\pi) - \frac{1}{n^2\pi^2}\Big].$$

We then conclude that $a_n = \dfrac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$, and $a_n = \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$, and $a_n = \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]$.

Finally, we must find the coefficients b_n.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$, and $a_n = \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]$.

Finally, we must find the coefficients b_n.

A similar calculation shows that $b_n = 0$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$, and $a_n = \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]$.

Finally, we must find the coefficients b_n.

A similar calculation shows that $b_n = 0$.

Then, the Fourier series of f is given by

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big] \cos(n\pi x).$$

◁

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]\, \cos(n\pi x).$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]\,\cos(n\pi x).$

We can obtain a simpler expression for the Fourier coefficients a_n.

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \big[1 - \cos(n\pi)\big] \cos(n\pi x).$

We can obtain a simpler expression for the Fourier coefficients a_n.

Recall the relations $\cos(n\pi) = (-1)^n$,

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \dfrac{1}{2} + \sum_{n=1}^{\infty} \dfrac{2}{n^2\pi^2} \big[1 - \cos(n\pi)\big] \cos(n\pi x).$

We can obtain a simpler expression for the Fourier coefficients a_n.

Recall the relations $\cos(n\pi) = (-1)^n$, then

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \big[1 - (-1)^n\big] \cos(n\pi x).$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \dfrac{2}{n^2 \pi^2} \big[1 - \cos(n\pi)\big] \cos(n\pi x).$

We can obtain a simpler expression for the Fourier coefficients a_n.

Recall the relations $\cos(n\pi) = (-1)^n$, then

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \big[1 - (-1)^n\big] \cos(n\pi x).$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \big[1 + (-1)^{n+1}\big] \cos(n\pi x).$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1 + (-1)^{n+1}\big] \cos(n\pi x).$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \dfrac{2}{n^2\pi^2}\left[1 + (-1)^{n+1}\right] \cos(n\pi x).$

If $n = 2k$,

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[1 + (-1)^{n+1}\right] \cos(n\pi x).$

If $n = 2k$, so n is even,

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left[1 + (-1)^{n+1}\right] \cos(n\pi x).$

If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd,

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[1 + (-1)^{n+1}\right] \cos(n\pi x)$.

If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2\pi^2}\,(1 - 1)$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \dfrac{1}{2} + \sum_{n=1}^{\infty} \dfrac{2}{n^2\pi^2}\left[1 + (-1)^{n+1}\right] \cos(n\pi x).$

If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2\pi^2}\,(1 - 1) \quad \Rightarrow \quad a_{2k} = 0.$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[1 + (-1)^{n+1}\right] \cos(n\pi x).$

If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2\pi^2}\,(1 - 1) \quad \Rightarrow \quad a_{2k} = 0.$$

If $n = 2k - 1$,

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \left[1 + (-1)^{n+1}\right] \cos(n\pi x)$.

If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2 \pi^2} (1 - 1) \quad \Rightarrow \quad a_{2k} = 0.$$

If $n = 2k - 1$, so n is odd,

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left[1 + (-1)^{n+1}\right] \cos(n\pi x).$

If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2\pi^2}(1 - 1) \quad \Rightarrow \quad a_{2k} = 0.$$

If $n = 2k - 1$, so n is odd, so $n + 1 = 2k$ is even,

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[1 + (-1)^{n+1}\right] \cos(n\pi x)$.

If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2\pi^2}(1 - 1) \quad \Rightarrow \quad a_{2k} = 0.$$

If $n = 2k - 1$, so n is odd, so $n + 1 = 2k$ is even, then

$$a_{2k-1} = \frac{2}{(2k-1)^2\pi^2}(1 + 1)$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left[1 + (-1)^{n+1}\right] \cos(n\pi x).$

If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2\pi^2} (1 - 1) \quad \Rightarrow \quad a_{2k} = 0.$$

If $n = 2k - 1$, so n is odd, so $n + 1 = 2k$ is even, then

$$a_{2k-1} = \frac{2}{(2k-1)^2\pi^2} (1 + 1) \quad \Rightarrow \quad a_{2k-1} = \frac{4}{(2k-1)^2\pi^2}.$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution:

Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left[1 + (-1)^{n+1}\right] \cos(n\pi x)$, and

$$a_{2k} = 0, \qquad a_{2k-1} = \frac{4}{(2k-1)^2\pi^2}.$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution:

Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[1 + (-1)^{n+1}\right] \cos(n\pi x)$, and

$$a_{2k} = 0, \qquad a_{2k-1} = \frac{4}{(2k-1)^2\pi^2}.$$

We conclude: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} \cos((2k-1)\pi x).$ $\triangleleft$

Overview of Fourier Series (Sect. 6.2).

- Periodic functions.
- Orthogonality of Sines and Cosines.
- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- **The Fourier Theorem: Piecewise continuous case.**
- Example: Using the Fourier Theorem.

The Fourier Theorem: Piecewise continuous case.

Recall:

Definition

A function $f : [a, b] \to \mathbb{R}$ is called *piecewise continuous* iff holds,

(a) $[a, b]$ can be partitioned in a finite number of sub-intervals such that f is continuous on the interior of these sub-intervals.

(b) f has finite limits at the endpoints of all sub-intervals.

The Fourier Theorem: Piecewise continuous case.

Theorem (Fourier Series)

If $f : [-L, L] \subset \mathbb{R} \to \mathbb{R}$ *is piecewise continuous, then the function*

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big]$$

where a_n *and* b_n *given by*

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

satisfies that:

(a) $f_F(x) = f(x)$ *for all* x *where* f *is continuous;*

(b) $f_F(x_0) = \frac{1}{2}\Big[\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x) \Big]$ *for all* x_0 *where* f *is discontinuous.*

Overview of Fourier Series (Sect. 6.2).

- Periodic functions.
- Orthogonality of Sines and Cosines.
- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- **Example: Using the Fourier Theorem.**

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: We start computing the Fourier coefficients b_n;

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: We start computing the Fourier coefficients b_n;

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx,$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: We start computing the Fourier coefficients b_n;

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad L = 1,$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: We start computing the Fourier coefficients b_n;

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad L = 1,$$

$$b_n = \int_{-1}^{0} (-1) \sin\big(n\pi x\big)\, dx + \int_{0}^{1} (1) \sin\big(n\pi x\big)\, dx,$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: We start computing the Fourier coefficients b_n;

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad L = 1,$$

$$b_n = \int_{-1}^{0} (-1) \sin\big(n\pi x\big)\, dx + \int_{0}^{1} (1) \sin\big(n\pi x\big)\, dx,$$

$$b_n = \frac{(-1)}{n\pi} \Big[-\cos(n\pi x)\Big|_{-1}^{0}\Big] + \frac{1}{n\pi} \Big[-\cos(n\pi x)\Big|_{0}^{1}\Big],$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: We start computing the Fourier coefficients b_n;

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \qquad L = 1,$$

$$b_n = \int_{-1}^{0} (-1) \sin\big(n\pi x\big)\, dx + \int_{0}^{1} (1) \sin\big(n\pi x\big)\, dx,$$

$$b_n = \frac{(-1)}{n\pi}\left[-\cos(n\pi x)\Big|_{-1}^{0}\right] + \frac{1}{n\pi}\left[-\cos(n\pi x)\Big|_{0}^{1}\right],$$

$$b_n = \frac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \frac{1}{n\pi}\big[-\cos(n\pi) + 1\big].$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: $b_n = \dfrac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \dfrac{1}{n\pi}\big[-\cos(n\pi) + 1\big]$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: $b_n = \dfrac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \dfrac{1}{n\pi}\big[-\cos(n\pi) + 1\big]$.

$$b_n = \frac{1}{n\pi}\big[1 - \cos(-n\pi) - \cos(n\pi) + 1\big]$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: $b_n = \dfrac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \dfrac{1}{n\pi}\big[-\cos(n\pi) + 1\big]$.

$$b_n = \frac{1}{n\pi}\big[1 - \cos(-n\pi) - \cos(n\pi) + 1\big] = \frac{2}{n\pi}\big[1 - \cos(n\pi)\big],$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: $b_n = \frac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \frac{1}{n\pi}\big[-\cos(n\pi) + 1\big]$.

$$b_n = \frac{1}{n\pi}\big[1 - \cos(-n\pi) - \cos(n\pi) + 1\big] = \frac{2}{n\pi}\big[1 - \cos(n\pi)\big],$$

We obtain: $b_n = \frac{2}{n\pi}\big[1 - (-1)^n\big]$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: $b_n = \dfrac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \dfrac{1}{n\pi}\big[-\cos(n\pi) + 1\big]$.

$$b_n = \frac{1}{n\pi}\big[1 - \cos(-n\pi) - \cos(n\pi) + 1\big] = \frac{2}{n\pi}\big[1 - \cos(n\pi)\big],$$

We obtain: $b_n = \dfrac{2}{n\pi}\big[1 - (-1)^n\big]$.

If $n = 2k$,

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: $b_n = \dfrac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \dfrac{1}{n\pi}\big[-\cos(n\pi) + 1\big].$

$$b_n = \frac{1}{n\pi}\big[1 - \cos(-n\pi) - \cos(n\pi) + 1\big] = \frac{2}{n\pi}\big[1 - \cos(n\pi)\big],$$

We obtain: $b_n = \dfrac{2}{n\pi}\big[1 - (-1)^n\big].$

If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[1 - (-1)^{2k}\big],$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: $b_n = \dfrac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \dfrac{1}{n\pi}\big[-\cos(n\pi) + 1\big]$.

$$b_n = \frac{1}{n\pi}\big[1 - \cos(-n\pi) - \cos(n\pi) + 1\big] = \frac{2}{n\pi}\big[1 - \cos(n\pi)\big],$$

We obtain: $b_n = \dfrac{2}{n\pi}\big[1 - (-1)^n\big]$.

If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[1 - (-1)^{2k}\big]$, hence $b_{2k} = 0$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: $b_n = \frac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \frac{1}{n\pi}\big[-\cos(n\pi) + 1\big].$

$$b_n = \frac{1}{n\pi}\big[1 - \cos(-n\pi) - \cos(n\pi) + 1\big] = \frac{2}{n\pi}\big[1 - \cos(n\pi)\big],$$

We obtain: $b_n = \frac{2}{n\pi}\big[1 - (-1)^n\big].$

If $n = 2k$, then $b_{2k} = \frac{2}{2k\pi}\big[1 - (-1)^{2k}\big]$, hence $b_{2k} = 0$.

If $n = 2k - 1$,

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: $b_n = \frac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \frac{1}{n\pi}\big[-\cos(n\pi) + 1\big].$

$$b_n = \frac{1}{n\pi}\big[1 - \cos(-n\pi) - \cos(n\pi) + 1\big] = \frac{2}{n\pi}\big[1 - \cos(n\pi)\big],$$

We obtain: $b_n = \frac{2}{n\pi}\big[1 - (-1)^n\big].$

If $n = 2k$, then $b_{2k} = \frac{2}{2k\pi}\big[1 - (-1)^{2k}\big]$, hence $b_{2k} = 0$.

If $n = 2k - 1$, then $b_{2k-1} = \frac{2}{(2k-1)\pi}\big[1 - (-1)^{2k-1}\big],$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: $b_n = \frac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \frac{1}{n\pi}\big[-\cos(n\pi) + 1\big].$

$$b_n = \frac{1}{n\pi}\big[1 - \cos(-n\pi) - \cos(n\pi) + 1\big] = \frac{2}{n\pi}\big[1 - \cos(n\pi)\big],$$

We obtain: $b_n = \frac{2}{n\pi}\big[1 - (-1)^n\big].$

If $n = 2k$, then $b_{2k} = \frac{2}{2k\pi}\big[1 - (-1)^{2k}\big]$, hence $b_{2k} = 0$.

If $n = 2k - 1$, then $b_{2k-1} = \frac{2}{(2k-1)\pi}\big[1 - (-1)^{2k-1}\big]$,

hence $b_{2k} = \frac{4}{(2k-1)\pi}.$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, and $b_{2k} = \dfrac{4}{(2k-1)\pi}$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, and $b_{2k} = \dfrac{4}{(2k-1)\pi}$.

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx,$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, and $b_{2k} = \dfrac{4}{(2k-1)\pi}$.

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \qquad L = 1,$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, and $b_{2k} = \dfrac{4}{(2k-1)\pi}$.

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad L = 1,$$

$$a_n = \int_{-1}^{0} (-1) \cos\big(n\pi x\big)\, dx + \int_{0}^{1} (1) \cos\big(n\pi x\big)\, dx,$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1,0), \\ 1 & x \in [0,1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, and $b_{2k} = \dfrac{4}{(2k-1)\pi}$.

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad L = 1,$$

$$a_n = \int_{-1}^{0} (-1) \cos\big(n\pi x\big)\, dx + \int_{0}^{1} (1) \cos\big(n\pi x\big)\, dx,$$

$$a_n = \frac{(-1)}{n\pi} \Big[\sin(n\pi x)\Big|_{-1}^{0} \Big] + \frac{1}{n\pi} \Big[\sin(n\pi x)\Big|_{0}^{1} \Big],$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, and $b_{2k} = \dfrac{4}{(2k-1)\pi}$.

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad L = 1,$$

$$a_n = \int_{-1}^{0} (-1) \cos\big(n\pi x\big)\, dx + \int_{0}^{1} (1) \cos\big(n\pi x\big)\, dx,$$

$$a_n = \frac{(-1)}{n\pi} \Big[\sin(n\pi x)\Big|_{-1}^{0}\Big] + \frac{1}{n\pi} \Big[\sin(n\pi x)\Big|_{0}^{1}\Big],$$

$$a_n = \frac{(-1)}{n\pi} \big[0 - \sin(-n\pi)\big] + \frac{1}{n\pi} \big[\sin(n\pi) - 0\big]$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, and $b_{2k} = \dfrac{4}{(2k-1)\pi}$.

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad L = 1,$$

$$a_n = \int_{-1}^{0} (-1) \cos\big(n\pi x\big)\, dx + \int_{0}^{1} (1) \cos\big(n\pi x\big)\, dx,$$

$$a_n = \frac{(-1)}{n\pi}\Big[\sin(n\pi x)\Big|_{-1}^{0}\Big] + \frac{1}{n\pi}\Big[\sin(n\pi x)\Big|_{0}^{1}\Big],$$

$$a_n = \frac{(-1)}{n\pi}\big[0 - \sin(-n\pi)\big] + \frac{1}{n\pi}\big[\sin(n\pi) - 0\big] \quad \Rightarrow \quad a_n = 0.$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, $b_{2k} = \dfrac{4}{(2k-1)\pi}$, and $a_n = 0$.

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, $b_{2k} = \dfrac{4}{(2k-1)\pi}$, and $a_n = 0$.

Therefore, we conclude that

$$f_F(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin\big((2k-1)\pi\, x\big).$$

$\triangleleft$

Solving the Heat Equation (Sect. 6.3).

- Review: The Stationary Heat Equation.
- The Heat Equation.
- The Initial-Boundary Value Problem.
- The separation of variables method.
- An example of separation of variables.

