Review for Exam 3.

- 5 or 6 problems, 60 minutes.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Integration table an LT table provided.
- Exam covers:
  - Power Series with Regular-Singular points (3.3).
  - Chapter 4: Laplace Transform methods.
    - Definition of Laplace Transform (4.1).
    - Solving IVP using LT (4.2).
    - Solving IVP with discontinuous sources using LT, (4.3).
    - Solving IVP with generalized sources using LT (4.4).
    - Convolutions and LT (4.5).
  - Systems of linear Differential Equations (5.1).
Regular-singular points (3.3).

Summary:

- Look for solutions \( y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)} \).
- Recall: Since \( r \neq 0 \), holds
  \[ y' = \sum_{n=0}^{\infty} (n+r) a_n (x-x_0)^{(n+r-1)} \neq \sum_{n=1}^{\infty} (n+r) a_n (x-x_0)^{(n+r-1)}, \]
- Find the indicial equation for \( r \), the recurrence relation for \( a_n \).
- Introduce the larger root \( r_+ \) of the indicial polynomial into the recurrence relation and solve for \( a_n \).
  - (a) If \( (r_+ - r_-) \) is not an integer, then each \( r_+ \) and \( r_- \) define linearly independent solutions.
  - (b) If \( (r_+ - r_-) \) is an integer, then both \( r_+ \) and \( r_- \) define proportional solutions.

Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( y = \sum_{n=0}^{\infty} a_n x^{(n+r)} \), \( y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{(n+r-2)} \),

\[ x^2 y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{(n+r)} \]

We also need to compute

\( \left( x^2 + \frac{1}{4} \right) y = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} \),
Example
Consider the equation \( x^2 y'' + \left( x^2 + \frac{1}{4} \right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution:

\[
\left( x^2 + \frac{1}{4} \right) y = \sum_{n=0}^{\infty} a_n x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},
\]

Re-label \( m = n + 2 \) in the first term and then switch back to \( n \),

\[
\left( x^2 + \frac{1}{4} \right) y = \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)},
\]

The equation is

\[
\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{(n+r)} = 0.
\]
Example
Consider the equation $x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$. Use a power series centered at the regular-singular point $x_0 = 0$ to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: 

$$\left[ r(r - 1) + \frac{1}{4}\right] a_0 = 0, \quad \left[ (r + 1)r + \frac{1}{4}\right] a_1 = 0,$$

$$\left[ (n + r)(n + r - 1) + \frac{1}{4}\right] a_n + a_{n-2} = 0.$$

The indicial equation $r^2 - r + \frac{1}{4} = 0$ implies $r = \frac{1}{2}$.

The indicial equation $r^2 + r + \frac{1}{4} = 0$ implies $r = -\frac{1}{2}$.

Choose $r = \frac{1}{2}$. That implies $a_0$ arbitrary and $a_1 = 0$.

$$n^2 a_n = -a_{n-2} \Rightarrow a_n = -\frac{a_{n-2}}{n^2} \Rightarrow \begin{cases} a_2 = -\frac{a_0}{4}, \\ a_4 = -\frac{a_2}{16} = \frac{a_0}{64}. \end{cases}$$
Example

Consider the equation \( x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0 \). Use a power series centered at the regular-singular point \( x_0 = 0 \) to find the three first terms of the solution corresponding to the larger root of the indicial polynomial.

Solution: \( r = \frac{1}{2} \), \( a_1 = 0 \), \( a_2 = -\frac{a_0}{4} \), and \( a_4 = \frac{a_0}{64} \). Then,

\[
y(x) = x^r \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right).
\]

Recall: \( a_1 = 0 \) and the recurrence relation imply \( a_n = 0 \) for \( n \) odd. Therefore,

\[
y(x) = a_0 x^{1/2} \left(1 - \frac{1}{4} x^2 + \frac{1}{64} x^4 + \cdots\right).
\]

\[
\triangleright
\]

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  - **Chapter 4: Laplace Transform methods.**
    - Definition of Laplace Transform (4.1).
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Laplace transforms (Chptr. 4).

Summary:

- Main Properties:
  \[
  \mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0); \quad (18)
  \]
  \[
  e^{-cs} \mathcal{L}[f(t)] = \mathcal{L}[u_c(t) f(t - c)]; \quad (13)
  \]
  \[
  \mathcal{L}[f(t)] \bigg|_{(s-c)} = \mathcal{L}[e^{ct} f(t)]. \quad (14)
  \]

- Convolutions:
  \[
  \mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)] \mathcal{L}[g(t)].
  \]

- Partial fraction decompositions, completing the squares.

Chapter 4: Laplace Transform methods.

