Complex, distinct eigenvalues (Sect. 5.8)

- Review: Classification of $2 \times 2$ diagonalizable systems.
- Review: The case of diagonalizable matrices.
- Real matrix with a pair of complex eigenvalues.
- Phase portraits for $2 \times 2$ systems.

Remark:
Diagonalizable $2 \times 2$ matrices $A$ with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, $A$ has two non-proportional eigenvectors $v_1, v_2$ (eigen-directions), (Section 5.7).

(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, $A$ has two non-proportional eigenvectors $v_1 = \overline{v}_2$, (Section 5.8).

(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $v_1, v_2$, (Section 5.9).

Remark:
(c-2) $\lambda_1 = \lambda_2$ real-valued with only one eigen-direction. Hence, $A$ is not diagonalizable, (Section 5.9).
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Review: The case of diagonalizable matrices.

**Theorem (Diagonalizable matrix)**

*If $n \times n$ matrix $A$ is diagonalizable, with a linearly independent eigenvectors set $\{v_1, \cdots, v_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $x$ to the homogeneous, constant coefficients, linear system

$$x'(t) = Ax(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$x(t) = c_1 v_1 e^{\lambda_1 t} + \cdots + c_n v_n e^{\lambda_n t}.$$*
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**Real matrix with a pair of complex eigenvalues.**

**Theorem**

*If* $\{\lambda, \mathbf{v}\}$ *is an eigen-pair of an* $n \times n$ *real-valued matrix* $A$, *then* $\{\bar{\lambda}, \mathbf{\bar{v}}\}$ *also is an eigen-pair of matrix* $A$.

**Proof:** By hypothesis $A \mathbf{v} = \lambda \mathbf{v}$ and $\bar{A} = A$. Then

$$\bar{A} \mathbf{v} = \bar{\lambda} \mathbf{v} \iff \bar{A} \mathbf{\bar{v}} = \bar{\lambda} \mathbf{\bar{v}} \iff A \mathbf{\bar{v}} = \bar{\lambda} \mathbf{\bar{v}}.$$ 

Therefore $\{\bar{\lambda}, \mathbf{\bar{v}}\}$ is an eigen-pair of matrix $A$. \[\square\]

**Remark:** The Theorem above is equivalent to the following:

If an $n \times n$ real-valued matrix $A$ has eigen pairs

$$\lambda_1 = \alpha + i\beta, \quad \mathbf{v}_1 = \mathbf{a} + i\mathbf{b},$$

with $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then so is

$$\lambda_2 = \alpha - i\beta, \quad \mathbf{v}_2 = \mathbf{a} - i\mathbf{b}.$$
Real matrix with a pair of complex eigenvalues.

Theorem (Complex pairs)

If an \( n \times n \) real-valued matrix \( A \) has eigen pairs
\[
\lambda_{\pm} = \alpha \pm i\beta, \quad v^{(\pm)} = a \pm ib,
\]
with \( \alpha, \beta \in \mathbb{R} \) and \( a, b \in \mathbb{R}^n \), then the differential equation
\[
x'(t) = Ax(t)
\]
has a linearly independent set of two complex-valued solutions
\[
x^{(+)} = v^{(+)} e^{\lambda_+ t}, \quad x^{(-)} = v^{(-)} e^{\lambda_- t},
\]
and it also has a linearly independent set of two real-valued solutions
\[
x^{(1)} = [a \cos(\beta t) - b \sin(\beta t)] e^{\alpha t},
\]
\[
x^{(2)} = [a \sin(\beta t) + b \cos(\beta t)] e^{\alpha t}.
\]

Proof: We know that one solution to the differential equation is
\[
x^{(+)} = v^{(+)} e^{\lambda_+ t} = (a + ib) e^{(\alpha+ib)t} = (a + ib) e^{\alpha t} e^{i\beta t}.
\]
Euler equation implies
\[
x^{(+)} = (a + ib) e^{\alpha t} \left[ \cos(\beta t) + i \sin(\beta t) \right],
\]
\[
x^{(+)} = [a \cos(\beta t) - b \sin(\beta t)] e^{\alpha t} + i [a \sin(\beta t) + b \cos(\beta t)] e^{\alpha t}
\]
A similar calculation done on \( x^{(-)} \) implies
\[
x^{(-)} = [a \cos(\beta t) - b \sin(\beta t)] e^{\alpha t} - i [a \sin(\beta t) + b \cos(\beta t)] e^{\alpha t}.
\]
Introduce \( x^{(1)} = (x^{(+)} + x^{(-)})/2, \quad x^{(2)} = (x^{(+)} - x^{(-)})/(2i) \), then
\[
x^{(1)} = [a \cos(\beta t) - b \sin(\beta t)] e^{\alpha t},
\]
\[
x^{(2)} = [a \sin(\beta t) + b \cos(\beta t)] e^{\alpha t}.
\]
\[\square\]
Real matrix with a pair of complex eigenvalues.