Review: The Stationary Heat Equation.

Review: The Stationary Heat Equation describes the temperature distribution in a solid material in thermal equilibrium. The temperature is time-independent.

Review: The Stationary Heat Equation.

Review: The Stationary Heat Equation describes the temperature distribution in a solid material in thermal equilibrium. The temperature is time-independent.

Problem: The time-independent temperature, T, of a bar of length L with insulated horizontal sides and vertical extremes kept at fixed temperatures T_0, T_L, is the solution of the BVP:

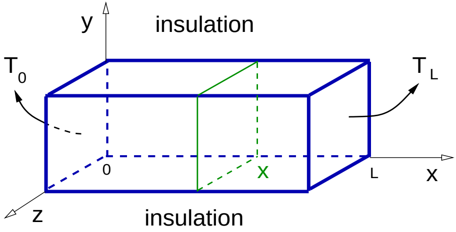
$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$

Review: The Stationary Heat Equation.

Review: The Stationary Heat Equation describes the temperature distribution in a solid material in thermal equilibrium. The temperature is time-independent.

Problem: The time-independent temperature, T, of a bar of length L with insulated horizontal sides and vertical extremes kept at fixed temperatures T_0, T_L, is the solution of the BVP:

$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$

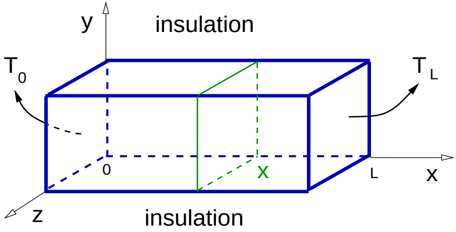


Review: The Stationary Heat Equation.

Review: The Stationary Heat Equation describes the temperature distribution in a solid material in thermal equilibrium. The temperature is time-independent.

Problem: The time-independent temperature, T, of a bar of length L with insulated horizontal sides and vertical extremes kept at fixed temperatures T_0, T_L, is the solution of the BVP:

$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$



Remark: The heat transfer occurs only along the x-axis.

Solving the Heat Equation (Sect. 6.3).

- Review: The Stationary Heat Equation.
- **The Heat Equation.**
- The Initial-Boundary Value Problem.
- The separation of variables method.
- An example of separation of variables.

The Heat Equation.

Remarks:

- The unknown of the problem is $u(t, x)$, the temperature of the bar at the time t and position x.

The Heat Equation.

Remarks:

- The unknown of the problem is $u(t, x)$, the temperature of the bar at the time t and position x.
- The temperature does not depend on y or z.

The Heat Equation.

Remarks:

- The unknown of the problem is $u(t,x)$, the temperature of the bar at the time t and position x.
- The temperature does not depend on y or z.
- The one-dimensional Heat Equation is:

$$\partial_t u(t,x) = k\,\partial_x^2 u(t,x),$$

The Heat Equation.

Remarks:

- The unknown of the problem is $u(t,x)$, the temperature of the bar at the time t and position x.
- The temperature does not depend on y or z.
- The one-dimensional Heat Equation is:

$$\partial_t u(t,x) = k\,\partial_x^2 u(t,x),$$

where $k > 0$ is the heat conductivity,

The Heat Equation.

Remarks:

- The unknown of the problem is $u(t,x)$, the temperature of the bar at the time t and position x.
- The temperature does not depend on y or z.
- The one-dimensional Heat Equation is:

$$\partial_t u(t,x) = k\, \partial_x^2 u(t,x),$$

where $k > 0$ is the heat conductivity, units: $[k] = \dfrac{(\text{distance})^2}{(\text{time})}$.

The Heat Equation.

Remarks:

- The unknown of the problem is $u(t,x)$, the temperature of the bar at the time t and position x.
- The temperature does not depend on y or z.
- The one-dimensional Heat Equation is:

$$\partial_t u(t,x) = k\,\partial_x^2 u(t,x),$$

where $k > 0$ is the heat conductivity, units: $[k] = \dfrac{(\text{distance})^2}{(\text{time})}$.

- The Heat Equation is a Partial Differential Equation, PDE.

The Heat Equation.

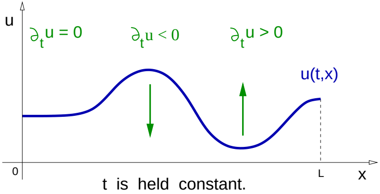
Remarks:

- The unknown of the problem is $u(t,x)$, the temperature of the bar at the time t and position x.
- The temperature does not depend on y or z.
- The one-dimensional Heat Equation is:

$$\partial_t u(t,x) = k\, \partial_x^2 u(t,x),$$

where $k > 0$ is the heat conductivity, units: $[k] = \frac{(\text{distance})^2}{(\text{time})}$.

- The Heat Equation is a Partial Differential Equation, PDE.



Solving the Heat Equation (Sect. 6.3).

- Review: The Stationary Heat Equation.
- The Heat Equation.
- **The Initial-Boundary Value Problem.**
- The separation of variables method.
- An example of separation of variables.

The Initial-Boundary Value Problem.

Definition

The IBVP for the one-dimensional Heat Equation is the following: Given a constant $k > 0$ and a function $f : [0, L] \to \mathbb{R}$ with $f(0) = f(L) = 0$, find $u : [0, \infty) \times [0, L] \to \mathbb{R}$ solution of

$$\partial_t u(t, x) = k\, \partial_x^2 u(t, x),$$

The Initial-Boundary Value Problem.

Definition

The IBVP for the one-dimensional Heat Equation is the following: Given a constant $k > 0$ and a function $f : [0, L] \to \mathbb{R}$ with $f(0) = f(L) = 0$, find $u : [0, \infty) \times [0, L] \to \mathbb{R}$ solution of

$$\partial_t u(t, x) = k\, \partial_x^2 u(t, x),$$

$$\text{I.C.:} \quad u(0, x) = f(x),$$

The Initial-Boundary Value Problem.

Definition

The IBVP for the one-dimensional Heat Equation is the following: Given a constant $k > 0$ and a function $f : [0, L] \to \mathbb{R}$ with $f(0) = f(L) = 0$, find $u : [0, \infty) \times [0, L] \to \mathbb{R}$ solution of

$$\partial_t u(t,x) = k\, \partial_x^2 u(t,x),$$

$$\text{I.C.:} \quad u(0,x) = f(x),$$

$$\text{B.C.:} \quad u(t,0) = 0, \quad u(t,L) = 0.$$

The Initial-Boundary Value Problem.

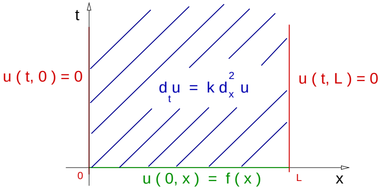
Definition

The IBVP for the one-dimensional Heat Equation is the following: Given a constant $k > 0$ and a function $f : [0, L] \to \mathbb{R}$ with $f(0) = f(L) = 0$, find $u : [0, \infty) \times [0, L] \to \mathbb{R}$ solution of

$$\partial_t u(t, x) = k\, \partial_x^2 u(t, x),$$

$$\text{I.C.:} \quad u(0, x) = f(x),$$

$$\text{B.C.:} \quad u(t, 0) = 0, \quad u(t, L) = 0.$$



Solving the Heat Equation (Sect. 6.3).

- Review: The Stationary Heat Equation.
- The Heat Equation.
- The Initial-Boundary Value Problem.
- **The separation of variables method.**
- An example of separation of variables.

The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x).$$

The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x).$$

where

- v_n: Solution of an IVP.

The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x).$$

where

- v_n: Solution of an IVP.
- w_n: Solution of a BVP, an eigenvalue-eigenfunction problem.

The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t, x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x).$$

where

- v_n: Solution of an IVP.
- w_n: Solution of a BVP, an eigenvalue-eigenfunction problem.
- c_n: Fourier Series coefficients.

The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x).$$

where

- v_n: Solution of an IVP.
- w_n: Solution of a BVP, an eigenvalue-eigenfunction problem.
- c_n: Fourier Series coefficients.

Remark:
The separation of variables method does not work for every PDE.

The separation of variables method.

Summary:

- The idea is to transform the PDE into infinitely many ODEs.

The separation of variables method.

Summary:

- The idea is to transform the PDE into infinitely many ODEs.
- We describe this method in 6 steps.

The separation of variables method.

Summary:

- The idea is to transform the PDE into infinitely many ODEs.
- We describe this method in 6 steps.

Step 1:

One looks for solutions u given by an infinite series of simpler functions, u_n,

The separation of variables method.

Summary:

- The idea is to transform the PDE into infinitely many ODEs.
- We describe this method in 6 steps.

Step 1:

One looks for solutions u given by an infinite series of simpler functions, u_n, that is,

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, u_n(t,x),$$

The separation of variables method.

Summary:

- The idea is to transform the PDE into infinitely many ODEs.
- We describe this method in 6 steps.

Step 1:

One looks for solutions u given by an infinite series of simpler functions, u_n, that is,

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, u_n(t,x),$$

where u_n is simpler than u is the sense,

$$u_n(t,x) = v_n(t)\, w_n(x).$$

The separation of variables method.

Summary:

- The idea is to transform the PDE into infinitely many ODEs.
- We describe this method in 6 steps.

Step 1:

One looks for solutions u given by an infinite series of simpler functions, u_n, that is,

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, u_n(t,x),$$

where u_n is simpler than u is the sense,

$$u_n(t,x) = v_n(t) \, w_n(x).$$

Here c_n are constants, $n = 1, 2, \cdots$.

The separation of variables method.

Step 2:

Introduce the series expansion for u into the Heat Equation,

$$\partial_t u - k\,\partial_x^2 u = 0$$

The separation of variables method.

Step 2:

Introduce the series expansion for u into the Heat Equation,

$$\partial_t u - k\,\partial_x^2 u = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} c_n \left[\partial_t u_n - k\,\partial_x^2 u_n\right] = 0.$$

The separation of variables method.

Step 2:

Introduce the series expansion for u into the Heat Equation,

$$\partial_t u - k\,\partial_x^2 u = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} c_n \left[\partial_t u_n - k\,\partial_x^2 u_n\right] = 0.$$

A sufficient condition for the equation above is:

The separation of variables method.

Step 2:

Introduce the series expansion for u into the Heat Equation,

$$\partial_t u - k\,\partial_x^2 u = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} c_n \left[\partial_t u_n - k\,\partial_x^2 u_n\right] = 0.$$

A sufficient condition for the equation above is: To find u_n, for $n = 1, 2, \cdots$,

The separation of variables method.

Step 2:

Introduce the series expansion for u into the Heat Equation,

$$\partial_t u - k\,\partial_x^2 u = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} c_n \left[\partial_t u_n - k\,\partial_x^2 u_n\right] = 0.$$

A sufficient condition for the equation above is: To find u_n, for $n = 1, 2, \cdots$, solutions of

$$\partial_t u_n - k\,\partial_x^2 u_n = 0.$$

The separation of variables method.

Step 2:

Introduce the series expansion for u into the Heat Equation,

$$\partial_t u - k\,\partial_x^2 u = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} c_n \left[\partial_t u_n - k\,\partial_x^2 u_n\right] = 0.$$

A sufficient condition for the equation above is: To find u_n, for $n = 1, 2, \cdots$, solutions of

$$\partial_t u_n - k\,\partial_x^2 u_n = 0.$$

Step 3:

Find $u_n(t,x) = v_n(t)\,w_n(x)$ solution of the IBVP

$$\partial_t u_n - k\,\partial_x^2 u_n = 0.$$

The separation of variables method.

Step 2:

Introduce the series expansion for u into the Heat Equation,

$$\partial_t u - k\,\partial_x^2 u = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} c_n \left[\partial_t u_n - k\,\partial_x^2 u_n\right] = 0.$$

A sufficient condition for the equation above is: To find u_n, for $n = 1, 2, \cdots$, solutions of

$$\partial_t u_n - k\,\partial_x^2 u_n = 0.$$

Step 3:

Find $u_n(t,x) = v_n(t)\,w_n(x)$ solution of the IBVP

$$\partial_t u_n - k\,\partial_x^2 u_n = 0.$$

$$\text{I.C.:} \quad u_n(0,x) = w_n(x),$$

The separation of variables method.

Step 2:

Introduce the series expansion for u into the Heat Equation,

$$\partial_t u - k\,\partial_x^2 u = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} c_n \left[\partial_t u_n - k\,\partial_x^2 u_n\right] = 0.$$

A sufficient condition for the equation above is: To find u_n, for $n = 1, 2, \cdots$, solutions of

$$\partial_t u_n - k\,\partial_x^2 u_n = 0.$$

Step 3:

Find $u_n(t,x) = v_n(t)\,w_n(x)$ solution of the IBVP

$$\partial_t u_n - k\,\partial_x^2 u_n = 0.$$

$$\text{I.C.:} \quad u_n(0,x) = w_n(x),$$

$$\text{B.C.:} \quad u_n(t,0) = 0, \quad u_n(t,L) = 0.$$

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into:

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n;

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

Notice:

$$\partial_t u_n(t,x)$$

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

Notice:

$$\partial_t u_n(t,x) = \partial_t \left[v_n(t)\, w_n(x) \right]$$

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

Notice:

$$\partial_t u_n(t,x) = \partial_t \big[v_n(t)\, w_n(x) \big] = w_n(x)\, \frac{dv_n}{dt}(t).$$

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

Notice:

$$\partial_t u_n(t,x) = \partial_t \big[v_n(t)\, w_n(x) \big] = w_n(x)\, \frac{dv_n}{dt}(t).$$

$$\partial_x^2 u_n(t,x)$$

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

Notice:

$$\partial_t u_n(t,x) = \partial_t \big[v_n(t)\, w_n(x) \big] = w_n(x)\, \frac{dv_n}{dt}(t).$$

$$\partial_x^2 u_n(t,x) = \partial_x^2 \big[v_n(t)\, w_n(x) \big]$$

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

Notice:

$$\partial_t u_n(t,x) = \partial_t \big[v_n(t)\, w_n(x) \big] = w_n(x)\, \frac{dv_n}{dt}(t).$$

$$\partial_x^2 u_n(t,x) = \partial_x^2 \big[v_n(t)\, w_n(x) \big] = v_n(t)\, \frac{d^2 w_n}{dx^2}(x).$$

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

Notice:

$$\partial_t u_n(t,x) = \partial_t \big[v_n(t)\, w_n(x) \big] = w_n(x)\, \frac{dv_n}{dt}(t).$$

$$\partial_x^2 u_n(t,x) = \partial_x^2 \big[v_n(t)\, w_n(x) \big] = v_n(t)\, \frac{d^2 w_n}{dx^2}(x).$$

Therefore, the equation $\partial_t u_n = k\partial_x^2 u_n$

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

Notice:

$$\partial_t u_n(t,x) = \partial_t \big[v_n(t)\, w_n(x) \big] = w_n(x)\, \frac{dv_n}{dt}(t).$$

$$\partial_x^2 u_n(t,x) = \partial_x^2 \big[v_n(t)\, w_n(x) \big] = v_n(t)\, \frac{d^2 w_n}{dx^2}(x).$$

Therefore, the equation $\partial_t u_n = k \partial_x^2 u_n$ is given by

$$w_n(x)\, \frac{dv_n}{dt}(t) = k\, v_n(t)\, \frac{d^2 w_n}{dx^2}(x)$$

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

Notice:

$$\partial_t u_n(t,x) = \partial_t \big[v_n(t)\, w_n(x) \big] = w_n(x)\, \frac{dv_n}{dt}(t).$$

$$\partial_x^2 u_n(t,x) = \partial_x^2 \big[v_n(t)\, w_n(x) \big] = v_n(t)\, \frac{d^2 w_n}{dx^2}(x).$$

Therefore, the equation $\partial_t u_n = k\partial_x^2 u_n$ is given by

$$w_n(x)\, \frac{dv_n}{dt}(t) = k\, v_n(t)\, \frac{d^2 w_n}{dx^2}(x)$$

$$\frac{1}{k\, v_n(t)}\, \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)}\, \frac{d^2 w_n}{dx^2}(x).$$

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

Notice:

$$\partial_t u_n(t,x) = \partial_t \big[v_n(t)\, w_n(x) \big] = w_n(x)\, \frac{dv_n}{dt}(t).$$

$$\partial_x^2 u_n(t,x) = \partial_x^2 \big[v_n(t)\, w_n(x) \big] = v_n(t)\, \frac{d^2 w_n}{dx^2}(x).$$

Therefore, the equation $\partial_t u_n = k \partial_x^2 u_n$ is given by

$$w_n(x)\, \frac{dv_n}{dt}(t) = k\, v_n(t)\, \frac{d^2 w_n}{dx^2}(x)$$

$$\frac{1}{k\, v_n(t)}\, \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)}\, \frac{d^2 w_n}{dx^2}(x).$$

Depends only on t

The separation of variables method.