Example
Use Laplace Transform to find \( y \) solution of
\[
y'' - 2y' + 2y = \delta(t - 2), \quad y(0) = 1, \quad y'(0) = 3.
\]

Solution: Compute the LT of the equation,
\[
\mathcal{L}[y''] - 2 \mathcal{L}[y'] + 2 \mathcal{L}[y] = \mathcal{L}[\delta(t - 2)] = e^{-2s}
\]
\[
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - s y(0) - y'(0), \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y(0).
\]
\[
(s^2 - 2s + 2) \mathcal{L}[y] - s y(0) - y'(0) + 2 y(0) = e^{-2s}
\]
\[
(s^2 - 2s + 2) \mathcal{L}[y] - s - 1 = e^{-2s}
\]
\[
\mathcal{L}[y] = \frac{(s + 1)}{(s^2 - 2s + 2)} + \frac{1}{(s^2 - 2s + 2)} e^{-2s}.
\]
Example
Use Laplace Transform to find $y$ solution of
$$y'' - 2y' + 2y = \delta(t - 2), \quad y(0) = 1, \quad y'(0) = 3.$$ 

Solution: Recall: $L[y] = \frac{(s + 1)}{(s^2 - 2s + 2)} + \frac{1}{(s^2 - 2s + 2)} e^{-2s}$. 

$s^2 - 2s + 2 = 0 \Rightarrow s_\pm = \frac{1}{2} [2 \pm \sqrt{4 - 8}], \quad \text{complex roots.}$

$s^2 - 2s + 2 = (s^2 - 2s + 1) - 1 + 2 = (s - 1)^2 + 1.$

$$L[y] = \frac{s + 1}{(s - 1)^2 + 1} + \frac{1}{(s - 1)^2 + 1} e^{-2s}.$$ 

$L[y] = \frac{(s - 1 + 1)}{(s - 1)^2 + 1} + \frac{1}{(s - 1)^2 + 1} e^{-2s}.$
Example
Use Laplace Transform to find $y$ solution of
\[ y'' - 2y' + 2y = \delta(t - 2), \quad y(0) = 1, \quad y'(0) = 3. \]

Solution: Recall:
\[
\mathcal{L}[y] = \mathcal{L}[\cos(t)]|_{(s-1)} + 2 \mathcal{L}[\sin(t)]|_{(s-1)} + e^{-2s} \mathcal{L}[\sin(t)]|_{(s-1)}
\]
and $\mathcal{L}[f(t)]|_{(s-c)} = \mathcal{L}[e^{ct} f(t)]$. Therefore,
\[
\mathcal{L}[y] = \mathcal{L}[e^t \cos(t)] + 2 \mathcal{L}[e^t \sin(t)] + e^{-2s} \mathcal{L}[e^t \sin(t)].
\]
Also recall: $e^{-cs} \mathcal{L}[f(t)] = \mathcal{L}[u_c(t) f(t - c)]$. Therefore,
\[
\mathcal{L}[y] = \mathcal{L}[e^t \cos(t)] + 2 \mathcal{L}[e^t \sin(t)] + \mathcal{L}[u_2(t) e^{(t-2)} \sin(t - 2)].
\]
\[
y(t) = \left[ \cos(t) + 2 \sin(t) \right] e^t + u_2(t) \sin(t - 2) e^{(t-2)}. \]

Example
Sketch the graph of $g$ and use LT to find $y$ solution of
\[ y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geq 2. \end{cases} \]

Solution:
Express $g$ using step functions,
\[ g(t) = u_2(t) e^{(t-2)}. \]
\[
\mathcal{L}[u_c(t) f(t - c)] = e^{-cs} \mathcal{L}[f(t)].
\]
Therefore,
\[
\mathcal{L}[g(t)] = e^{-2s} \mathcal{L}[e^t].
\]

We obtain: $\mathcal{L}[g(t)] = \frac{e^{-2s}}{(s - 1)}$. 
Chapter 4: Laplace Transform methods.

Example
Sketch the graph of $g$ and use LT to find $y$ solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geq 2. \end{cases}$$

Solution: Recall: $\mathcal{L}[g(t)] = \frac{e^{-2s}}{(s - 1)}$.

$$\mathcal{L}[y''] + 3 \mathcal{L}[y] = \mathcal{L}[g(t)] = \frac{e^{-2s}}{(s - 1)}.$$

$$(s^2 + 3) \mathcal{L}[y] = \frac{e^{-2s}}{(s - 1)} \Rightarrow \mathcal{L}[y] = e^{-2s} \frac{1}{(s - 1)(s^2 + 3)}.$$

$$H(s) = \frac{1}{(s - 1)(s^2 + 3)} = \frac{a}{(s - 1)} + \frac{(bs + c)}{(s^2 + 3)}.$$

$$1 = a(s^2 + 3) + (bs + c)(s - 1).$$

Chapter 4: Laplace Transform methods.