Example
Find a real-valued set of fundamental solutions to the equation
\[ x' = Ax, \quad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}. \]

Solution: (1) Find the eigenvalues of matrix \( A \) above,
\[ p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 3 \\ -3 & (2 - \lambda) \end{vmatrix} = (\lambda - 2)^2 + 9. \]
The roots of the characteristic polynomial are
\[ (\lambda - 2)^2 + 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} - 2 = \pm 3i \quad \Rightarrow \quad \lambda_{\pm} = 2 \pm 3i. \]
(2) Find the eigenvectors of matrix \( A \) above. For \( \lambda_+ \),
\[ A - \lambda_+ I = A - (2 + 3i)I = \begin{bmatrix} 2 - (2 + 3i) & 3 \\ -3 & 2 - (2 + 3i) \end{bmatrix}. \]
We need to solve \((A - \lambda_+ I) \mathbf{v}^{(+)} = 0\) for \( \mathbf{v}^{(+)} \). Gauss operations
\[ \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}. \]
So, the eigenvector \( \mathbf{v}^{(+)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \) is given by \( v_1 = -iv_2 \). Choose
\[ v_2 = 1, \quad v_1 = -i, \quad \Rightarrow \quad \mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \lambda_+ = 2 + 3i. \]
Real matrix with a pair of complex eigenvalues.

Example
Find a real-valued set of fundamental solutions to the equation

\[ x' = Ax, \quad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}. \]

Solution: Recall: eigenvalues \( \lambda_\pm = 2 \pm 3i \), and \( \mathbf{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix} \).

The second eigenvector is \( \mathbf{v}^{(-)} = \overline{\mathbf{v}}^{(+)} \), that is, \( \mathbf{v}^{(-)} = \begin{bmatrix} i \\ 1 \end{bmatrix} \).

Notice that \( \mathbf{v}^{(\pm)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i \).

The notation \( \lambda_\pm = \alpha \pm \beta i \) and \( \mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b} i \) implies

\[ \alpha = 2, \quad \beta = 3, \quad \mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \]

Real matrix with a pair of complex eigenvalues.

Example
Find a real-valued set of fundamental solutions to the equation

\[ x' = Ax, \quad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}. \]

Solution: Recall: \( \alpha = 2, \quad \beta = 3, \quad \mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \) and \( \mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \).

Real-valued solutions are \( x^{(1)} = [\mathbf{a} \cos(\beta t) - \mathbf{b} \sin(\beta t)] e^{\alpha t} \), and \( x^{(2)} = [\mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t)] e^{\alpha t} \). That is

\[ x^{(1)} = \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \right) e^{2t} \Rightarrow x^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}. \]

\[ x^{(2)} = \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(3t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(3t) \right) e^{2t} \Rightarrow x^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t}. \]

\( \triangleright \)
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Phase portraits for $2 \times 2$ systems.

**Example**

Sketch a phase portrait for solutions of $x' = Ax$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

**Solution:**
The phase portrait of the vectors

$$\tilde{x}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix},$$

$$\tilde{x}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix},$$

is a radius one circle.
Phase portraits for $2 \times 2$ systems.

Example

Sketch a phase portrait for solutions of $x' = Ax$, $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Solution:
The phase portrait of the solutions

$$\tilde{x}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t},$$

$$\tilde{x}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t},$$

are outgoing spirals.

Phase portraits for $2 \times 2$ systems.

Example

Given any vectors $a$ and $b$, sketch qualitative phase portraits of

$$x^{(1)} = [a \cos(\beta t) - b \sin(\beta t)] e^{\alpha t}, \quad x^{(2)} = [a \sin(\beta t) + b \cos(\beta t)] e^{\alpha t}.$$ 

for the cases $\alpha = 0$, $\alpha > 0$, and $\alpha < 0$, where $\beta > 0$.

Solution: