Real, distinct eigenvalues (Sect. 5.7)

- Review: $n \times n$ linear differential systems.
- The case of diagonalizable matrices.
- Examples: 2×2 linear systems.
- Classification of 2×2 diagonalizable systems.
- Phase portraits for 2×2 systems.

Review: $n \times n$ linear differential systems.

Recall:

- Given an $n \times n$ matrix $A(t)$, n-vector $\mathbf{b}(t)$, find $\mathbf{x}(t)$ solution
$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t).$$
- The system is *homogeneous* iff $\mathbf{b} = 0$, that is,
$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t).$$
- The system has *constant coefficients* iff matrix A does not depend on t, that is,
$$\mathbf{x}'(t) = A\,\mathbf{x}(t) + \mathbf{b}(t).$$
- We study homogeneous, constant coefficient systems, that is,
$$\mathbf{x}'(t) = A\,\mathbf{x}(t).$$

Review: $n \times n$ linear differential systems.

Recall:

- Given continuous functions A, $\mathbf{b}$ on $(t_1, t_2) \subset \mathbb{R}$, a constant $t_0 \in (t_1, t_2)$ and a vector $\mathbf{x}_0$, there exists a unique function $\mathbf{x}$ solution of the IVP

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Today we learn to find such solution in the case of homogeneous, constant coefficients, $n \times n$ linear systems,

$$\mathbf{x}'(t) = A\,\mathbf{x}(t).$$

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The case of diagonalizable matrices.

Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to the homogeneous, constant coefficients, linear system

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

Remark:

- The differential system for the variable $\mathbf{x}$ is coupled, that is, A is not diagonal.
- We transform the system into a system for a variable $\mathbf{y}$ such that the system for $\mathbf{y}$ is decoupled, that is, $\mathbf{y}'(t) = D\,\mathbf{y}(t)$, where D is a diagonal matrix.
- We solve for $\mathbf{y}(t)$ and we transform back to $\mathbf{x}(t)$.

The case of diagonalizable matrices.

Proof: Since A is diagonalizable, we know that $A = PDP^{-1}$, with

$$P = \big[\mathbf{v}_1, \cdots, \mathbf{v}_n\big], \qquad D = \operatorname{diag}\big[\lambda_1, \cdots, \lambda_n\big].$$

Equivalently, $P^{-1}AP = D$. Multiply $\mathbf{x}' = A\,\mathbf{x}$ by P^{-1} on the left

$$P^{-1}\mathbf{x}'(t) = P^{-1}A\,\mathbf{x}(t) \quad \Leftrightarrow \quad \big(P^{-1}\mathbf{x}\big)' = \big(P^{-1}AP\big)\,\big(P^{-1}\mathbf{x}\big).$$

Introduce the new unknown $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, then

$$\mathbf{y}'(t) = D\,\mathbf{y}(t) \;\Leftrightarrow\; \begin{cases} y_1'(t) = \lambda_1\, y_1(t), \\ \qquad \vdots \\ y_n'(t) = \lambda_n\, y_n(t), \end{cases} \quad \Rightarrow\; \mathbf{y}(t) = \begin{bmatrix} c_1\, e^{\lambda_1 t} \\ \vdots \\ c_n\, e^{\lambda_n t} \end{bmatrix}.$$

The case of diagonalizable matrices.

Proof: Recall: $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, and $\mathbf{y}(t) = \begin{bmatrix} c_1\, e^{\lambda_1 t} \\ \vdots \\ c_n\, e^{\lambda_n t} \end{bmatrix}$.

Transform back to $\mathbf{x}(t)$, that is,

$$\mathbf{x}(t) = P\,\mathbf{y}(t) = \big[\mathbf{v}_1, \cdots, \mathbf{v}_n\big] \begin{bmatrix} c_1\, e^{\lambda_1 t} \\ \vdots \\ c_n\, e^{\lambda_n t} \end{bmatrix}$$

We conclude: $\mathbf{x}(t) = c_1\mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n\mathbf{v}_n\, e^{\lambda_n t}$. $\square$

Remark:

- $A\,\mathbf{v}_i = \lambda_i\mathbf{v}_i$.
- The eigenvalues and eigenvectors of A are crucial to solve the differential linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$.