Example
Sketch the graph of $g$ and use LT to find $y$ solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geq 2. \end{cases}$$

Solution: Recall: $1 = a(s^2 + 3) + (bs + c)(s - 1)$.

$$1 = as^2 + 3a + bs^2 + cs - bs - c$$

$$1 = (a + b) s^2 + (c - b) s + (3a - c)$$

$$a + b = 0, \quad c - b = 0, \quad 3a - c = 1.$$

$$a = -b, \quad c = b, \quad -3b - b = 1 \Rightarrow b = -\frac{1}{4}, \quad a = \frac{1}{4}, \quad c = -\frac{1}{4}.$$

$$H(s) = \frac{1}{4}\left[\frac{1}{s - 1} - \frac{s + 1}{s^2 + 3}\right].$$
Example
Sketch the graph of $g$ and use LT to find $y$ solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geq 2. \end{cases}$$

Solution: Recall: $H(s) = \frac{1}{4} \left[ \frac{1}{s-1} - \frac{s+1}{s^2+3} \right], \quad \mathcal{L}[y] = e^{-2s} H(s).$

$$H(s) = \frac{1}{4} \left[ \frac{1}{s-1} - \frac{s}{s^2+3} - \frac{1}{\sqrt{3}} \frac{s}{s^2+3} \right],$$

$$H(s) = \frac{1}{4} \left[ \mathcal{L}[e^t] - \mathcal{L}[\cos(\sqrt{3} t)] - \frac{1}{\sqrt{3}} \mathcal{L}[\sin(\sqrt{3} t)] \right].$$

We conclude: $y(t) = u_2(t) h(t - 2).$ Equivalently,

$$y(t) = \frac{u_2(t)}{4} \left[ e^{(t-2)} - \cos(\sqrt{3} (t - 2)) - \frac{1}{\sqrt{3}} \sin(\sqrt{3} (t - 2)) \right].$$
Chapter 4: Laplace Transform methods.

Example

Use convolutions to find \( f \) satisfying \( \mathcal{L}[f(t)] = \frac{e^{-2s}}{(s - 1)(s^2 + 3)} \).

Solution: One way to solve this is with the splitting

\[
\mathcal{L}[f(t)] = e^{-2s} \frac{1}{s^2 + 3} \left( \frac{1}{s - 1} \right) = e^{-2s} \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{s^2 + 3} \frac{1}{s - 1},
\]

\[
\mathcal{L}[f(t)] = e^{-2s} \frac{1}{\sqrt{3}} \mathcal{L}[\sin(\sqrt{3}t)] \mathcal{L}[e^t]
\]

\[
\mathcal{L}[f(t)] = \frac{1}{\sqrt{3}} \mathcal{L}[u_2(t) \sin(\sqrt{3}(t - 2))] \mathcal{L}[e^t].
\]

\[
f(t) = \frac{1}{\sqrt{3}} \int_0^t u_2(\tau) \sin(\sqrt{3}(\tau - 2)) e^{(t - \tau)} d\tau. \quad \Box
\]

Chapter 4: Laplace Transform methods.

Example

Sketch the graph of \( g \) and use LT to find \( y \) solution of

\[
y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geq \pi. \end{cases}
\]

Solution:

Express \( g \) using step functions,

\[
g(t) = u_\pi(t) \sin(t - \pi).
\]

\[
\mathcal{L}[u_c(t) f(t - c)] = e^{-cs} \mathcal{L}[f(t)].
\]

Therefore,

\[
\mathcal{L}[g(t)] = e^{-\pi s} \mathcal{L}[\sin(t)].
\]

We obtain: \( \mathcal{L}[g(t)] = \frac{e^{-\pi s}}{s^2 + 1} \).
Chapter 4: Laplace Transform methods.

Example
Sketch the graph of \( g \) and use LT to find \( y \) solution of
\[
y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 
0, & t < \pi, \\
\sin(t - \pi), & t \geq \pi.
\end{cases}
\]

Solution: 
\[
\mathcal{L}[g(t)] = \frac{e^{-\pi s}}{s^2 + 1}.
\]

\[
\mathcal{L}[y''] - 6 \mathcal{L}[y] = \mathcal{L}[g(t)] = \frac{e^{-\pi s}}{s^2 + 1}.
\]

\[
(s^2 - 6) \mathcal{L}[y] = \frac{e^{-\pi s}}{s^2 + 1} \implies \mathcal{L}[y] = e^{-\pi s} \frac{1}{(s^2 + 1)(s^2 - 6)}.
\]

\[
H(s) = \frac{1}{(s^2 + 1)(s^2 - 6)} = \frac{1}{(s^2 + 1)(s + \sqrt{6})(s - \sqrt{6})}
\]

\[
H(s) = \frac{a}{(s + \sqrt{6})} + \frac{b}{(s - \sqrt{6})} + \frac{cs + d}{(s^2 + 1)}.
\]

The solution is:
\[
a = -\frac{1}{14\sqrt{6}}, \quad b = \frac{1}{14\sqrt{6}}, \quad c = 0, \quad d = -\frac{1}{7}.
\]
Chapter 4: Laplace Transform methods.