Step 4: (Key step.)
Transform the IBVP for u_n into: (a) IVP for v_n; (b) BVP for w_n.

Notice:

$$\partial_t u_n(t,x) = \partial_t \big[v_n(t)\, w_n(x)\big] = w_n(x)\, \frac{dv_n}{dt}(t).$$

$$\partial_x^2 u_n(t,x) = \partial_x^2 \big[v_n(t)\, w_n(x)\big] = v_n(t)\, \frac{d^2 w_n}{dx^2}(x).$$

Therefore, the equation $\partial_t u_n = k\partial_x^2 u_n$ is given by

$$w_n(x)\, \frac{dv_n}{dt}(t) = k\, v_n(t)\, \frac{d^2 w_n}{dx^2}(x)$$

$$\frac{1}{k\, v_n(t)}\, \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)}\, \frac{d^2 w_n}{dx^2}(x).$$

Depends only on t = Depends only on x.

The separation of variables method.

Recall:
$$\frac{1}{k\, v_n(t)}\, \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)}\, \frac{d^2 w_n}{dx^2}(x).$$

Depends only on t $\quad = \quad$ Depends only on x.

The separation of variables method.

Recall:
$$\frac{1}{k\,v_n(t)}\,\frac{dv_n}{dt}(t) = \frac{1}{w_n(x)}\,\frac{d^2w_n}{dx^2}(x).$$

Depends only on t $=$ Depends only on x.

- The Heat Equation has the following property:
  The left-hand side depends only on t, while the right-hand side depends only on x.

The separation of variables method.

Recall:
$$\frac{1}{k\,v_n(t)}\,\frac{dv_n}{dt}(t) = \frac{1}{w_n(x)}\,\frac{d^2 w_n}{dx^2}(x).$$

Depends only on t $=$ Depends only on x.

- The Heat Equation has the following property:
  The left-hand side depends only on t, while the right-hand side depends only on x.
- When this happens in a PDE, one can use the separation of variables method on that PDE.

The separation of variables method.

Recall:
$$\frac{1}{k\, v_n(t)}\,\frac{dv_n}{dt}(t) = \frac{1}{w_n(x)}\,\frac{d^2 w_n}{dx^2}(x).$$

Depends only on t = Depends only on x.

- The Heat Equation has the following property:
  The left-hand side depends only on t, while the right-hand side depends only on x.
- When this happens in a PDE, one can use the separation of variables method on that PDE.
- We conclude that for appropriate constants λ_m holds

$$\frac{1}{k\, v_n(t)}\,\frac{dv_n}{dt}(t) = -\lambda_n,$$

The separation of variables method.

Recall: $$\frac{1}{k\, v_n(t)}\, \frac{dv_n}{dt}(t) = \frac{1}{w_n(x)}\, \frac{d^2 w_n}{dx^2}(x).$$

Depends only on t = Depends only on x.

- The Heat Equation has the following property:
  The left-hand side depends only on t, while the right-hand side depends only on x.
- When this happens in a PDE, one can use the separation of variables method on that PDE.
- We conclude that for appropriate constants λ_m holds

$$\frac{1}{k\, v_n(t)}\, \frac{dv_n}{dt}(t) = -\lambda_n, \qquad \frac{1}{w_n(x)}\, \frac{d^2 w_n}{dx^2}(x) = -\lambda_n.$$

The separation of variables method.

Recall: $$\frac{1}{k\,v_n(t)}\,\frac{dv_n}{dt}(t) = \frac{1}{w_n(x)}\,\frac{d^2w_n}{dx^2}(x).$$

$$\text{Depends only on } t \quad = \quad \text{Depends only on } x.$$

- The Heat Equation has the following property:
  The left-hand side depends only on t, while the right-hand side depends only on x.
- When this happens in a PDE, one can use the separation of variables method on that PDE.
- We conclude that for appropriate constants λ_m holds

$$\frac{1}{k\,v_n(t)}\,\frac{dv_n}{dt}(t) = -\lambda_n, \qquad \frac{1}{w_n(x)}\,\frac{d^2w_n}{dx^2}(x) = -\lambda_n.$$

- We have transformed the original PDE into infinitely many ODEs parametrized by n, positive integer.

The separation of variables method.

Summary Step 4: The original *IBVP* for the Heat Equation, PDE, can transformed into:

The separation of variables method.

Summary Step 4: The original *IBVP* for the Heat Equation, PDE, can transformed into:

(a) We choose to solve the following IVP for v_n,

$$\frac{1}{k\, v_n(t)} \frac{dv_n}{dt}(t) = -\lambda_n,$$

The separation of variables method.

Summary Step 4: The original *IBVP* for the Heat Equation, PDE, can transformed into:

(a) We choose to solve the following IVP for v_n,

$$\frac{1}{k\, v_n(t)}\,\frac{dv_n}{dt}(t) = -\lambda_n, \quad \text{I.C.:} \quad v_n(0) = 1.$$

The separation of variables method.

Summary Step 4: The original *IBVP* for the Heat Equation, PDE, can transformed into:

(a) We choose to solve the following IVP for v_n,

$$\frac{1}{k\,v_n(t)}\,\frac{dv_n}{dt}(t) = -\lambda_n, \quad \text{I.C.:} \quad v_n(0) = 1.$$

Remark: This choice of I.C. simplifies the problem.

The separation of variables method.

Summary Step 4: The original *IBVP* for the Heat Equation, PDE, can transformed into:

(a) We choose to solve the following IVP for v_n,

$$\frac{1}{k\, v_n(t)}\,\frac{dv_n}{dt}(t) = -\lambda_n, \quad \text{I.C.:} \quad v_n(0) = 1.$$

Remark: This choice of I.C. simplifies the problem.

(b) The BVP for w_n,

$$\frac{1}{w_n(x)}\,\frac{d^2 w_n}{dx^2}(x) = -\lambda_n,$$

The separation of variables method.

Summary Step 4: The original *IBVP* for the Heat Equation, PDE, can transformed into:

(a) We choose to solve the following IVP for v_n,

$$\frac{1}{k\, v_n(t)}\, \frac{dv_n}{dt}(t) = -\lambda_n, \quad \text{I.C.:} \quad v_n(0) = 1.$$

Remark: This choice of I.C. simplifies the problem.

(b) The BVP for w_n,

$$\frac{1}{w_n(x)}\, \frac{d^2 w_n}{dx^2}(x) = -\lambda_n, \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0.$$

The separation of variables method.

Summary Step 4: The original *IBVP* for the Heat Equation, PDE, can transformed into:

(a) We choose to solve the following IVP for v_n,

$$\frac{1}{k\, v_n(t)}\,\frac{dv_n}{dt}(t) = -\lambda_n, \quad \text{I.C.:} \quad v_n(0) = 1.$$

Remark: This choice of I.C. simplifies the problem.

(b) The BVP for w_n,

$$\frac{1}{w_n(x)}\,\frac{d^2 w_n}{dx^2}(x) = -\lambda_n, \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0.$$

Step 5:

(a) Solve the IVP for v_n.

(b) Solve the BVP for w_n.

The separation of variables method.

Step 5(a): Solving the IVP for v_n.

$$v_n'(t) + k\lambda_n\, v_n(t) = 0,$$

The separation of variables method.

Step 5(a): Solving the IVP for v_n.

$$v_n'(t) + k\lambda_n \, v_n(t) = 0, \quad \text{I.C.:} \quad v_n(0) = 1.$$

The separation of variables method.

Step 5(a): Solving the IVP for v_n.

$$v_n'(t) + k\lambda_n \, v_n(t) = 0, \quad \text{I.C.:} \quad v_n(0) = 1.$$

The integrating factor method implies that $\mu(t) = e^{k\lambda_n t}$.

$$e^{k\lambda_n t} v_n'(t) + k\lambda_n \, e^{k\lambda_n t} \, v_n(t) = 0$$

The separation of variables method.

Step 5(a): Solving the IVP for v_n.

$$v_n'(t) + k\lambda_n\, v_n(t) = 0, \quad \text{I.C.:} \quad v_n(0) = 1.$$

The integrating factor method implies that $\mu(t) = e^{k\lambda_n t}$.

$$e^{k\lambda_n t} v_n'(t) + k\lambda_n\, e^{k\lambda_n t}\, v_n(t) = 0 \quad \Rightarrow \quad \left[e^{k\lambda_n t} v_n(t)\right]' = 0.$$

The separation of variables method.

Step 5(a): Solving the IVP for v_n.

$$v_n'(t) + k\lambda_n\, v_n(t) = 0, \quad \text{I.C.:} \quad v_n(0) = 1.$$

The integrating factor method implies that $\mu(t) = e^{k\lambda_n t}$.

$$e^{k\lambda_n t} v_n'(t) + k\lambda_n\, e^{k\lambda_n t}\, v_n(t) = 0 \quad \Rightarrow \quad \Big[e^{k\lambda_n t} v_n(t)\Big]' = 0.$$

$$e^{k\lambda_n t} v_n(t) = c_n$$

The separation of variables method.

Step 5(a): Solving the IVP for v_n.

$$v_n'(t) + k\lambda_n\, v_n(t) = 0, \quad \text{I.C.:} \quad v_n(0) = 1.$$

The integrating factor method implies that $\mu(t) = e^{k\lambda_n t}$.

$$e^{k\lambda_n t} v_n'(t) + k\lambda_n\, e^{k\lambda_n t}\, v_n(t) = 0 \quad \Rightarrow \quad \Big[e^{k\lambda_n t} v_n(t)\Big]' = 0.$$

$$e^{k\lambda_n t} v_n(t) = c_n \quad \Rightarrow \quad v_n(t) = c_n\, e^{-k\lambda_n t}.$$

The separation of variables method.

Step 5(a): Solving the IVP for v_n.

$$v_n'(t) + k\lambda_n\, v_n(t) = 0, \quad \text{I.C.:} \quad v_n(0) = 1.$$

The integrating factor method implies that $\mu(t) = e^{k\lambda_n t}$.

$$e^{k\lambda_n t} v_n'(t) + k\lambda_n\, e^{k\lambda_n t}\, v_n(t) = 0 \quad \Rightarrow \quad \left[e^{k\lambda_n t} v_n(t)\right]' = 0.$$

$$e^{k\lambda_n t} v_n(t) = c_n \quad \Rightarrow \quad v_n(t) = c_n\, e^{-k\lambda_n t}.$$

$$1 = v_n(0) = c$$

The separation of variables method.

Step 5(a): Solving the IVP for v_n.

$$v_n'(t) + k\lambda_n \, v_n(t) = 0, \quad \text{I.C.:} \quad v_n(0) = 1.$$

The integrating factor method implies that $\mu(t) = e^{k\lambda_n t}$.

$$e^{k\lambda_n t} v_n'(t) + k\lambda_n \, e^{k\lambda_n t} \, v_n(t) = 0 \quad \Rightarrow \quad \Big[e^{k\lambda_n t} v_n(t) \Big]' = 0.$$

$$e^{k\lambda_n t} v_n(t) = c_n \quad \Rightarrow \quad v_n(t) = c_n \, e^{-k\lambda_n t}.$$

$$1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-k\lambda_n t}.$$

The separation of variables method.

Step 5(a): Recall: $v_n(t) = e^{-k\lambda_n t}$.

The separation of variables method.

Step 5(a): Recall: $v_n(t) = e^{-k\lambda_n t}$.

Step 5(b): Eigenvalue-eigenvector problem for w_n:

The separation of variables method.

Step 5(a): Recall: $v_n(t) = e^{-k\lambda_n t}$.

Step 5(b): Eigenvalue-eigenvector problem for w_n:
Find the eigenvalues λ_n and the non-zero eigenfunctions w_n solutions of the BVP

$$w_n''(x) + \lambda_n \, w_n(x) = 0$$

The separation of variables method.

Step 5(a): Recall: $v_n(t) = e^{-k\lambda_n t}$.

Step 5(b): Eigenvalue-eigenvector problem for w_n:
Find the eigenvalues λ_n and the non-zero eigenfunctions w_n solutions of the BVP

$$w_n''(x) + \lambda_n w_n(x) = 0 \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0.$$

The separation of variables method.

Step 5(a): Recall: $v_n(t) = e^{-k\lambda_n t}$.

Step 5(b): Eigenvalue-eigenvector problem for w_n:
Find the eigenvalues λ_n and the non-zero eigenfunctions w_n solutions of the BVP

$$w_n''(x) + \lambda_n \, w_n(x) = 0 \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0.$$

We know that this problem has solution only for $\lambda_n > 0$.

The separation of variables method.

Step 5(a): Recall: $v_n(t) = e^{-k\lambda_n t}$.

Step 5(b): Eigenvalue-eigenvector problem for w_n:
Find the eigenvalues λ_n and the non-zero eigenfunctions w_n solutions of the BVP

$$w_n''(x) + \lambda_n \, w_n(x) = 0 \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0.$$

We know that this problem has solution only for $\lambda_n > 0$.
Denote: $\lambda_n = \mu_n^2$.

The separation of variables method.

Step 5(a): Recall: $v_n(t) = e^{-k\lambda_n t}$.

Step 5(b): Eigenvalue-eigenvector problem for w_n:
Find the eigenvalues λ_n and the non-zero eigenfunctions w_n solutions of the BVP

$$w_n''(x) + \lambda_n w_n(x) = 0 \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0.$$

We know that this problem has solution only for $\lambda_n > 0$.
Denote: $\lambda_n = \mu_n^2$. Proposing $w_n(x) = e^{r_n x}$,

The separation of variables method.

Step 5(a): Recall: $v_n(t) = e^{-k\lambda_n t}$.

Step 5(b): Eigenvalue-eigenvector problem for w_n:
Find the eigenvalues λ_n and the non-zero eigenfunctions w_n solutions of the BVP

$$w_n''(x) + \lambda_n w_n(x) = 0 \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0.$$

We know that this problem has solution only for $\lambda_n > 0$.
Denote: $\lambda_n = \mu_n^2$. Proposing $w_n(x) = e^{r_n x}$, we get that

$$p(r_n) = r_n^2 + \mu_n^2 = 0$$

The separation of variables method.

Step 5(a): Recall: $v_n(t) = e^{-k\lambda_n t}$.

Step 5(b): Eigenvalue-eigenvector problem for w_n:
Find the eigenvalues λ_n and the non-zero eigenfunctions w_n solutions of the BVP

$$w_n''(x) + \lambda_n\, w_n(x) = 0 \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0.$$

We know that this problem has solution only for $\lambda_n > 0$.
Denote: $\lambda_n = \mu_n^2$. Proposing $w_n(x) = e^{r_n x}$, we get that

$$p(r_n) = r_n^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i$$

The separation of variables method.

Step 5(a): Recall: $v_n(t) = e^{-k\lambda_n t}$.

Step 5(b): Eigenvalue-eigenvector problem for w_n:
Find the eigenvalues λ_n and the non-zero eigenfunctions w_n solutions of the BVP

$$w_n''(x) + \lambda_n \, w_n(x) = 0 \quad \text{B.C.:} \quad w_n(0) = 0, \quad w_n(L) = 0.$$

We know that this problem has solution only for $\lambda_n > 0$.
Denote: $\lambda_n = \mu_n^2$. Proposing $w_n(x) = e^{r_n x}$, we get that

$$p(r_n) = r_n^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm \mu_n i$$

The real-valued general solution is

$$w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x).$$

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0)$$

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0) = c_1$$

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0) = c_1 \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0) = c_1 \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(L)$$

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0) = c_1 \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(L) = c_2 \sin(\mu_n L),$$

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0) = c_1 \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(L) = c_2 \sin(\mu_n L), \quad c_2 \neq 0,$$

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0) = c_1 \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(L) = c_2 \sin(\mu_n L), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n L) = 0.$$

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0) = c_1 \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(L) = c_2 \sin(\mu_n L), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n L) = 0.$$

$$\mu_n L = n\pi$$

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0) = c_1 \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(L) = c_2 \sin(\mu_n L), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n L) = 0.$$

$$\mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L}$$

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0) = c_1 \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(L) = c_2 \sin(\mu_n L), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n L) = 0.$$

$$\mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L} \quad \Rightarrow \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0) = c_1 \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(L) = c_2 \sin(\mu_n L), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n L) = 0.$$

$$\mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L} \quad \Rightarrow \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

Choosing $c_2 = 1$, we get $w_n(x) = \sin\left(\frac{n\pi x}{L}\right)$.

The separation of variables method.

Recall: $v_n(t) = e^{-k\lambda_n t}$, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply,

$$0 = w_n(0) = c_1 \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(L) = c_2 \sin(\mu_n L), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n L) = 0.$$

$$\mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L} \quad \Rightarrow \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

Choosing $c_2 = 1$, we get $w_n(x) = \sin\left(\frac{n\pi x}{L}\right)$.

We conclude that: $u_n(t, x) = e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, \cdots$.

The separation of variables method.

Step 6: Recall: $u_n(t,x) = e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi x}{L}\right).$

The separation of variables method.

Step 6: Recall: $u_n(t,x) = e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big)$.

Compute the solution to the IBVP for the Heat Equation,

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, u_n(t,x).$$

The separation of variables method.

Step 6: Recall: $u_n(t,x) = e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi x}{L}\right)$.

Compute the solution to the IBVP for the Heat Equation,

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, u_n(t,x).$$

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi x}{L}\right).$$

The separation of variables method.

Step 6: Recall: $u_n(t,x) = e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi x}{L}\right)$.

Compute the solution to the IBVP for the Heat Equation,

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, u_n(t,x).$$

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi x}{L}\right).$$

By construction, this solution satisfies the boundary conditions,

$$u(t,0) = 0, \qquad u(t,L) = 0.$$

The separation of variables method.

Step 6: Recall: $u_n(t,x) = e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big)$.

Compute the solution to the IBVP for the Heat Equation,

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, u_n(t,x).$$

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big).$$

By construction, this solution satisfies the boundary conditions,

$$u(t,0) = 0, \qquad u(t,L) = 0.$$

Given a function f with $f(0) = f(L) = 0$, the solution u above satisfies the initial condition $f(x) = u(0,x)$ iff holds

The separation of variables method.

Step 6: Recall: $u_n(t,x) = e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big)$.

Compute the solution to the IBVP for the Heat Equation,

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, u_n(t,x).$$

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big).$$

By construction, this solution satisfies the boundary conditions,

$$u(t,0) = 0, \qquad u(t,L) = 0.$$

Given a function f with $f(0) = f(L) = 0$, the solution u above satisfies the initial condition $f(x) = u(0,x)$ iff holds

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi x}{L}\Big).$$

The separation of variables method.

Recall:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-k(\frac{n\pi}{L})^2 t} \, \sin\Big(\frac{n\pi x}{L}\Big), \quad f(x) = \sum_{n=1}^{\infty} c_n \, \sin\Big(\frac{n\pi x}{L}\Big).$$

The separation of variables method.

Recall:

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big), \quad f(x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi x}{L}\Big).$$

This is a Sine Series for f. The coefficients c_n are computed in the usual way.

The separation of variables method.

Recall:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-k(\frac{n\pi}{L})^2 t} \, \sin\Big(\frac{n\pi x}{L}\Big), \quad f(x) = \sum_{n=1}^{\infty} c_n \, \sin\Big(\frac{n\pi x}{L}\Big).$$

This is a Sine Series for f. The coefficients c_n are computed in the usual way. Recall the orthogonality relation

$$\int_0^L \sin\Big(\frac{n\pi x}{L}\Big) \, \sin\Big(\frac{m\pi x}{L}\Big) \, dx = \begin{cases} 0, & m \neq n, \\ \dfrac{L}{2}, & m = n. \end{cases}$$

The separation of variables method.

Recall:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \, \sin\left(\frac{n\pi x}{L}\right), \quad f(x) = \sum_{n=1}^{\infty} c_n \, \sin\left(\frac{n\pi x}{L}\right).$$

This is a Sine Series for f. The coefficients c_n are computed in the usual way. Recall the orthogonality relation

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n, \\ \frac{L}{2}, & m = n. \end{cases}$$

Multiply the equation for u by $\sin\left(\frac{m\pi x}{L}\right)$ nd integrate,

$$\sum_{n=1}^{\infty} c_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx.$$

The separation of variables method.

Recall:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-k(\frac{n\pi}{L})^2 t} \, \sin\Big(\frac{n\pi x}{L}\Big), \quad f(x) = \sum_{n=1}^{\infty} c_n \, \sin\Big(\frac{n\pi x}{L}\Big).$$

This is a Sine Series for f. The coefficients c_n are computed in the usual way. Recall the orthogonality relation

$$\int_0^L \sin\Big(\frac{n\pi x}{L}\Big) \, \sin\Big(\frac{m\pi x}{L}\Big)\, dx = \begin{cases} 0, & m \neq n, \\ \frac{L}{2}, & m = n. \end{cases}$$

Multiply the equation for u by $\sin\big(\frac{m\pi x}{L}\big)$ nd integrate,

$$\sum_{n=1}^{\infty} c_n \int_0^L \sin\Big(\frac{n\pi x}{L}\Big) \, \sin\Big(\frac{m\pi x}{L}\Big)\, dx = \int_0^L f(x) \, \sin\Big(\frac{m\pi x}{L}\Big)\, dx.$$

$$c_n = \frac{2}{L} \int_0^L f(x) \, \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \quad u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-k(\frac{n\pi}{L})^2 t} \, \sin\Big(\frac{n\pi x}{L}\Big).$$

The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, v_n(t)\, w_n(x).$$

The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x).$$

where

- v_n: Solution of an IVP.

The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x).$$

where

- v_n: Solution of an IVP.
- w_n: Solution of a BVP, an eigenvalue-eigenfunction problem.

The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, v_n(t) \, w_n(x).$$

where

- v_n: Solution of an IVP.
- w_n: Solution of a BVP, an eigenvalue-eigenfunction problem.
- c_n: Fourier Series coefficients.

The separation of variables method.

Summary: IBVP for the Heat Equation.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, v_n(t)\, w_n(x).$$

where

- v_n: Solution of an IVP.
- w_n: Solution of a BVP, an eigenvalue-eigenfunction problem.
- c_n: Fourier Series coefficients.

Remark:
The separation of variables method does not work for every PDE.

Solving the Heat Equation (Sect. 6.3).

- Review: The Stationary Heat Equation.
- The Heat Equation.
- The Initial-Boundary Value Problem.
- The separation of variables method.
- **An example of separation of variables.**

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)\, w_n(x)$.

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv}{dt}(t) = v_n(t)\,\frac{d^2w}{dx^2}(x)$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)\, w_n(x)$. Then

$$4 w_n(x)\,\frac{dv}{dt}(t) = v_n(t)\,\frac{d^2 w}{dx^2}(x) \quad \Rightarrow \quad \frac{4 v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Let $u_n(t,x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv}{dt}(t) = v_n(t)\,\frac{d^2w}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n'(t) + \frac{\lambda_n}{4}\,v_n(t) = 0,$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv}{dt}(t) = v_n(t)\,\frac{d^2w}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n'(t) + \frac{\lambda_n}{4}\, v_n(t) = 0, \qquad w_n''(x) + \lambda_n\, w_n(x) = 0.$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Let $u_n(t,x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv}{dt}(t) = v_n(t)\,\frac{d^2 w}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n'(t) + \frac{\lambda_n}{4}\, v_n(t) = 0, \qquad w_n''(x) + \lambda_n\, w_n(x) = 0.$$

We solve for v_n with the initial condition $v_n(0) = 1$.

$$e^{\frac{\lambda_n}{4}t}\, v_n'(t) + \frac{\lambda_n}{4}\, e^{\frac{\lambda_n}{4}t}\, v_n(t) = 0$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)\, w_n(x)$. Then

$$4w_n(x)\,\frac{dv}{dt}(t) = v_n(t)\,\frac{d^2w}{dx^2}(x) \quad \Rightarrow \quad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n'(t) + \frac{\lambda_n}{4}\, v_n(t) = 0, \qquad w_n''(x) + \lambda_n\, w_n(x) = 0.$$

We solve for v_n with the initial condition $v_n(0) = 1$.

$$e^{\frac{\lambda_n}{4}t}\, v_n'(t) + \frac{\lambda_n}{4}\, e^{\frac{\lambda_n}{4}t}\, v_n(t) = 0 \quad \Rightarrow \quad \Big[e^{\frac{\lambda_n}{4}t}\, v_n(t)\Big]' = 0.$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0.$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t},$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0.$ Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0)$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\Big[e^{\frac{\lambda_n}{4}t}\, v_n(t)\Big]' = 0.$ Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4} t} v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4} t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4} t}.$$

An example of separation of variables.

Example

Find the solution to the IBVP $\;4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(L) = 0$.

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0.$ Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(L) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$.

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0.$ Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(L) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0.$ Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(L) = 0.$

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(L) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0.$ Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(L) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution, $w_n(x) = c_1\, \cos(\mu_n x) + c_2\, \sin(\mu_n x)$.

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(L) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply

$$0 = w_n(0) = c_1,$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $\left[e^{\frac{\lambda_n}{4}t}\, v_n(t)\right]' = 0$. Therefore,

$$v_n(t) = c\, e^{-\frac{\lambda_n}{4}t}, \quad 1 = v_n(0) = c \quad \Rightarrow \quad v_n(t) = e^{-\frac{\lambda_n}{4}t}.$$

Next the BVP: $w_n''(x) + \lambda_n\, w_n(x) = 0$, with $w_n(0) = w_n(L) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.

The boundary conditions imply

$$0 = w_n(0) = c_1, \quad \Rightarrow \quad w_n(x) = c_2 \sin(\mu_n x).$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2),$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4} t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0,$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$,

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$.

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2]$,

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n\pi}{2}\right)^2, \qquad w_n(x) = \sin\left(\frac{n\pi x}{2}\right).$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $v_n(t) = e^{-\frac{\lambda_n}{4}t}$, and $w_n(x) = c_2 \sin(\mu_n x)$.

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n\pi}{2}\right)^2, \qquad w_n(x) = \sin\left(\frac{n\pi x}{2}\right).$$

$$u(t, x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \sin\left(\frac{n\pi x}{2}\right).$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $u(t, x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \, \sin\Big(\frac{n\pi x}{2}\Big).$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $u(t, x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \sin\Big(\frac{n\pi x}{2}\Big).$

The initial condition is $3 \sin\Big(\frac{\pi x}{2}\Big) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi x}{2}\Big).$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $u(t, x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \sin\left(\frac{n\pi x}{2}\right).$

The initial condition is $3 \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right).$

The orthogonality of the sine functions implies

$$3 \int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx.$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $u(t, x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \, \sin\Big(\frac{n\pi x}{2}\Big).$

The initial condition is $3 \, \sin\Big(\frac{\pi x}{2}\Big) = \sum_{n=1}^{\infty} c_n \, \sin\Big(\frac{n\pi x}{2}\Big).$

The orthogonality of the sine functions implies

$$3 \int_0^2 \sin\Big(\frac{\pi x}{2}\Big) \, \sin\Big(\frac{m\pi x}{2}\Big) \, dx = \sum_{n=1}^{\infty} \int_0^2 \sin\Big(\frac{n\pi x}{2}\Big) \, \sin\Big(\frac{m\pi x}{2}\Big) \, dx.$$

If $m \neq 1$, then $0 = c_m \, \frac{2}{2}$,

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2]$,

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \, \sin\left(\frac{n\pi x}{2}\right)$.

The initial condition is $3 \, \sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \, \sin\left(\frac{n\pi x}{2}\right)$.

The orthogonality of the sine functions implies

$$3 \int_0^2 \sin\left(\frac{\pi x}{2}\right) \, \sin\left(\frac{m\pi x}{2}\right) dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \, \sin\left(\frac{m\pi x}{2}\right) dx.$$

If $m \neq 1$, then $0 = c_m \, \frac{2}{2}$, that is, $c_m = 0$ for $m \neq 1$.

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0, x) = 3\sin(\pi x/2), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Recall: $u(t, x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{4})^2 t} \, \sin\Big(\frac{n\pi x}{2}\Big).$

The initial condition is $3 \sin\Big(\frac{\pi x}{2}\Big) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi x}{2}\Big).$

The orthogonality of the sine functions implies

$$3 \int_0^2 \sin\Big(\frac{\pi x}{2}\Big) \sin\Big(\frac{m\pi x}{2}\Big) \, dx = \sum_{n=1}^{\infty} \int_0^2 \sin\Big(\frac{n\pi x}{2}\Big) \sin\Big(\frac{m\pi x}{2}\Big) \, dx.$$

If $m \neq 1$, then $0 = c_m \frac{2}{2}$, that is, $c_m = 0$ for $m \neq 1$. Therefore,

$$3\sin\Big(\frac{\pi x}{2}\Big) = c_1 \sin\Big(\frac{\pi x}{2}\Big)$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: Recall: $u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-(\frac{n\pi}{4})^2 t} \sin\left(\frac{n\pi x}{2}\right).$

The initial condition is $3\sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right).$

The orthogonality of the sine functions implies

$$3\int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx = \sum_{n=1}^{\infty} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx.$$

If $m \neq 1$, then $0 = c_m \frac{2}{2}$, that is, $c_m = 0$ for $m \neq 1$. Therefore,

$$3\sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right) \quad \Rightarrow \quad c_1 = 3.$$

An example of separation of variables.

Example

Find the solution to the IBVP $4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: We conclude that

$$u(t,x) = 3\, e^{-(\frac{\pi}{4})^2 t} \sin\left(\frac{\pi x}{2}\right).$$

Review for Final Exam.

- Exam is cumulative.
- Heat equation and Fourier Series not included.
- 10-12 problems.
- Two hours.
- Integration and Laplace Transform tables included.

- **Not in the exam:** Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Power Series Methods (Chptr. 3).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic,

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series,

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

$$b_n = 2 \int_0^1 (-1)\, \sin(n\pi x)\, dx$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

$$b_n = 2 \int_0^1 (-1) \,\sin(n\pi x)\, dx = (-2)\, \frac{(-1)}{n\pi}\, \cos(n\pi x)\Big|_0^1,$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

$$b_n = 2 \int_0^1 (-1)\, \sin(n\pi x)\, dx = (-2)\, \frac{(-1)}{n\pi}\, \cos(n\pi x)\Big|_0^1,$$

$$b_n = \frac{2}{n\pi} \big[\cos(n\pi) - 1 \big]$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

$$b_n = 2 \int_0^1 (-1)\, \sin(n\pi x)\, dx = (-2)\, \frac{(-1)}{n\pi}\, \cos(n\pi x)\Big|_0^1,$$

$$b_n = \frac{2}{n\pi} \big[\cos(n\pi) - 1\big] \quad \Rightarrow \quad b_n = \frac{2}{n\pi} \big[(-1)^n - 1\big].$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \dfrac{2}{n\pi}\,\big[(-1)^n - 1\big]$.

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \dfrac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$,

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \dfrac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[(-1)^{2k} - 1\big]$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \dfrac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[(-1)^{2k} - 1\big] = 0$.

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \dfrac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[(-1)^{2k} - 1\big] = 0$.

If $n = 2k - 1$,

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \dfrac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[(-1)^{2k} - 1\big] = 0$.

If $n = 2k - 1$,

$$b_{(2k-1)} = \frac{2}{(2k-1)\pi}\big[(-1)^{2k-1} - 1\big]$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \dfrac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[(-1)^{2k} - 1\big] = 0.$

If $n = 2k - 1$,

$$b_{(2k-1)} = \frac{2}{(2k-1)\pi}\big[(-1)^{2k-1} - 1\big] = -\frac{4}{(2k-1)\pi}.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \frac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \frac{2}{2k\pi}\big[(-1)^{2k} - 1\big] = 0$.

If $n = 2k - 1$,

$$b_{(2k-1)} = \frac{2}{(2k-1)\pi}\big[(-1)^{2k-1} - 1\big] = -\frac{4}{(2k-1)\pi}.$$

We conclude: $f(x) = -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\sin[(2k-1)\pi x]$. $\triangleleft$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic,

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series,

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx,$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \;\; L = 2,$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \;\; L = 2,$$

$$b_n = \int_0^2 (2 - x)\, \sin\Big(\frac{n\pi x}{2}\Big)\, dx.a$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2 \int_0^2 \sin\Big(\frac{n\pi x}{2}\Big)\, dx - \int_0^2 x \sin\Big(\frac{n\pi x}{2}\Big)\, dx.$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2 \int_0^2 \sin\Big(\frac{n\pi x}{2}\Big)\, dx - \int_0^2 x \sin\Big(\frac{n\pi x}{2}\Big)\, dx.$

$$\int \sin\Big(\frac{n\pi x}{2}\Big)\, dx = \frac{-2}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big),$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2 \int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx.$

$$\int \sin\left(\frac{n\pi x}{2}\right) dx = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right),$$

The other integral is done by parts,

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2 \int_0^2 \sin\Big(\frac{n\pi x}{2}\Big)\, dx - \int_0^2 x \, \sin\Big(\frac{n\pi x}{2}\Big)\, dx.$

$$\int \sin\Big(\frac{n\pi x}{2}\Big)\, dx = \frac{-2}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

$$I = \int x \sin\Big(\frac{n\pi x}{2}\Big)\, dx,$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2 \int_0^2 \sin\Big(\frac{n\pi x}{2}\Big)\, dx - \int_0^2 x \sin\Big(\frac{n\pi x}{2}\Big)\, dx.$

$$\int \sin\Big(\frac{n\pi x}{2}\Big)\, dx = \frac{-2}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

$$I = \int x \sin\Big(\frac{n\pi x}{2}\Big)\, dx, \quad \left\{ \quad u = x, \quad v' = \sin\Big(\frac{n\pi x}{2}\Big) \right.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2\int_0^2 \sin\Big(\frac{n\pi x}{2}\Big)\,dx - \int_0^2 x\,\sin\Big(\frac{n\pi x}{2}\Big)\,dx.$

$$\int \sin\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{-2}{n\pi}\,\cos\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

$$I = \int x \sin\Big(\frac{n\pi x}{2}\Big)\,dx, \qquad \begin{cases} u = x, & v' = \sin\Big(\frac{n\pi x}{2}\Big) \\ u' = 1, & v = -\frac{2}{n\pi}\,\cos\Big(\frac{n\pi x}{2}\Big) \end{cases}$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2 \int_0^2 \sin\Big(\frac{n\pi x}{2}\Big)\,dx - \int_0^2 x \sin\Big(\frac{n\pi x}{2}\Big)\,dx.$

$$\int \sin\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{-2}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

$$I = \int x \sin\Big(\frac{n\pi x}{2}\Big)\,dx, \qquad \begin{cases} u = x, & v' = \sin\Big(\frac{n\pi x}{2}\Big) \\ u' = 1, & v = -\frac{2}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big) \end{cases}$$

$$I = \frac{-2x}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big) - \int \Big(\frac{-2}{n\pi}\Big) \cos\Big(\frac{n\pi x}{2}\Big)\,dx.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \dfrac{-2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) - \displaystyle\int \Big(\dfrac{-2}{n\pi}\Big) \cos\Big(\dfrac{n\pi x}{2}\Big)\, dx.$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \dfrac{-2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) - \displaystyle\int \Big(\dfrac{-2}{n\pi}\Big) \cos\Big(\dfrac{n\pi x}{2}\Big)\, dx.$

$$I = -\frac{2x}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big) + \Big(\frac{2}{n\pi}\Big)^2 \sin\Big(\frac{n\pi x}{2}\Big).$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \dfrac{-2x}{n\pi} \cos\left(\dfrac{n\pi x}{2}\right) - \displaystyle\int \left(\dfrac{-2}{n\pi}\right) \cos\left(\dfrac{n\pi x}{2}\right) dx.$

$I = -\dfrac{2x}{n\pi} \cos\left(\dfrac{n\pi x}{2}\right) + \left(\dfrac{2}{n\pi}\right)^2 \sin\left(\dfrac{n\pi x}{2}\right).$ So, we get

$$b_n = 2\left[\frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 + \left[\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 - \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)\Big|_0^2$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \dfrac{-2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) - \displaystyle\int \Big(\dfrac{-2}{n\pi}\Big) \cos\Big(\dfrac{n\pi x}{2}\Big)\, dx.$

$I = -\dfrac{2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) + \Big(\dfrac{2}{n\pi}\Big)^2 \sin\Big(\dfrac{n\pi x}{2}\Big).$ So, we get

$$b_n = 2\Big[\frac{-2}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 + \Big[\frac{2x}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big(\frac{2}{n\pi}\Big)^2 \sin\Big(\frac{n\pi x}{2}\Big)\Big|_0^2$$

$$b_n = \frac{-4}{n\pi}\big[\cos(n\pi) - 1\big] + \Big[\frac{4}{n\pi}\cos(n\pi) - 0\Big]$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \dfrac{-2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) - \displaystyle\int \Big(\dfrac{-2}{n\pi}\Big) \cos\Big(\dfrac{n\pi x}{2}\Big)\, dx.$

$I = -\dfrac{2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) + \Big(\dfrac{2}{n\pi}\Big)^2 \sin\Big(\dfrac{n\pi x}{2}\Big).$ So, we get

$$b_n = 2\Big[\frac{-2}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 + \Big[\frac{2x}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big(\frac{2}{n\pi}\Big)^2 \sin\Big(\frac{n\pi x}{2}\Big)\Big|_0^2$$

$$b_n = \frac{-4}{n\pi}\big[\cos(n\pi) - 1\big] + \Big[\frac{4}{n\pi} \cos(n\pi) - 0\Big] \quad \Rightarrow \quad b_n = \frac{4}{n\pi}.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \dfrac{-2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) - \displaystyle\int \Big(\dfrac{-2}{n\pi}\Big) \cos\Big(\dfrac{n\pi x}{2}\Big)\, dx.$

$I = -\dfrac{2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) + \Big(\dfrac{2}{n\pi}\Big)^2 \sin\Big(\dfrac{n\pi x}{2}\Big).$ So, we get

$$b_n = 2\Big[\frac{-2}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 + \Big[\frac{2x}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big(\frac{2}{n\pi}\Big)^2 \sin\Big(\frac{n\pi x}{2}\Big)\Big|_0^2$$

$$b_n = \frac{-4}{n\pi}\big[\cos(n\pi) - 1\big] + \Big[\frac{4}{n\pi}\cos(n\pi) - 0\Big] \quad \Rightarrow \quad b_n = \frac{4}{n\pi}.$$

We conclude: $f(x) = \dfrac{4}{\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n} \sin\Big(\dfrac{n\pi x}{2}\Big).$ ◁

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic,

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series,

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = \int_{0}^{2} (2 - x)\, dx$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\,dx = \int_0^2 (2 - x)\,dx = \frac{\text{base x height}}{2}$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = \int_0^2 (2 - x)\, dx = \frac{\text{base x height}}{2} \;\Rightarrow\; a_0 = 2.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = \int_0^2 (2 - x)\, dx = \frac{\text{base x height}}{2} \Rightarrow \ a_0 = 2.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = \int_0^2 (2 - x)\, dx = \frac{\text{base x height}}{2} \;\Rightarrow\; a_0 = 2.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx,$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = \int_0^2 (2 - x)\, dx = \frac{\text{base x height}}{2} \;\Rightarrow\; a_0 = 2.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \;\; L = 2,$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = \int_0^2 (2 - x)\, dx = \frac{\text{base x height}}{2} \;\Rightarrow\; a_0 = 2.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \;\; L = 2,$$

$$a_n = \int_0^2 (2 - x)\, \cos\Big(\frac{n\pi x}{2}\Big)\, dx.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2\int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx.$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2 \int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx.$

$$\int \cos\left(\frac{n\pi x}{2}\right) dx = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right),$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2 \int_0^2 \cos\Big(\frac{n\pi x}{2}\Big)\, dx - \int_0^2 x \cos\Big(\frac{n\pi x}{2}\Big)\, dx.$

$$\int \cos\Big(\frac{n\pi x}{2}\Big)\, dx = \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2\int_0^2 \cos\Big(\frac{n\pi x}{2}\Big)\,dx - \int_0^2 x\,\cos\Big(\frac{n\pi x}{2}\Big)\,dx.$

$$\int \cos\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{2}{n\pi}\,\sin\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

$$I = \int x \cos\Big(\frac{n\pi x}{2}\Big)\,dx,$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2 \int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx.$

$$\int \cos\left(\frac{n\pi x}{2}\right) dx = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right),$$

The other integral is done by parts,

$$I = \int x \cos\left(\frac{n\pi x}{2}\right) dx, \qquad \left\{ \quad u = x, \quad v' = \cos\left(\frac{n\pi x}{2}\right) \right.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2 \int_0^2 \cos\Big(\frac{n\pi x}{2}\Big)\, dx - \int_0^2 x \cos\Big(\frac{n\pi x}{2}\Big)\, dx.$

$$\int \cos\Big(\frac{n\pi x}{2}\Big)\, dx = \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

$$I = \int x \cos\Big(\frac{n\pi x}{2}\Big)\, dx, \qquad \begin{cases} u = x, & v' = \cos\Big(\frac{n\pi x}{2}\Big) \\ u' = 1, & v = \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) \end{cases}$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2 \int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx.$

$$\int \cos\left(\frac{n\pi x}{2}\right) dx = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right),$$

The other integral is done by parts,

$$I = \int x \cos\left(\frac{n\pi x}{2}\right) dx, \quad \begin{cases} u = x, & v' = \cos\left(\frac{n\pi x}{2}\right) \\ u' = 1, & v = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \end{cases}$$

$$I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \int \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) dx.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $I = \frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) - \int \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\, dx.$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $I = \frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) - \int \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\, dx.$

$$I = \frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) + \Big(\frac{2}{n\pi}\Big)^2 \cos\Big(\frac{n\pi x}{2}\Big).$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $I = \dfrac{2x}{n\pi} \sin\Big(\dfrac{n\pi x}{2}\Big) - \displaystyle\int \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\, dx.$

$$I = \frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) + \Big(\frac{2}{n\pi}\Big)^2 \cos\Big(\frac{n\pi x}{2}\Big).$$ So, we get

$$a_n = 2\Big[\frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big[\frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big(\frac{2}{n\pi}\Big)^2 \cos\Big(\frac{n\pi x}{2}\Big)\Big|_0^2$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \int \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) dx.$

$$I = \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right).$$ So, we get

$$a_n = 2\left[\frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 - \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 - \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right)\Big|_0^2$$

$$a_n = 0 - 0 - \frac{4}{n^2\pi^2}\left[\cos(n\pi) - 1\right]$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $I = \dfrac{2x}{n\pi} \sin\Big(\dfrac{n\pi x}{2}\Big) - \displaystyle\int \dfrac{2}{n\pi} \sin\Big(\dfrac{n\pi x}{2}\Big)\, dx.$

$$I = \frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) + \Big(\frac{2}{n\pi}\Big)^2 \cos\Big(\frac{n\pi x}{2}\Big).$$ So, we get

$$a_n = 2\Big[\frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big[\frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big(\frac{2}{n\pi}\Big)^2 \cos\Big(\frac{n\pi x}{2}\Big)\Big|_0^2$$

$$a_n = 0 - 0 - \frac{4}{n^2\pi^2}\big[\cos(n\pi) - 1\big] \quad \Rightarrow \quad a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n].$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0$, $a_0 = 2$, $a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n]$.

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0$, $a_0 = 2$, $a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n]$.

If $n = 2k$,

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0, \quad a_0 = 2, \quad a_n = \dfrac{4}{n^2\pi^2}[1 - (-1)^n]$.

If $n = 2k$, then $a_{2k} = \dfrac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right]$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0$, $a_0 = 2$, $a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n]$.

If $n = 2k$, then $a_{2k} = \frac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right] = 0.$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0$, $a_0 = 2$, $a_n = \dfrac{4}{n^2\pi^2}[1 - (-1)^n]$.

If $n = 2k$, then $a_{2k} = \dfrac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right] = 0.$

If $n = 2k - 1$,

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0$, $a_0 = 2$, $a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n]$.

If $n = 2k$, then $a_{2k} = \frac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right] = 0$.

If $n = 2k - 1$, then we obtain

$$a_{(2k-1)} = \frac{4}{(2k-1)^2\pi^2}\left[1 - (-1)^{2k-1}\right]$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0$, $a_0 = 2$, $a_n = \dfrac{4}{n^2\pi^2}[1 - (-1)^n]$.

If $n = 2k$, then $a_{2k} = \dfrac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right] = 0.$

If $n = 2k - 1$, then we obtain

$$a_{(2k-1)} = \frac{4}{(2k-1)^2\pi^2}\left[1 - (-1)^{2k-1}\right] = \frac{8}{(2k-1)^2\pi^2}.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0, \quad a_0 = 2, \quad a_n = \dfrac{4}{n^2\pi^2}[1 - (-1)^n].$

If $n = 2k$, then $a_{2k} = \dfrac{4}{(2k)^2\pi^2}\left[1 - (-1)^{2k}\right] = 0.$

If $n = 2k - 1$, then we obtain

$$a_{(2k-1)} = \frac{4}{(2k-1)^2\pi^2}\left[1 - (-1)^{2k-1}\right] = \frac{8}{(2k-1)^2\pi^2}.$$

We conclude: $f(x) = 1 + \dfrac{8}{\pi^2}\displaystyle\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\Big(\frac{(2k-1)\pi x}{2}\Big). \triangleleft$

Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- **Eigenvalue-Eigenfunction BVP (Chptr. 6).**
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Power Series Methods (Chptr. 3).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$,

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$,

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0)$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8),$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$\mu = \frac{n\pi}{8},$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$\mu = \frac{n\pi}{8}, \quad \lambda = \left(\frac{n\pi}{8}\right)^2,$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$\mu = \frac{n\pi}{8}, \quad \lambda = \left(\frac{n\pi}{8}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{8}\right), \quad n = 1, 2, \cdots \quad \triangleleft$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0)$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8),$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

$$8\mu = (2n+1)\frac{\pi}{2},$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

$$8\mu = (2n+1)\frac{\pi}{2}, \quad \Rightarrow \quad \mu = \frac{(2n+1)\pi}{16}.$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

$$8\mu = (2n+1)\frac{\pi}{2}, \quad \Rightarrow \quad \mu = \frac{(2n+1)\pi}{16}.$$

Then, for $n = 1, 2, \cdots$ holds

$$\lambda = \Big[\frac{(2n+1)\pi}{16}\Big]^2,$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

$$8\mu = (2n+1)\frac{\pi}{2}, \quad \Rightarrow \quad \mu = \frac{(2n+1)\pi}{16}.$$

Then, for $n = 1, 2, \cdots$ holds

$$\lambda = \left[\frac{(2n+1)\pi}{16}\right]^2, \quad y_n(x) = \sin\left(\frac{(2n+1)\pi x}{16}\right). \qquad \triangleleft$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$.

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$.

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1 \mu \sin(\mu x) + c_2 \mu \cos(\mu x)$. The B.C. imply:

$0 = y'(0)$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1 \mu \sin(\mu x) + c_2 \mu \cos(\mu x)$. The B.C. imply:

$0 = y'(0) = c_2$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1 \mu \sin(\mu x) + c_2 \mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1\mu \sin(\mu 8),$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1\mu \sin(\mu 8), \quad c_1 \neq 0$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1\,\cos(\mu x) + c_2\,\sin(\mu x)$.

Then, $y'(x) = -c_1\mu\,\sin(\mu x) + c_2\mu\,\cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1\cos(\mu x),\; y'(x) = -c_1\mu\,\sin(\mu x).$$

$$0 = y'(8) = c_1\mu\sin(\mu 8), \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1\mu \sin(\mu 8), \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$8\mu = n\pi,$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1\mu \sin(\mu 8), \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$8\mu = n\pi, \quad \Rightarrow \quad \mu = \frac{n\pi}{8}.$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1\mu \sin(\mu 8), \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$8\mu = n\pi, \quad \Rightarrow \quad \mu = \frac{n\pi}{8}.$$

Then, choosing $c_1 = 1$, for $n = 1, 2, \cdots$ holds

$$\lambda = \left(\frac{n\pi}{8}\right)^2,$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x),\; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1\mu \sin(\mu 8), \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$8\mu = n\pi, \quad \Rightarrow \quad \mu = \frac{n\pi}{8}.$$

Then, choosing $c_1 = 1$, for $n = 1, 2, \cdots$ holds

$$\lambda = \Big(\frac{n\pi}{8}\Big)^2, \quad y_n(x) = \cos\Big(\frac{n\pi x}{8}\Big).$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0)$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1,$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

$$0 = y'(8)$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

$$0 = y'(8) = 0.$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

$$0 = y'(8) = 0.$$

Then, choosing $c_1 = 1$, holds,

$$\lambda = 0,$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

$$0 = y'(8) = 0.$$

Then, choosing $c_1 = 1$, holds,

$$\lambda = 0, \qquad y_0(x) = 1.$$

$\lhd$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0$$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$.

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0)$$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2$$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3)$$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3)$$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \;\Rightarrow\; c_1 = -\frac{\sin(\pi/3)}{\cos(\pi/3)}.$$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \;\Rightarrow\; c_1 = -\frac{\sin(\pi/3)}{\cos(\pi/3)}.$$

$$c_1 = -\frac{\sqrt{3}/2}{1/2}$$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \Rightarrow c_1 = -\frac{\sin(\pi/3)}{\cos(\pi/3)}.$$

$$c_1 = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3}$$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \ \Rightarrow \ c_1 = -\frac{\sin(\pi/3)}{\cos(\pi/3)}.$$

$$c_1 = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3} \ \Rightarrow \ y(x) = -\sqrt{3}\cos(x) + \sin(x). \quad \triangleleft$$

Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- **Systems of linear Equations (Chptr. 5).**
- Laplace transforms (Chptr. 4).
- Power Series Methods (Chptr. 3).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real,

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent,

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex,

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$,

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big] \pm i e^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big] \pm i e^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Real-valued fundamental solutions are

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big] \pm i e^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Real-valued fundamental solutions are

$$\mathbf{x}^{(1)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big],$$

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big] \pm i e^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Real-valued fundamental solutions are

$$\mathbf{x}^{(1)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big],$$

$$\mathbf{x}^{(2)} = e^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real,

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent,

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real,

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$,

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$, then find $\mathbf{w}$ solution of $(A - \lambda I)\mathbf{w} = \mathbf{v}$.

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$, then find $\mathbf{w}$ solution of $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then fundamental solutions to the differential equation are given by

$$\mathbf{x}^{(1)} = \mathbf{v}\,e^{\lambda t},$$

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$, then find $\mathbf{w}$ solution of $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then fundamental solutions to the differential equation are given by

$$\mathbf{x}^{(1)} = \mathbf{v}\,e^{\lambda t}, \quad \mathbf{x}^{(2)} = (\mathbf{v}\,t + \mathbf{w})\,e^{\lambda t}.$$

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$, then find $\mathbf{w}$ solution of $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then fundamental solutions to the differential equation are given by

$$\mathbf{x}^{(1)} = \mathbf{v}\,e^{\lambda t}, \quad \mathbf{x}^{(2)} = (\mathbf{v}\,t + \mathbf{w})\,e^{\lambda t}.$$

Then, the general solution is

$$\mathbf{x} = c_1\,\mathbf{v}\,e^{\lambda t} + c_2\,(\mathbf{v}\,t + \mathbf{w})\,e^{\lambda t}.$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix}$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_\pm = \pm 3.$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix}$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_\pm = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = 2v_2$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \;\Rightarrow\; v_1 = 2v_2 \;\Rightarrow\; \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_\pm = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

$$A + 3I$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

$$A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_\pm = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = 2v_2 \;\Rightarrow\; \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

$$A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

$$A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -v_2$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

$$A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -v_2 \Rightarrow \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_{\pm} = \pm 3, \;\; \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \;\; \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_\pm = \pm 3$, $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}$, $\quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$.

Solution: Recall: $\lambda_{\pm} = \pm 3$, $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$.
The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0)$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_{\pm} = \pm 3, \ \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$
The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_\pm = \pm 3, \;\; \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \;\; \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$

The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_{\pm} = \pm 3, \ \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$
The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2+1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_{\pm} = \pm 3, \;\; \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \;\; \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$

The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2+1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_{\pm} = \pm 3, \;\; \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \;\; \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$

The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2+1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

We conclude: $\mathbf{x}(t) = \frac{5}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$ $\lhd$

Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- **Laplace transforms (Chptr. 4).**
- Power Series Methods (Chptr. 3).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

Laplace transforms.

Summary:

- Main Properties:

Laplace transforms.

Summary:

- Main Properties:

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n \,\mathcal{L}[f(t)] - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0); \quad (18)$$

Laplace transforms.

Summary:

- Main Properties:

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \qquad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \qquad (13)$$

Laplace transforms.

Summary:

- Main Properties:

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n \mathcal{L}[f(t)] - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0); \qquad (18)$$

$$e^{-cs} \mathcal{L}[f(t)] = \mathcal{L}[u_c(t) f(t-c)]; \qquad (13)$$

$$\mathcal{L}[f(t)]\Big|_{(s-c)} = \mathcal{L}[e^{ct} f(t)]. \qquad (14)$$

Laplace transforms.

Summary:

- Main Properties:

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \qquad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \qquad (13)$$

$$\mathcal{L}[f(t)]\Big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]. \qquad (14)$$

- Convolutions:

Laplace transforms.

Summary:

- Main Properties:

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \quad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \quad (13)$$

$$\mathcal{L}[f(t)]\Big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]. \quad (14)$$

- Convolutions:

$$\mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)].$$

Laplace transforms.

Summary:

- Main Properties:

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \quad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \quad (13)$$

$$\mathcal{L}[f(t)]\Big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]. \quad (14)$$

- Convolutions:

$$\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)].$$

- Partial fraction decompositions, completing the squares.

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)]$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$,

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad \mathcal{L}[y''] = s^2\,\mathcal{L}[y] - 3s - 2.$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad \mathcal{L}[y''] = s^2\,\mathcal{L}[y] - 3s - 2.$$

$$(s^2 + 9)\,\mathcal{L}[y] - 3s - 2 = \frac{e^{-5s}}{s}$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad \mathcal{L}[y''] = s^2\,\mathcal{L}[y] - 3s - 2.$$

$$(s^2 + 9)\,\mathcal{L}[y] - 3s - 2 = \frac{e^{-5s}}{s}$$

$$\mathcal{L}[y] = \frac{(3s+2)}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad \mathcal{L}[y''] = s^2\,\mathcal{L}[y] - 3s - 2.$$

$$(s^2 + 9)\,\mathcal{L}[y] - 3s - 2 = \frac{e^{-5s}}{s}$$

$$\mathcal{L}[y] = \frac{(3s+2)}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

$$\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}$.

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}$.

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}$.

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)}$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}$.

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)}$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}$.

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs = (a+b)\,s^2 + cs + 9a$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}$.

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs = (a+b)\,s^2 + cs + 9a$$

$$a = \frac{1}{9},$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\dfrac{s}{(s^2+9)} + \dfrac{2}{3}\,\dfrac{3}{(s^2+9)} + e^{-5s}\,\dfrac{1}{s(s^2+9)}.$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs = (a+b)\,s^2 + cs + 9a$$

$$a = \frac{1}{9}, \quad c = 0,$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\dfrac{s}{(s^2+9)} + \dfrac{2}{3}\,\dfrac{3}{(s^2+9)} + e^{-5s}\,\dfrac{1}{s(s^2+9)}.$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs = (a+b)\,s^2 + cs + 9a$$

$$a = \frac{1}{9}, \quad c = 0, \quad b = -a$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}$.

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs = (a+b)\,s^2 + cs + 9a$$

$$a = \frac{1}{9}, \quad c = 0, \quad b = -a \quad \Rightarrow \quad b = -\frac{1}{9}.$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: So, $\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,H(s)$, and

$$H(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\Big[\frac{1}{s} - \frac{s}{s^2+9}\Big]$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: So, $\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,H(s)$, and

$$H(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\Big[\frac{1}{s} - \frac{s}{s^2+9}\Big] = \frac{1}{9}\Big(\mathcal{L}[u(t)] - \mathcal{L}[\cos(3t)]\Big)$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: So, $\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,H(s)$, and

$$H(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\Big[\frac{1}{s} - \frac{s}{s^2+9}\Big] = \frac{1}{9}\Big(\mathcal{L}[u(t)] - \mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(e^{-5s}\,\mathcal{L}[u(t)] - e^{-5s}\,\mathcal{L}[\cos(3t)]\Big)$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: So, $\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \dfrac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,H(s)$, and

$$H(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\Big[\frac{1}{s} - \frac{s}{s^2+9}\Big] = \frac{1}{9}\Big(\mathcal{L}[u(t)] - \mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(e^{-5s}\,\mathcal{L}[u(t)] - e^{-5s}\,\mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: So, $\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,H(s)$, and

$$H(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\Big[\frac{1}{s} - \frac{s}{s^2+9}\Big] = \frac{1}{9}\Big(\mathcal{L}[u(t)] - \mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(e^{-5s}\,\mathcal{L}[u(t)] - e^{-5s}\,\mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution:

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution:

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

Therefore, we conclude that,

$$y(t) = 3\cos(3t) + \frac{2}{3}\,\sin(3t) + \frac{u_5(t)}{9}\Big[1 - \cos(3(t-5))\Big].$$

$\triangleleft$

Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- **Power Series Methods (Chptr. 3).**
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

Power series solutions (Chptr. 3).

Summary: Solve: $a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$ near x_0.

Power series solutions (Chptr. 3).

Summary: Solve: $a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$ near x_0.

(a) If x_0 is a regular point,

Power series solutions (Chptr. 3).

Summary: Solve: $a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^n$.

Power series solutions (Chptr. 3).

Summary: Solve: $a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^n$.

Find a recurrence relation for a_n.

Power series solutions (Chptr. 3).

Summary: Solve: $a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point,

Power series solutions (Chptr. 3).

Summary: Solve: $a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point, $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^{(n+r)}$.

Power series solutions (Chptr. 3).

Summary: Solve: $a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point, $y(x) = \sum_{n=0}^{\infty} a_n\, (x - x_0)^{(n+r)}$.

Find a recurrence relation for a_n and indicial equation for r.

Power series solutions (Chptr. 3).

Summary: Solve: $a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point, $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^{(n+r)}$.

Find a recurrence relation for a_n and indicial equation for r.

(c) Euler equation: $(x - x_0)^2\,y'' + \alpha\,(x - x_0)\,y' + \beta\,y = 0$.

Power series solutions (Chptr. 3).

Summary: Solve: $a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n\,(x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point, $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n\,(x - x_0)^{(n+r)}$.

Find a recurrence relation for a_n and indicial equation for r.

(c) Euler equation: $(x - x_0)^2\,y'' + \alpha\,(x - x_0)\,y' + \beta\,y = 0$.

Solutions: If $y(x) = |x - x_0|^r$,

Power series solutions (Chptr. 3).

Summary: Solve: $a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$ near x_0.

(a) If x_0 is a regular point, then $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$.

Find a recurrence relation for a_n.

(b) If x_0 is a regular-singular point, $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^{(n+r)}$.

Find a recurrence relation for a_n and indicial equation for r.

(c) Euler equation: $(x - x_0)^2\,y'' + \alpha\,(x - x_0)\,y' + \beta\,y = 0$.

Solutions: If $y(x) = |x - x_0|^r$, then r is solution of the indicial equation $p(r) = r(r-1) + \alpha r + \beta = 0$.

Power series solutions (Chptr. 3).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\,(x - x_0)\, y' + \beta\, y = 0.$$

Power series solutions (Chptr. 3).

Summary: Solving the Euler equation

$$(x - x_0)^2 \, y'' + \alpha \, (x - x_0) \, y' + \beta \, y = 0.$$

(i) If $r_1 \neq r_2$, reals,

Power series solutions (Chptr. 3).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\,(x - x_0)\,y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\,|x - x_0|^{r_1} + c_2\,|x - x_0|^{r_2}.$$

Power series solutions (Chptr. 3).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\, (x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\, |x - x_0|^{r_1} + c_2\, |x - x_0|^{r_2}.$$

(ii) If $r_1 \neq r_2$, complex,

Power series solutions (Chptr. 3).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\,(x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\,|x - x_0|^{r_1} + c_2\,|x - x_0|^{r_2}.$$

(ii) If $r_1 \neq r_2$, complex, denote them as $r_\pm = \lambda \pm \mu i$.

Power series solutions (Chptr. 3).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\,(x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\,|x - x_0|^{r_1} + c_2\,|x - x_0|^{r_2}.$$

(ii) If $r_1 \neq r_2$, complex, denote them as $r_\pm = \lambda \pm \mu i$. Then, the real-valued general solution is

$$y(x) = c_1\,|x - x_0|^{\lambda}\,\cos\big(\mu \ln|x - x_0|\big) + c_2\,|x - x_0|^{\lambda}\,\sin\big(\mu \ln|x - x_0|\big).$$

Power series solutions (Chptr. 3).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\,(x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\,|x - x_0|^{r_1} + c_2\,|x - x_0|^{r_2}.$$

(ii) If $r_1 \neq r_2$, complex, denote them as $r_\pm = \lambda \pm \mu i$. Then, the real-valued general solution is

$$y(x) = c_1\,|x - x_0|^{\lambda}\,\cos\big(\mu \ln |x - x_0|\big) + c_2\,|x - x_0|^{\lambda}\,\sin\big(\mu \ln |x - x_0|\big).$$

(iii) If $r_1 = r_2 = r$, real,

Power series solutions (Chptr. 3).

Summary: Solving the Euler equation

$$(x - x_0)^2\, y'' + \alpha\,(x - x_0)\, y' + \beta\, y = 0.$$

(i) If $r_1 \neq r_2$, reals, then the general solution is

$$y(x) = c_1\,|x - x_0|^{r_1} + c_2\,|x - x_0|^{r_2}.$$

(ii) If $r_1 \neq r_2$, complex, denote them as $r_{\pm} = \lambda \pm \mu i$. Then, the real-valued general solution is

$$y(x) = c_1\,|x - x_0|^{\lambda} \cos\big(\mu \ln|x - x_0|\big) + c_2\,|x - x_0|^{\lambda} \sin\big(\mu \ln|x - x_0|\big).$$

(iii) If $r_1 = r_2 = r$, real, then the general solution is

$$y(x) = \big(c_1 + c_2 \ln|x - x_0|\big)\,|x - x_0|^{r}.$$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n x^n$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad xy = \sum_{n=0}^{\infty} a_n x^{(n+1)}$.

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad xy = \sum_{n=0}^{\infty} a_n x^{(n+1)}$.

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{(n-1)}$$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad xy = \sum_{n=0}^{\infty} a_n x^{(n+1)}$.

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{(n-1)} \quad \Rightarrow \quad -3y = \sum_{n=0}^{\infty} (-3n) a_n x^{(n-1)}.$$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad xy = \sum_{n=0}^{\infty} a_n x^{(n+1)}$.

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{(n-1)} \quad \Rightarrow \quad -3y = \sum_{n=0}^{\infty} (-3n) a_n x^{(n-1)}.$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{(n-2)}.$$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution: $x_0 = 0$ is a regular point of the differential equation.

Therefore, $y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad xy = \sum_{n=0}^{\infty} a_n x^{(n+1)}$.

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{(n-1)} \quad \Rightarrow \quad -3y = \sum_{n=0}^{\infty} (-3n) a_n x^{(n-1)}.$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{(n-2)}.$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{(n-2)} + \sum_{n=0}^{\infty} (-3n) a_n x^{(n-1)} + \sum_{n=0}^{\infty} a_n x^{(n+1)} = 0.$$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=0}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=0}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=1}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0.$

Solution:

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=0}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=1}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$$m = n - 2$$

$$m \to n$$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=0}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=1}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$m = n - 2$ $\qquad$ $m = n - 1$

$m \to n$ $\qquad$ $m \to n$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=0}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=1}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$m = n-2$ $\quad$ $m = n-1$ $\quad$ $m = n+1$

$m \to n$ $\quad$ $m \to n$ $\quad$ $m \to n$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=0}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{(n-2)} + \sum_{n=1}^{\infty}(-3n)a_n\, x^{(n-1)} + \sum_{n=0}^{\infty} a_n\, x^{(n+1)} = 0.$$

$m = n-2$ $\qquad$ $m = n-1$ $\qquad$ $m = n+1$

$m \to n$ $\qquad$ $m \to n$ $\qquad$ $m \to n$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n + \sum_{n=0}^{\infty}(-3)(n+1)a_{n+1}\, x^n + \sum_{n=1}^{\infty} a_{n-1}\, x^n = 0.$$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + \sum_{n=0}^{\infty}(-3)(n+1)a_{n+1}\,x^n + \sum_{n=1}^{\infty} a_{n-1}\,x^n = 0.$$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + \sum_{n=0}^{\infty}(-3)(n+1)a_{n+1}\,x^n + \sum_{n=1}^{\infty} a_{n-1}\,x^n = 0.$$

$$(2)(1)a_2 + (-3)(1)a_1 +$$

$$\sum_{n=1}^{\infty}\big[(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1}\big]\,x^n = 0$$

Power series solutions (Chptr. 3).

Example

Find the recurrence relation for the coefficients of the power series solution centered at $x_0 = 0$ of the equation $y'' - 3y' + xy = 0$.

Solution:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + \sum_{n=0}^{\infty}(-3)(n+1)a_{n+1}\,x^n + \sum_{n=1}^{\infty} a_{n-1}\,x^n = 0.$$

$$(2)(1)a_2 + (-3)(1)a_1 +$$

$$\sum_{n=1}^{\infty}\big[(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1}\big]\,x^n = 0$$

We conclude: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1. \quad \triangleleft$$

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\, a_1$,

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \dfrac{3}{2}\, a_1$, and $n = 1$ in the other equation implies

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\, a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0$$

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\,a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\,a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$

Using the equation for a_2 we obtain

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\,a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$

Using the equation for a_2 we obtain $a_3 = \frac{3}{2}\,a_1 - \frac{a_0}{6}$.

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\,a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$

Using the equation for a_2 we obtain $a_3 = \frac{3}{2}\,a_1 - \frac{a_0}{6}$.

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $2a_2 - 3a_1 = 0$, and

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + a_{n-1} = 0, \quad n \geqslant 1.$$

Therefore, $a_2 = \frac{3}{2}\,a_1$, and $n = 1$ in the other equation implies

$$(3)(2)a_3 - 3(2)a_2 + a_0 = 0 \quad \Rightarrow \quad a_3 = a_2 - \frac{a_0}{6}.$$

Using the equation for a_2 we obtain $a_3 = \frac{3}{2}\,a_1 - \frac{a_0}{6}$.

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y(x) = a_0 + a_1 x + \frac{3}{2}\,a_1 x^2 + \left(\frac{3}{2}a_1 - \frac{a_0}{6}\right) x^3 + \cdots$$

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $y(x) = a_0 + a_1 x + \frac{3}{2}\, a_1 x^2 + \left(\frac{3}{2} a_1 - \frac{a_0}{6}\right) x^3 + \cdots.$

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $y(x) = a_0 + a_1 x + \frac{3}{2}\, a_1 x^2 + \left(\frac{3}{2} a_1 - \frac{a_0}{6}\right) x^3 + \cdots$.

$$y(x) = a_0\Big(1 - \frac{1}{6}\, x^3 + \cdots\Big) + a_1\Big(x + \frac{3}{2}\, x^2 + \frac{3}{2}\, x^3 + \cdots\Big),$$

Power series solutions (Chptr. 3).

Example

Find the first two terms on the power series expansion around $x_0 = 0$ of each fundamental solution of $y'' - 3y' + xy = 0$.

Solution: Recall: $y(x) = a_0 + a_1 x + \frac{3}{2}\, a_1 x^2 + \left(\frac{3}{2} a_1 - \frac{a_0}{6}\right) x^3 + \cdots.$

$$y(x) = a_0\Big(1 - \frac{1}{6}\, x^3 + \cdots\Big) + a_1\Big(x + \frac{3}{2}\, x^2 + \frac{3}{2}\, x^3 + \cdots\Big),$$

We conclude that:

$$y_1(x) = 1 - \frac{1}{6}\, x^3 + \cdots\,,$$

$$y_2(x) = x + \frac{3}{2}\, x^2 + \frac{3}{2}\, x^3 + \cdots\,. \qquad \triangleleft$$

Review for Final Exam.

- Fourier Series expansions (Chptr. 6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Power Series Methods (Chptr. 3).
- **Second order linear equations (Chptr. 2).**
- First order differential equations (Chptr. 1).

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$,

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

Second order linear equations.

Summary: Solve $y'' + a_1 y' + a_0 y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real,

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex,

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$,

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t}$$

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t} \left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_{\pm} = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_{\pm}(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_{\pm}(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

and real-valued fundamental solutions are

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

and real-valued fundamental solutions are

$$y_1(t) = e^{\alpha t} \cos(\beta t),$$

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

and real-valued fundamental solutions are

$$y_1(t) = e^{\alpha t} \cos(\beta t), \qquad y_2(t) = e^{\alpha t} \sin(\beta t).$$

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

and real-valued fundamental solutions are

$$y_1(t) = e^{\alpha t}\, \cos(\beta t), \qquad y_2(t) = e^{\alpha t}\, \sin(\beta t).$$

If $r_1 = r_2 = r$, real,

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

and real-valued fundamental solutions are

$$y_1(t) = e^{\alpha t} \cos(\beta t), \qquad y_2(t) = e^{\alpha t} \sin(\beta t).$$

If $r_1 = r_2 = r$, real, then the general solution is

$$y(t) = (c_1 + c_2 t)\, e^{rt}.$$

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*.

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook.

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients:

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters:

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If y_1 and y_2 are fundamental solutions to the homogeneous equation,

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If y_1 and y_2 are fundamental solutions to the homogeneous equation, and W is their Wronskian,

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If y_1 and y_2 are fundamental solutions to the homogeneous equation, and W is their Wronskian, then $y_p = u_1 y_1 + u_2 y_2$,

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If y_1 and y_2 are fundamental solutions to the homogeneous equation, and W is their Wronskian, then $y_p = u_1 y_1 + u_2 y_2$, where

$$u_1' = -\frac{y_2 g}{W},$$

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If y_1 and y_2 are fundamental solutions to the homogeneous equation, and W is their Wronskian, then $y_p = u_1 y_1 + u_2 y_2$, where

$$u_1' = -\frac{y_2 g}{W}, \qquad u_2' = \frac{y_1 g}{W}.$$

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\,y'' - 4x\,y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method.

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\,y'' - 4x\,y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$,

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v,$$

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv,$$

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\, v' + (2x^2 - 8x^2 + 6x^2)\, v = 0.$$

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\, v' + (2x^2 - 8x^2 + 6x^2)\, v = 0.$$

$$v'' = 0$$

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\, v' + (2x^2 - 8x^2 + 6x^2)\, v = 0.$$

$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x$$

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\, v' + (2x^2 - 8x^2 + 6x^2)\, v = 0.$$

$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2 x\, y_1.$$

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\, v' + (2x^2 - 8x^2 + 6x^2)\, v = 0.$$

$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2 x\, y_1.$$

Choose $c_1 = 0$, $c_2 = 1$.

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\, v' + (2x^2 - 8x^2 + 6x^2)\, v = 0.$$

$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2 x\, y_1.$$

Choose $c_1 = 0$, $c_2 = 1$. Hence $y_2(x) = x^3$, and $y_1(x) = x^2$. $\lhd$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt},$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big]$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big]$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4 + 12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_\pm = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p. Since $g(t) = 3\,e^{-t}$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p. Since $g(t) = 3\,e^{-t} \quad \Rightarrow \quad y_p(t) = k\,e^{-t}$.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p. Since $g(t) = 3\,e^{-t} \quad \Rightarrow \quad y_p(t) = k\,e^{-t}$.

But this $y_p = k\,e^{-t}$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p. Since $g(t) = 3\,e^{-t} \quad \Rightarrow \quad y_p(t) = k\,e^{-t}$.

But this $y_p = k\,e^{-t}$ is solution of the homogeneous equation.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_{\pm} = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p. Since $g(t) = 3\,e^{-t} \quad \Rightarrow \quad y_p(t) = k\,e^{-t}$.

But this $y_p = k\,e^{-t}$ is solution of the homogeneous equation.