The case of diagonalizable matrices.

Remark: Here is another argument useful to understand why the vector $\mathbf{x}(t) = c_1\mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n\mathbf{v}_n\, e^{\lambda_n t}$ is solution of the linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$. On the one hand, derivate $\mathbf{x}$,

$$\mathbf{x}'(t) = c_1\lambda_1\,\mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n\lambda_n\,\mathbf{v}_n\, e^{\lambda_n t}.$$

On the other hand, compute $A\,\mathbf{x}(t)$,

$$A\,\mathbf{x}(t) = c_1(A\,\mathbf{v}_1)\, e^{\lambda_1 t} + \cdots + c_n(A\,\mathbf{v}_n)\, e^{\lambda_n t},$$

$$A\,\mathbf{x}(t) = c_1\lambda_1\,\mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n\lambda_n\,\mathbf{v}_n\, e^{\lambda_n t}.$$

We conclude: $\mathbf{x}'(t) = A\,\mathbf{x}(t)$.

Remark: Unlike the proof of the Theorem, this second argument does not show that $\mathbf{x}(t) = c_1\mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n\mathbf{v}_n\, e^{\lambda_n t}$ are all possible solutions to the system.

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Examples: 2×2 linear systems.

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: Find eigenvalues and eigenvectors of A. We found that:

$$\lambda_1 = 4, \quad \mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_2 = -2, \quad \mathbf{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Fundamental solutions are

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

The general solution is $\mathbf{x}(t) = c_1\, \mathbf{x}^{(1)}(t) + c_2\, \mathbf{x}^{(2)}(t)$, that is,

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}, \qquad c_1, c_2 \in \mathbb{R}.$$

$\lhd$

Examples: 2×2 linear systems.

Remark:

Re-writing the solution vector $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$ in components $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, then

$$x_1(t) = c_1\, e^{4t} - c_2\, e^{-2t}, \quad x_2(t) = c_1\, e^{4t} + c_2\, e^{-2t}.$$

Introducing the fundamental matrix $X(t) = \big[\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t)\big]$ and the vector $\mathbf{c}$,

$$X(t) = \begin{bmatrix} e^{4t} & -e^{-2t} \\ e^{4t} & e^{-2t} \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

then the general solution above can be expressed as follows

$$\mathbf{x}(t) = X(t)\mathbf{c} \quad \Leftrightarrow \quad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{4t} & -e^{-2t} \\ e^{4t} & e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Examples: 2×2 linear systems.

Example

Solve the IVP $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: The general solution: $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$.

The initial condition is,

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We need to solve the linear system

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, hence $\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$. $\triangleleft$

Real, distinct eigenvalues (Sect. 5.7)

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Classification of 2×2 diagonalizable systems.

Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) Matrix A has two different, real eigenvalues $\lambda_1 \neq \lambda_2$, so it has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions). (Section 5.7)

(b) Matrix A has two different, complex eigenvalues $\lambda_1 = \overline{\lambda}_2$, so it has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$. (Section 5.8)

(c-1) Matrix A has repeated, real eigenvalues, $\lambda_1 = \lambda_2 \in \mathbb{R}$ with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$. (Section 5.9)

Remark:

(c-2) We will also study in Section 5.9 how to obtain solutions to a 2×2 system $\mathbf{x}' = A\,\mathbf{x}$ in the case that A is not diagonalizable and A has only one eigen-direction.

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Phase portraits for 2×2 systems.

Remark:

- There are two main types of graphs for solutions of 2×2 linear systems:
  (i) The graphs of the vector components;
  (ii) The phase portrait.
- Case (i): Express the solution in vector components $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, and graph x_1 and x_2 as functions of t.
  (Recall the solution in the IVP of the previous Example: $x_1(t) = 3\,e^{4t} - e^{-2t}$ and $x_2(t) = 3\,e^{4t} + e^{-2t}$.)
- Case (ii): Express the solution as a vector-valued function,
  $$\mathbf{x}(t) = c_1\,\mathbf{v}_1\,e^{\lambda_1 t} + c_2\,\mathbf{v}_2\,e^{\lambda_2 t},$$
  and plot the vector $\mathbf{x}(t)$ for different values of t.
- Case (ii) is called a *phase portrait*.

Phase portraits for 2×2 systems.

Example

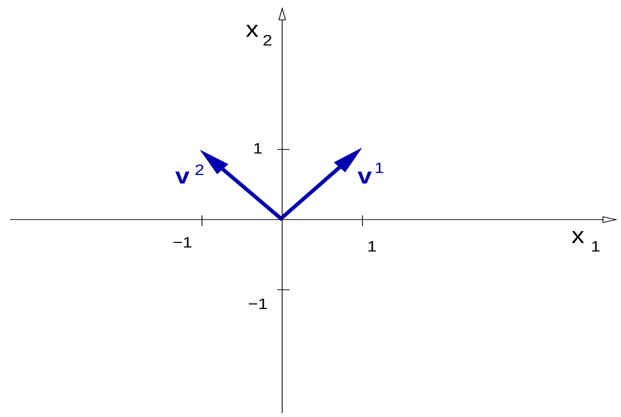
Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:
We start plotting the vectors

$$\mathbf{v}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\mathbf{v}^2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$



Phase portraits for 2×2 systems.

Example

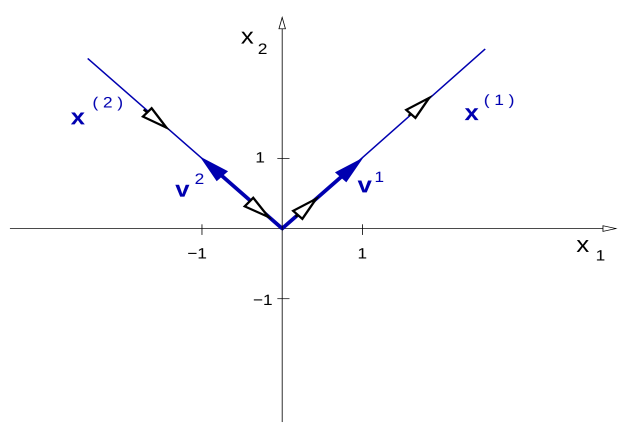
Plot the phase portrait of several linear combinations of the fundamental solutions found above,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:
We now plot the functions

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t},$$

$$\mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$



Phase portraits for 2×2 systems.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

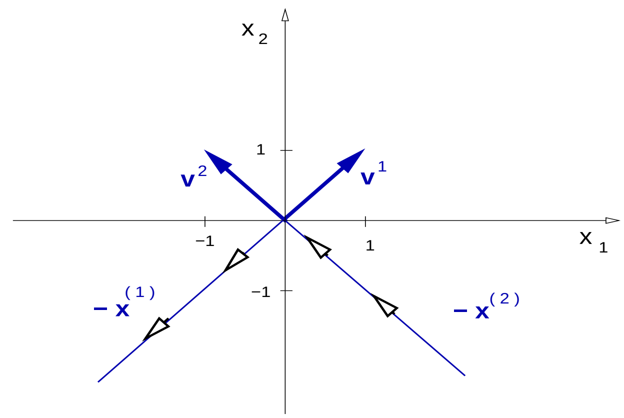
$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:

We now plot the functions

$$-\mathbf{x}^{(1)} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t},$$

$$-\mathbf{x}^{(2)} = -\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$



Phase portraits for 2×2 systems.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

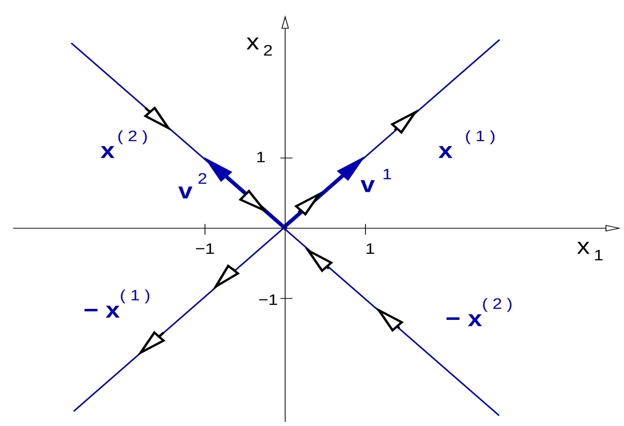
$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:

We now plot the four functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

$$\mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)}.$$



Phase portraits for 2×2 systems.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

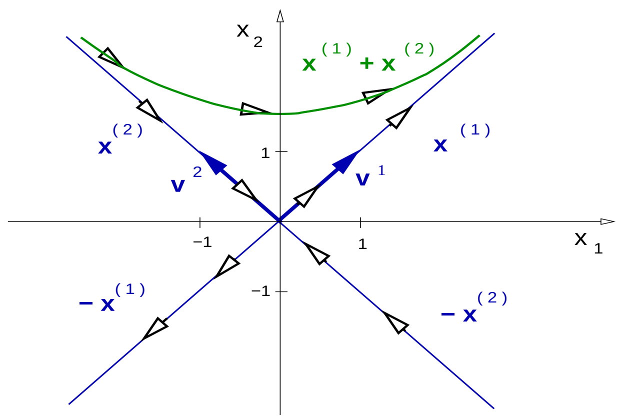
$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:
We now plot the four functions

$\mathbf{x}^{(1)}, \ -\mathbf{x}^{(1)}, \ \mathbf{x}^{(2)}, \ -\mathbf{x}^{(2)},$

and $\mathbf{x}^{(1)} + \mathbf{x}^{(2)}$,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$



Phase portraits for 2×2 systems.

Example

Plot the phase portrait of several linear combinations of the fundamental solutions found above,

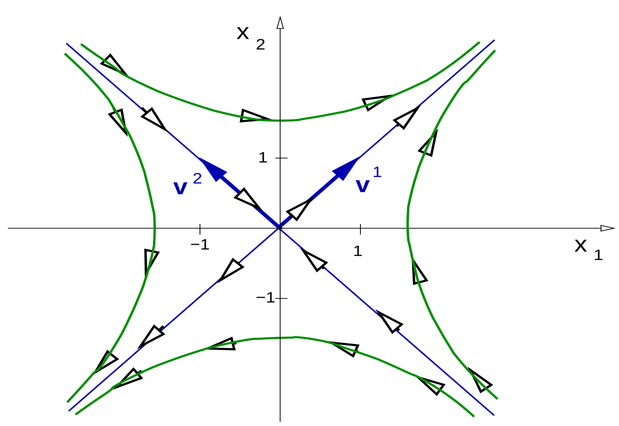
$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Solution:
We now plot the eight functions

$\mathbf{x}^{(1)}, \ -\mathbf{x}^{(1)}, \ \mathbf{x}^{(2)}, \ -\mathbf{x}^{(2)},$

$\mathbf{x}^{(1)} + \mathbf{x}^{(2)}, \quad -\mathbf{x}^{(1)} + \mathbf{x}^{(2)},$

$\mathbf{x}^{(1)} - \mathbf{x}^{(2)}, \quad -\mathbf{x}^{(1)} - \mathbf{x}^{(2)}.$



Phase portraits for 2×2 systems.

Problem:
Case (a): Consider a 2×2 matrix A having two different, real eigenvalues $\lambda_1 \neq \lambda_2$, so A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions).

Given a solution $\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}$, to $\mathbf{x}'(t) = A\mathbf{x}(t)$, plot different solution vectors $\mathbf{x}(t)$ on the plane as function of t for different choices of the constants c_1 and c_2.