Example
Sketch the graph of $g$ and use LT to find $y$ solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geq \pi. \end{cases}$$

Solution: $H(s) = \frac{1}{14\sqrt{6}} \left[ -\frac{1}{s + \sqrt{6}} + \frac{1}{s - \sqrt{6}} - \frac{2\sqrt{6}}{(s^2 + 1)} \right]$.

$$H(s) = \frac{1}{14\sqrt{6}} \left[ -L[e^{-\sqrt{6}t}] + L[e^{\sqrt{6}t}] - 2\sqrt{6}L[\sin(t)] \right]$$

$$H(s) = L\left[ \frac{1}{14\sqrt{6}} \left( -e^{-\sqrt{6}t} + e^{\sqrt{6}t} - 2\sqrt{6}\sin(t) \right) \right].$$

$$h(t) = \frac{1}{14\sqrt{6}} \left[ -e^{-\sqrt{6}t} + e^{\sqrt{6}t} - 2\sqrt{6}\sin(t) \right] \Rightarrow H(s) = L[h(t)].$$

Chapter 4: Laplace Transform methods.

Example
Sketch the graph of $g$ and use LT to find $y$ solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geq \pi. \end{cases}$$

Solution: Recall: $L[y] = e^{-\pi s} H(s)$, where $H(s) = L[h(t)]$, and

$$h(t) = \frac{1}{14\sqrt{6}} \left[ -e^{-\sqrt{6}t} + e^{\sqrt{6}t} - 2\sqrt{6}\sin(t) \right].$$

$L[y] = e^{-\pi s} L[h(t)] = L[u_{\pi}(t) h(t - \pi)] \Rightarrow y(t) = u_{\pi}(t) h(t - \pi)$.

Equivalently:

$$y(t) = \frac{u_{\pi}(t)}{14\sqrt{6}} \left[ -e^{-\sqrt{6}(t - \pi)} + e^{\sqrt{6}(t - \pi)} - 2\sqrt{6}\sin(t - \pi) \right]. \quad \triangle$
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Second order equations and first order systems.

Theorem (Reduction to first order)

Every solution \( y \) to the second order linear equation

\[
y'' + p(t)y' + q(t)y = g(t),
\]

defines a solution \( x_1 = y \) and \( x_2 = y' \) of the \( 2 \times 2 \) first order linear differential system

\[
x_1' = x_2, \tag{2}
\]
\[
x_2' = -q(t)x_1 - p(t)x_2 + g(t). \tag{3}
\]

Conversely, every solution \( x_1, x_2 \) of the \( 2 \times 2 \) first order linear system in Eqs. (2)-(3) defines a solution \( y = x_1 \) of the second order differential equation in (1).
Second order equations and first order systems.

**Remark:** Systems of first order equations can, sometimes, be transformed into a second order single equation.

**Example**
Express as a single second order equation the $2 \times 2$ system and solve it, 

\[
\begin{align*}
x_1' &= -x_1 + 3x_2, \\
x_2' &= x_1 - x_2.
\end{align*}
\]

**Solution:** Compute $x_1$ from the second equation: 

\[
x_1 = x_2' + x_2.
\]

Introduce this expression into the first equation,

\[
(x_2' + x_2)' = -(x_2' + x_2) + 3x_2,
\]

\[
x_2'' + x_2' = -x_2' - x_2 + 3x_2,
\]

\[
x_2'' + 2x_2' - 2x_2 = 0.
\]

Second order equations and first order systems.

**Example**
Express as a single second order equation the $2 \times 2$ system and solve it, 

\[
\begin{align*}
x_1' &= -x_1 + 3x_2, \\
x_2' &= x_1 - x_2.
\end{align*}
\]

**Solution:** Recall: 

\[
x_2'' + 2x_2' - 2x_2 = 0.
\]

\[
r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} \left[ -2 \pm \sqrt{4 + 8} \right] \quad \Rightarrow \quad r_{\pm} = -1 \pm \sqrt{3}.
\]

Therefore, 

\[
x_2 = c_1 e^{r_+ t} + c_2 e^{r_- t}.
\]

Since $x_1 = x_2' + x_2,$

\[
x_1 = (c_1 r_+ e^{r_+ t} + c_2 r_- e^{r_- t}) + (c_1 e^{r_+ t} + c_2 e^{r_- t}),
\]

We conclude: 

\[
x_1 = c_1 (1 + r_+) e^{r_+ t} + c_2 (1 + r_-) e^{r_- t}.
\]