

## The Laplace Transform of step functions (Sect. 4.3).

### Last Lecture

- ▶ Overview and notation.
- ▶ The definition of a step function.
- ▶ Piecewise discontinuous functions.
- ▶ The Laplace Transform of discontinuous functions.
- ▶ Properties of the Laplace Transform.

### This Lecture

- ▶ Differential equations with discontinuous sources.

## Equations with discontinuous sources (Sect. 4.3).

- ▶ Differential equations with discontinuous sources.
- ▶ We solve the IVPs:
  - Example 1:

$$y' + 2y = u(t - 4), \quad y(0) = 3.$$

- Example 2:

$$y'' + y' + \frac{5}{4}y = b(t), \quad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

- Example 3:

$$y'' + y' + \frac{5}{4}y = g(t), \quad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \quad g(t) = \begin{cases} \sin(t), & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

## Equations with discontinuous sources (Sect. 4.3).

- ▶ Differential equations with discontinuous sources.
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- (b) Example 2:

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- (c) Example 3:

$$y'' + y' + \frac{5}{4}y = g(t), \quad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \quad g(t) = \begin{cases} \sin(t), & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

## Differential equations with discontinuous sources.

### Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t - 4), \quad y(0) = 3.$$

**Solution:** Compute the Laplace transform of the whole equation,

$$\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[u(t - 4)] = \frac{e^{-4s}}{s}.$$

From the previous Section we know that

$$[s\mathcal{L}[y] - y(0)] + 2\mathcal{L}[y] = \frac{e^{-4s}}{s} \quad \Rightarrow \quad (s+2)\mathcal{L}[y] = y(0) + \frac{e^{-4s}}{s}.$$

Introduce the initial condition,  $\mathcal{L}[y] = \frac{3}{(s+2)} + e^{-4s} \frac{1}{s(s+2)}$ ,

Use the table:  $\mathcal{L}[y] = 3\mathcal{L}[e^{-2t}] + e^{-4s} \frac{1}{s(s+2)}$ .

## Differential equations with discontinuous sources.

### Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t - 4), \quad y(0) = 3.$$

Solution: Recall:  $\mathcal{L}[y] = 3\mathcal{L}[e^{-2t}] + e^{-4s} \frac{1}{s(s+2)}$ .

We need to invert the Laplace transform on the last term.

Partial fractions:

$$\frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{s+2} = \frac{a(s+2) + bs}{s(s+2)} = \frac{(a+b)s + (2a)}{s(s+2)}$$

We get,  $a + b = 0$ ,  $2a = 1$ . We obtain:  $a = \frac{1}{2}$ ,  $b = -\frac{1}{2}$ . Hence,

$$\frac{1}{s(s+2)} = \frac{1}{2} \left[ \frac{1}{s} - \frac{1}{s+2} \right].$$

## Differential equations with discontinuous sources.

### Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t - 4), \quad y(0) = 3.$$

Solution: Recall:  $\frac{1}{s(s+2)} = \frac{1}{2} \left[ \frac{1}{s} - \frac{1}{s+2} \right]$ .

The algebraic equation for  $\mathcal{L}[y]$  has the form,

$$\mathcal{L}[y] = 3\mathcal{L}[e^{-2t}] + \frac{1}{2} \left[ e^{-4s} \frac{1}{s} - e^{-4s} \frac{1}{s+2} \right].$$

$$\mathcal{L}[y] = 3\mathcal{L}[e^{-2t}] + \frac{1}{2} \left( \mathcal{L}[u(t-4)] - \mathcal{L}[u(t-4)e^{-2(t-4)}] \right).$$

We conclude that

$$y(t) = 3e^{-2t} + \frac{1}{2} u(t-4) \left[ 1 - e^{-2(t-4)} \right].$$

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## Equations with discontinuous sources (Sect. 4.3).

- ▶ Differential equations with discontinuous sources.
- ▶ We solve the IVPs:

(a) Example 1:

$$y' + 2y = u(t - 4), \quad y(0) = 3.$$

(b) **Example 2:**

$$y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

(c) Example 3:

$$y'' + y' + \frac{5}{4}y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t), & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

## Differential equations with discontinuous sources.

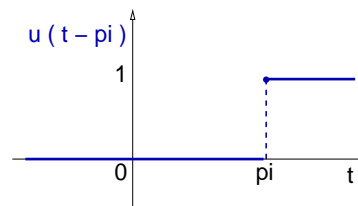
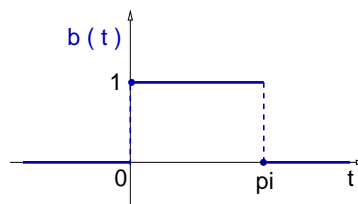
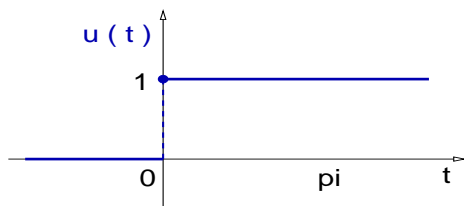
### Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

### Solution:

Rewrite the source function using step functions.



## Differential equations with discontinuous sources.

### Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

**Solution:** The graphs imply:  $b(t) = u(t) - u(t - \pi)$

Now is simple to find  $\mathcal{L}[b]$ , since

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t - \pi)] = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

So, the source is  $\mathcal{L}[b(t)] = (1 - e^{-\pi s}) \frac{1}{s}$ , and the equation is

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s}.$$

## Differential equations with discontinuous sources.

### Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

**Solution:** So:  $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s}$ .

The initial conditions imply:  $\mathcal{L}[y''] = s^2 \mathcal{L}[y]$  and  $\mathcal{L}[y'] = s \mathcal{L}[y]$ .

Therefore,  $\left(s^2 + s + \frac{5}{4}\right) \mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s}$ .

We arrive at the expression:  $\mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s \left(s^2 + s + \frac{5}{4}\right)}$ .

## Differential equations with discontinuous sources.

### Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall:  $\mathcal{L}[y] = (1 - e^{-\pi s}) \frac{1}{s(s^2 + s + \frac{5}{4})}$ .

Denoting:  $H(s) = \frac{1}{s(s^2 + s + \frac{5}{4})}$ ,

we obtain,  $\mathcal{L}[y] = (1 - e^{-\pi s}) H(s)$ .

In other words:  $y(t) = \mathcal{L}^{-1}[H(s)] - \mathcal{L}^{-1}[e^{-\pi s} H(s)]$ .

## Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall:  $y(t) = \mathcal{L}^{-1}[H(s)] - \mathcal{L}^{-1}[e^{-\pi s} H(s)]$ .

Denoting:  $h(t) = \mathcal{L}^{-1}[H(s)]$ , the  $\mathcal{L}[\ ]$  properties imply

$$\mathcal{L}^{-1}[e^{-\pi s} H(s)] = u(t - \pi) h(t - \pi).$$

Therefore, the solution has the form

$$y(t) = h(t) - u(t - \pi) h(t - \pi).$$

We only need to find  $h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2 + s + \frac{5}{4})}\right]$ .

## Differential equations with discontinuous sources.

### Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$
$$y'(0) = 0,$$

Solution: Recall:  $h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2 + s + \frac{5}{4})}\right]$ .

Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}[-1 \pm \sqrt{1-5}] \Rightarrow \text{Complex roots.}$$

The partial fraction decomposition is:

$$H(s) = \frac{1}{(s^2 + s + \frac{5}{4})s} = \frac{a}{s} + \frac{(bs + c)}{(s^2 + s + \frac{5}{4})}$$

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Solution: Recall:  $H(s) = \frac{1}{(s^2 + s + \frac{5}{4})s} = \frac{a}{s} + \frac{(bs + c)}{(s^2 + s + \frac{5}{4})}$ .

The partial fraction decomposition is:

$$1 = a\left(s^2 + s + \frac{5}{4}\right) + s(bs + c) = (a + b)s^2 + (a + c)s + \frac{5}{4}a.$$

This equation implies that  $a$ ,  $b$ , and  $c$ , are solutions of

$$a + b = 0, \quad a + c = 0, \quad \frac{5}{4}a = 1.$$

## Differential equations with discontinuous sources.

### Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad y'(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So:  $a = \frac{4}{5}$ ,  $b = -\frac{4}{5}$ ,  $c = -\frac{4}{5}$ .

Hence, we have found that,

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right) s} = \frac{4}{5} \left[ \frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)} \right]$$

We have to compute the inverse Laplace Transform

$$h(t) = \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)} \right]$$

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Solution: Recall:  $h(t) = \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)} \right]$ .

In this case we complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \left[ s^2 + 2\left(\frac{1}{2}\right)s + \frac{1}{4} \right] - \frac{1}{4} + \frac{5}{4} = \left( s + \frac{1}{2} \right)^2 + 1.$$

So:  $h(t) = \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{1}{s} - \frac{(s+1)}{\left[ \left( s + \frac{1}{2} \right)^2 + 1 \right]} \right]$ .

That is,  $h(t) = \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{1}{s} \right] - \frac{4}{5} \mathcal{L}^{-1} \left[ \frac{\left( s + \frac{1}{2} \right) + \frac{1}{2}}{\left[ \left( s + \frac{1}{2} \right)^2 + 1 \right]} \right]$ .



## Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = b(t), \quad y(0) = 0, \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$
$$y'(0) = 0,$$

Solution: Recall:  $h(t) = \frac{4}{5} \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{4}{5} \mathcal{L}^{-1}\left[\frac{(s + \frac{1}{2}) + \frac{1}{2}}{[(s + \frac{1}{2})^2 + 1]}\right].$

$$h(t) = \frac{4}{5} \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{4}{5} \mathcal{L}^{-1}\left[\frac{(s + \frac{1}{2})}{[(s + \frac{1}{2})^2 + 1]}\right] - \frac{2}{5} \mathcal{L}^{-1}\left[\frac{1}{[(s + \frac{1}{2})^2 + 1]}\right].$$

Recall:  $\mathcal{L}^{-1}[F(s - c)] = e^{ct} f(t)$ . Hence,

$$h(t) = \frac{4}{5} \left[ 1 - e^{-t/2} \cos(t) - \frac{1}{2} e^{-t/2} \sin(t) \right].$$

We conclude:  $y(t) = h(t) + u(t - \pi)h(t - \pi)$ .  $\triangleleft$

## Equations with discontinuous sources (Sect. 4.3).

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$$y'' + y' + \frac{5}{4}y = g(t), \quad y(0) = 0, \quad g(t) = \begin{cases} \sin(t), & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$
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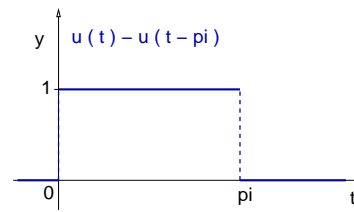
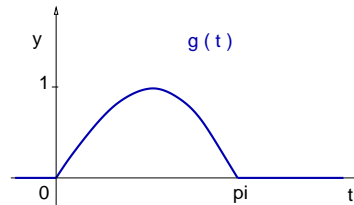
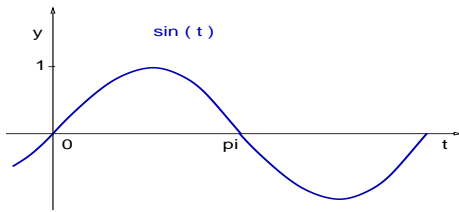
### Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

### Solution:

Rewrite the source function using step functions.



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**Solution:** The graphs imply:  $g(t) = [u(t) - u(t - \pi)] \sin(t)$ .

Recall the identity:  $\sin(t) = -\sin(t - \pi)$ . Then,

$$g(t) = u(t) \sin(t) - u(t - \pi) \sin(t),$$

$$g(t) = u(t) \sin(t) + u(t - \pi) \sin(t - \pi).$$

Now is simple to find  $\mathcal{L}[g]$ , since

$$\mathcal{L}[g(t)] = \mathcal{L}[u(t) \sin(t)] + \mathcal{L}[u(t - \pi) \sin(t - \pi)].$$

## Differential equations with discontinuous sources.

### Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So:  $\mathcal{L}[g(t)] = \mathcal{L}[u(t) \sin(t)] + \mathcal{L}[u(t - \pi) \sin(t - \pi)]$ .

$$\mathcal{L}[g(t)] = \frac{1}{(s^2 + 1)} + e^{-\pi s} \frac{1}{(s^2 + 1)}.$$

Recall the Laplace transform of the differential equation

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4} \mathcal{L}[y] = \mathcal{L}[g].$$

The initial conditions imply:  $\mathcal{L}[y''] = s^2 \mathcal{L}[y]$  and  $\mathcal{L}[y'] = s \mathcal{L}[y]$ .

$$\text{Therefore, } \left(s^2 + s + \frac{5}{4}\right) \mathcal{L}[y] = (1 + e^{-\pi s}) \frac{1}{(s^2 + 1)}.$$

## Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall:  $\left(s^2 + s + \frac{5}{4}\right) \mathcal{L}[y] = (1 + e^{-\pi s}) \frac{1}{(s^2 + 1)}$ .

$$\mathcal{L}[y] = (1 + e^{-\pi s}) \frac{1}{\left(s^2 + s + \frac{5}{4}\right) (s^2 + 1)}.$$

Introduce the function  $H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right) (s^2 + 1)}$ .

Then,  $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s} H(s)]$ .

## Differential equations with discontinuous sources.

### Example

Use the Laplace transform to find the solution of the IVP

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Solution: Recall:  $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s} H(s)]$ , and

$$H(s) = \frac{1}{(s^2 + s + \frac{5}{4})(s^2 + 1)}.$$

Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}[-1 \pm \sqrt{1 - 5}] \Rightarrow \text{Complex roots.}$$

The partial fraction decomposition is:

$$\frac{1}{(s^2 + s + \frac{5}{4})(s^2 + 1)} = \frac{(as + b)}{(s^2 + s + \frac{5}{4})} + \frac{(cs + d)}{(s^2 + 1)}.$$

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Solution: So: 
$$\frac{1}{(s^2 + s + \frac{5}{4})(s^2 + 1)} = \frac{(as + b)}{(s^2 + s + \frac{5}{4})} + \frac{(cs + d)}{(s^2 + 1)}.$$

Therefore, we get

$$1 = (as + b)(s^2 + 1) + (cs + d)\left(s^2 + s + \frac{5}{4}\right),$$

$$1 = (a + c)s^3 + (b + c + d)s^2 + \left(a + \frac{5}{4}c + d\right)s + \left(b + \frac{5}{4}d\right).$$

This equation implies that  $a$ ,  $b$ ,  $c$ , and  $d$ , are solutions of

$$a + c = 0, \quad b + c + d = 0, \quad a + \frac{5}{4}c + d = 0, \quad b + \frac{5}{4}d = 1.$$

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$$y'' + y' + \frac{5}{4}y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So:  $a = \frac{16}{17}$ ,  $b = \frac{12}{17}$ ,  $c = -\frac{16}{17}$ ,  $d = \frac{4}{17}$ .

We have found:  $H(s) = \frac{4}{17} \left[ \frac{(4s+3)}{(s^2+s+\frac{5}{4})} + \frac{(-4s+1)}{(s^2+1)} \right]$ .

Complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \left[ s^2 + 2\left(\frac{1}{2}\right)s + \frac{1}{4} \right] - \frac{1}{4} + \frac{5}{4} = \left( s + \frac{1}{2} \right)^2 + 1.$$

$$H(s) = \frac{4}{17} \left[ \frac{(4s+3)}{\left[ \left( s + \frac{1}{2} \right)^2 + 1 \right]} + \frac{(-4s+1)}{(s^2+1)} \right].$$

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Solution: So:  $H(s) = \frac{4}{17} \left[ \frac{(4s+3)}{\left[ \left( s + \frac{1}{2} \right)^2 + 1 \right]} + \frac{(-4s+1)}{(s^2+1)} \right]$ .

Rewrite the polynomial in the numerator,

$$(4s+3) = 4\left(s + \frac{1}{2} - \frac{1}{2}\right) + 3 = 4\left(s + \frac{1}{2}\right) + 1,$$

$$H(s) = \frac{4}{17} \left[ 4 \frac{\left( s + \frac{1}{2} \right)}{\left[ \left( s + \frac{1}{2} \right)^2 + 1 \right]} + \frac{1}{\left[ \left( s + \frac{1}{2} \right)^2 + 1 \right]} - 4 \frac{s}{(s^2+1)} + \frac{1}{(s^2+1)} \right],$$

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Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution:

$$H(s) = \frac{4}{17} \left[ 4 \frac{(s + \frac{1}{2})}{[(s + \frac{1}{2})^2 + 1]} + \frac{1}{[(s + \frac{1}{2})^2 + 1]} - 4 \frac{s}{(s^2 + 1)} + \frac{1}{(s^2 + 1)} \right],$$

Use the Laplace Transform table to get  $H(s)$  equal to

$$H(s) = \frac{4}{17} \left[ 4 \mathcal{L}[e^{-t/2} \cos(t)] + \mathcal{L}[e^{-t/2} \sin(t)] - 4 \mathcal{L}[\cos(t)] + \mathcal{L}[\sin(t)] \right].$$

$$H(s) = \mathcal{L} \left[ \frac{4}{17} \left( 4e^{-t/2} \cos(t) + e^{-t/2} \sin(t) - 4 \cos(t) + \sin(t) \right) \right].$$

## Differential equations with discontinuous sources.

### Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall:

$$H(s) = \mathcal{L} \left[ \frac{4}{17} \left( 4e^{-t/2} \cos(t) + e^{-t/2} \sin(t) - 4 \cos(t) + \sin(t) \right) \right].$$

Denote:

$$h(t) = \frac{4}{17} \left[ 4e^{-t/2} \cos(t) + e^{-t/2} \sin(t) - 4 \cos(t) + \sin(t) \right].$$

Then,  $H(s) = \mathcal{L}[h(t)]$ . Recalling:  $\mathcal{L}[y(t)] = H(s) + e^{-\pi s} H(s)$ ,

$$\mathcal{L}[y(t)] = \mathcal{L}[h(t)] + e^{-\pi s} \mathcal{L}[h(t)].$$

We conclude:  $y(t) = h(t) + u(t - \pi)h(t - \pi)$ . ◁