Then propose $y_p(t) = kt\,e^{-t}$.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t},$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

$$(-2k\,e^{-t} + kt\,e^{-t}) - 2(k\,e^{-t} - kt\,e^{-t}) - 3(kt\,e^{-t}) = 3\,e^{-t}$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

$$(-2k\,e^{-t} + kt\,e^{-t}) - 2(k\,e^{-t} - kt\,e^{-t}) - 3(kt\,e^{-t}) = 3\,e^{-t}$$

$$(-2 + t - 2 + 2t - 3t)\,k\,e^{-t} = 3\,e^{-t}$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

$$(-2k\,e^{-t} + kt\,e^{-t}) - 2(k\,e^{-t} - kt\,e^{-t}) - 3(kt\,e^{-t}) = 3\,e^{-t}$$

$$(-2 + t - 2 + 2t - 3t)\,k\,e^{-t} = 3\,e^{-t} \;\Rightarrow\; -4k = 3$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

$$(-2k\,e^{-t} + kt\,e^{-t}) - 2(k\,e^{-t} - kt\,e^{-t}) - 3(kt\,e^{-t}) = 3\,e^{-t}$$

$$(-2 + t - 2 + 2t - 3t)\,k\,e^{-t} = 3\,e^{-t} \;\Rightarrow\; -4k = 3 \;\Rightarrow\; k = -\frac{3}{4}.$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

$$(-2k\,e^{-t} + kt\,e^{-t}) - 2(k\,e^{-t} - kt\,e^{-t}) - 3(kt\,e^{-t}) = 3\,e^{-t}$$

$$(-2 + t - 2 + 2t - 3t)\,k\,e^{-t} = 3\,e^{-t} \;\Rightarrow\; -4k = 3 \;\Rightarrow\; k = -\frac{3}{4}.$$

We obtain: $y_p(t) = -\frac{3}{4}t\,e^{-t}$.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution:

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0)$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0) = c_1 + c_2,$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0) = c_1 + c_2, \qquad \frac{1}{4} = y'(0)$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0) = c_1 + c_2, \qquad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}.$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0) = c_1 + c_2, \qquad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}.$$

$$\left.\begin{aligned} c_1 + c_2 &= 1, \\ 3_1 - c_2 &= 1 \end{aligned}\right\}$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0) = c_1 + c_2, \qquad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}.$$

$$\left.\begin{aligned} c_1 + c_2 &= 1, \\ 3_1 - c_2 &= 1 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Since $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$,

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Since $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$, we obtain,

$$y(t) = \frac{1}{2}\left(e^{3t} + e^{-t}\right) - \frac{3}{4}t\,e^{-t}.$$

◁

Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Power Series Methods (Chptr. 3).
- Second order linear equations (Chptr. 2).
- **First order differential equations (Chptr. 1).**

First order differential equations.

Summary:

- Linear, first order equations: $y' + p(t)\,y = q(t)$.

First order differential equations.

Summary:

- Linear, first order equations: $y' + p(t)\,y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\,dt}$.

First order differential equations.

Summary:

- Linear, first order equations: $y' + p(t)\,y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\,dt}$.

- Separable, non-linear equations: $h(y)\,y' = g(t)$.

First order differential equations.

Summary:

- Linear, first order equations: $y' + p(t)\,y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\,dt}$.

- Separable, non-linear equations: $h(y)\,y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\,dt$,

First order differential equations.

Summary:

- Linear, first order equations: $y' + p(t)\,y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\,dt}$.

- Separable, non-linear equations: $h(y)\,y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\,dt$, that is,

$$\int h(u)\,du = \int g(t)\,dt + c.$$

First order differential equations.

Summary:

- Linear, first order equations: $y' + p(t)\,y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\,dt}$.

- Separable, non-linear equations: $h(y)\,y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\,dt$, that is,

  $$\int h(u)\,du = \int g(t)\,dt + c.$$

  The solution can be found in implicit of explicit form.

First order differential equations.

Summary:

- Linear, first order equations: $y' + p(t)\,y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\,dt}$.

- Separable, non-linear equations: $h(y)\,y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\,dt$, that is,

  $$\int h(u)\,du = \int g(t)\,dt + c.$$

  The solution can be found in implicit of explicit form.

- Homogeneous equations can be converted into separable equations.

First order differential equations.

Summary:

- Linear, first order equations: $y' + p(t)\,y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\,dt}$.

- Separable, non-linear equations: $h(y)\,y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\,dt$, that is,

  $$\int h(u)\,du = \int g(t)\,dt + c.$$

  The solution can be found in implicit of explicit form.

- Homogeneous equations can be converted into separable equations.

- Applications: Modeling problems from Sect. 2.3.

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook,

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

$$N(x,y)\, y' + M(x,y) = 0.$$

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

$$N(x,y)\, y' + M(x,y) = 0.$$

  The equation is exact iff $\partial_x N = \partial_y M$.

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

$$N(x,y)\,y' + M(x,y) = 0.$$

  The equation is exact iff $\partial_x N = \partial_y M$.

  If the equation is exact, then there is a potential function ψ,

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

  $$N(x,y)\,y' + M(x,y) = 0.$$

  The equation is exact iff $\partial_x N = \partial_y M$.

  If the equation is exact, then there is a potential function ψ, such that $N = \partial_y \psi$

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

  $$N(x,y)\,y' + M(x,y) = 0.$$

  The equation is exact iff $\partial_x N = \partial_y M$.

  If the equation is exact, then there is a potential function ψ, such that $N = \partial_y \psi$ and $M = \partial_x \psi$.

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\, y = q(t)\, y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

  $$N(x,y)\, y' + M(x,y) = 0.$$

  The equation is exact iff $\partial_x N = \partial_y M$.

  If the equation is exact, then there is a potential function ψ, such that $N = \partial_y \psi$ and $M = \partial_x \psi$.

  The solution of the differential equation is

  $$\psi\big(x, y(x)\big) = c.$$

First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)$.)

First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)\, y^n$.)

First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)\, y^n$.)
3. Separable equations.
   (Few manipulations: $h(y)\, y' = g(t)$.)

First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)\, y^n$.)
3. Separable equations.
   (Few manipulations: $h(y)\, y' = g(t)$.)
4. Homogeneous equations.
   (Several manipulations: $y' = F(y/t)$.)

First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)\, y^n$.)
3. Separable equations.
   (Few manipulations: $h(y)\, y' = g(t)$.)
4. Homogeneous equations.
   (Several manipulations: $y' = F(y/t)$.)
5. Exact equations.
   (Check one equation: $N\, y' + M = 0$, and $\partial_t N = \partial_y M$.)

First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\,y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\,y = b(t)\,y^n$.)
3. Separable equations.
   (Few manipulations: $h(y)\,y' = g(t)$.)
4. Homogeneous equations.
   (Several manipulations: $y' = F(y/t)$.)
5. Exact equations.
   (Check one equation: $N\,y' + M = 0$, and $\partial_t N = \partial_y M$.)
6. Exact equation with integrating factor.
   (Very complicated to check.)

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number,

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example.

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)}$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}.$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy}\,\frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}.$$

$$v(x) = \frac{y}{x}$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\,v,$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\,v, \quad y' = x\,v' + v$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\,v, \quad y' = x\,v' + v \quad x\,v' + v = \frac{1 + v + v^2}{v}.$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\,v, \quad y' = x\,v' + v \quad x\,v' + v = \frac{1 + v + v^2}{v}.$$

$$x\,v' = \frac{1 + v + v^2}{v} - v$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \, \frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\,v, \quad y' = x\,v' + v \quad x\,v' + v = \frac{1 + v + v^2}{v}.$$

$$x\,v' = \frac{1 + v + v^2}{v} - v = \frac{1 + v + v^2 - v^2}{v}$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy}\,\frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\,v, \quad y' = x\,v' + v \quad x\,v' + v = \frac{1 + v + v^2}{v}.$$

$$x\,v' = \frac{1 + v + v^2}{v} - v = \frac{1 + v + v^2 - v^2}{v} \quad \Rightarrow \quad x\,v' = \frac{1 + v}{v}.$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$.

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1 + v}{v}$. This is a separable equation.

$$\frac{v(x)}{1 + v(x)}\, v'(x) = \frac{1}{x}$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$,

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \Big(1 - \frac{1}{u}\Big)\, du = \int \frac{dx}{x} + c$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \Big(1 - \frac{1}{u}\Big)\, du = \int \frac{dx}{x} + c$$

$$u - \ln|u| = \ln|x| + c$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \Big(1 - \frac{1}{u}\Big)\, du = \int \frac{dx}{x} + c$$

$$u - \ln|u| = \ln|x| + c \quad \Rightarrow \quad 1 + v - \ln|1+v| = \ln|x| + c.$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \Big(1 - \frac{1}{u}\Big)\, du = \int \frac{dx}{x} + c$$

$$u - \ln|u| = \ln|x| + c \quad \Rightarrow \quad 1 + v - \ln|1+v| = \ln|x| + c.$$

$$v = \frac{y}{x}$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \Big(1 - \frac{1}{u}\Big)\, du = \int \frac{dx}{x} + c$$

$$u - \ln|u| = \ln|x| + c \quad \Rightarrow \quad 1 + v - \ln|1+v| = \ln|x| + c.$$

$$v = \frac{y}{x} \quad \Rightarrow \quad 1 + \frac{y(x)}{x} - \ln\Big|1 + \frac{y(x)}{x}\Big| = \ln|x| + c. \qquad \triangleleft$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\,y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation,

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$,

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3.

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$.

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$,

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n, \;\; n = 3.$

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}.$

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\, v' + v = -e^{2x}.$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\, v' + v = -e^{2x}$.

We obtain the linear equation $v' - 2v = 2e^{2x}$.

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\, v' + v = -e^{2x}$.

We obtain the linear equation $v' - 2v = 2e^{2x}$.

Use the integrating factor method.

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\,y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\,y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\,v' + v = -e^{2x}$.

We obtain the linear equation $v' - 2v = 2e^{2x}$.

Use the integrating factor method. $\mu(x) = e^{-2x}$.

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\, v' + v = -e^{2x}$.

We obtain the linear equation $v' - 2v = 2e^{2x}$.

Use the integrating factor method. $\mu(x) = e^{-2x}$.

$$e^{-2x}\, v' - 2\, e^{-2x}\, v = 2$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\, v' + v = -e^{2x}$.

We obtain the linear equation $v' - 2v = 2e^{2x}$.

Use the integrating factor method. $\mu(x) = e^{-2x}$.

$$e^{-2x}\, v' - 2\, e^{-2x}\, v = 2 \quad \Rightarrow \quad \big(e^{-2x}\, v\big)' = 2.$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2$.

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x} v\right)' = 2.$

$$e^{-2x} v = 2x + c$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x}$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x} v\right)' = 2.$

$$e^{-2x} v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\,y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\,v\right)' = 2.$

$$e^{-2x}\,v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\,e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\,e^{2x}.$$

$$y^2 = \frac{1}{e^2x\,(2x + c)}$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2x\,(2x + c)} \quad \Rightarrow \quad y_{\pm}(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\,(2x + c)} \quad \Rightarrow \quad y_{\pm}(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\,(2x + c)} \quad \Rightarrow \quad y_{\pm}(x) = \pm\frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\,y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\,v\right)' = 2.$

$$e^{-2x}\,v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\,e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\,e^{2x}.$$

$$y^2 = \frac{1}{e^2x\,(2x + c)} \quad\Rightarrow\quad y_{\pm}(x) = \pm\frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies:

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\big(e^{-2x}\, v\big)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\,(2x + c)} \quad \Rightarrow \quad y_\pm(x) = \pm\frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies: Choose y_+.

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2$.

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x+c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x+c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2x\,(2x+c)} \quad \Rightarrow \quad y_{\pm}(x) = \pm \frac{e^{-x}}{\sqrt{2x+c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies: Choose y_+.

$$\frac{1}{3} = y_+(0) = \frac{1}{\sqrt{c}}$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\,(2x + c)} \quad\Rightarrow\quad y_{\pm}(x) = \pm\frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies: Choose y_+.

$$\frac{1}{3} = y_+(0) = \frac{1}{\sqrt{c}} \quad\Rightarrow\quad c = 9$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\, v\right)' = 2.$

$$e^{-2x}\, v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\, e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\, e^{2x}.$$

$$y^2 = \frac{1}{e^2 x\,(2x + c)} \quad \Rightarrow \quad y_{\pm}(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies: Choose y_+.

$$\frac{1}{3} = y_+(0) = \frac{1}{\sqrt{c}} \quad \Rightarrow \quad c = 9 \quad \Rightarrow \quad y(x) = \frac{e^{-x}}{\sqrt{2x + 9}}. \quad \triangleleft$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0$.

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0$.

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$N = [2x^2y + 2x]$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.$$
$$M = [2xy^2 + 2y]$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.$$

$$M = [2xy^2 + 2y] \quad \Rightarrow \quad \partial_y M = 4xy + 2.$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \partial_x N = \partial_y M.$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \partial_x N = \partial_y M.$$

The equation is exact.

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \ \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N,$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0$.

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{aligned} N = [2x^2y + 2x] \quad &\Rightarrow \quad \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] \quad &\Rightarrow \quad \partial_y M = 4xy + 2. \end{aligned}\right\} \Rightarrow \ \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0$.

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \ \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0$.

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{aligned} N = [2x^2y + 2x] \quad &\Rightarrow \quad \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] \quad &\Rightarrow \quad \partial_y M = 4xy + 2. \end{aligned}\right\} \Rightarrow \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x, y) = x^2y^2 + 2xy + g(x).$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{aligned} N = [2x^2y + 2x] \quad &\Rightarrow \quad \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] \quad &\Rightarrow \quad \partial_y M = 4xy + 2. \end{aligned}\right\} \Rightarrow \ \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x, y) = x^2y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x,y) = x^2y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi = M$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2 y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2 y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2 y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \ \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2 y + 2x \quad \Rightarrow \quad \psi(x,y) = x^2 y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{aligned} N = [2x^2y + 2x] \quad &\Rightarrow \quad \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] \quad &\Rightarrow \quad \partial_y M = 4xy + 2. \end{aligned}\right\} \;\Rightarrow\; \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x,y) = x^2y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y \;\Rightarrow\; g'(x) = 0.$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x,y) = x^2y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y \;\Rightarrow\; g'(x) = 0.$$

$$\psi(x,y) = x^2y^2 + 2xy + c,$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2 y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2 y + 2x]\, y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{array}{lcl} N = [2x^2 y + 2x] & \Rightarrow & \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] & \Rightarrow & \partial_y M = 4xy + 2. \end{array}\right\} \Rightarrow \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2 y + 2x \quad \Rightarrow \quad \psi(x, y) = x^2 y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y \;\Rightarrow\; g'(x) = 0.$$

$$\psi(x, y) = x^2 y^2 + 2xy + c, \quad x^2\, y^2(x) + 2x\, y(x) + c = 0. \quad \triangleleft$$