The plots are different depending on the eigenvalues signs.
We have the following three sub-cases:

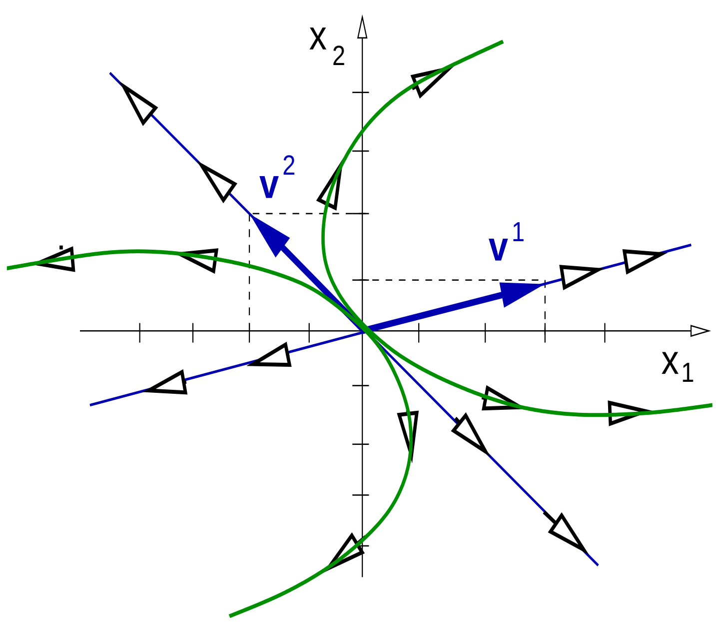
(i) $0 < \lambda_2 < \lambda_1$, both positive;

(ii) $\lambda_2 < 0 < \lambda_1$, one positive the other negative;

(iii) $\lambda_2 < \lambda_1 < 0$, both negative.

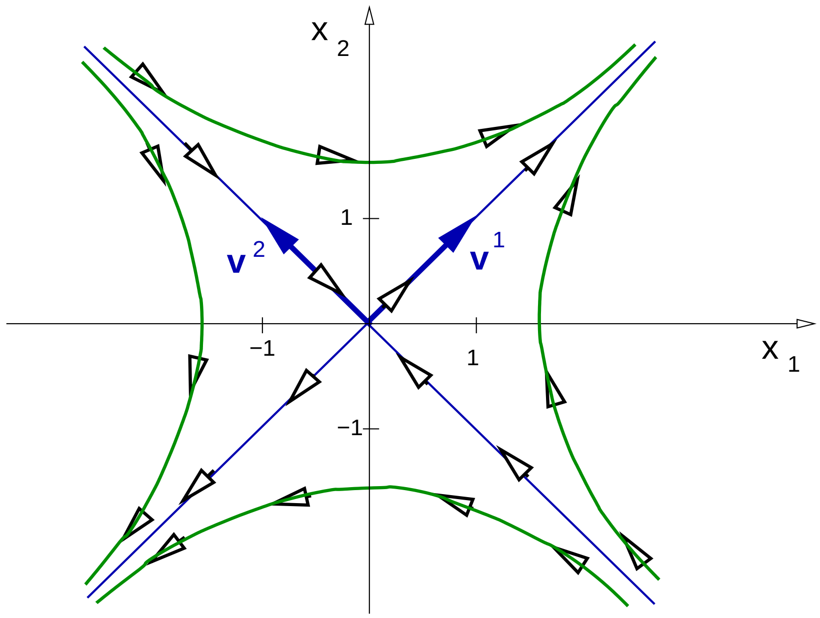
Phase portraits for 2×2 systems.

Phase portrait: Case (a), two different, real eigenvalues $\lambda_1 \neq \lambda_2$, sub-case $0 < \lambda_2 < \lambda_1$, both eigenvalue positive.



Phase portraits for 2×2 systems.

Phase portrait: Case (a), two different, real eigenvalues $\lambda_1 \neq \lambda_2$, sub-case $\lambda_2 < 0 < \lambda_1$, one eigenvalue positive the other negative.



Phase portraits for 2×2 systems.

Phase portrait: Case (a), two different, real eigenvalues $\lambda_1 \neq \lambda_2$, sub-case $\lambda_2 < \lambda_1 < 0$, both eigenvalues negative.