Overview and notation.

Overview: The Laplace Transform method can be used to solve constant coefficients differential equations with \textit{discontinuous source functions}.

Notation:
If $\mathcal{L}[f(t)] = F(s)$, then we denote $\mathcal{L}^{-1}[F(s)] = f(t)$.

Remark: One can show that for a particular type of functions $f$, that includes all functions we work with in this Section, the notation above is well-defined.

Example
From the Laplace Transform table we know that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$.
Then also holds that $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$. \hspace{1cm} \triangle
The Laplace Transform of step functions (Sect. 4.3).

- Overview and notation.
- **The definition of a step function.**
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.

### The definition of a step function.

**Definition**
A function $u$ is called a *step function* at $t = 0$ iff holds

$$u(t) = \begin{cases} 
0 & \text{for } t < 0, \\
1 & \text{for } t \geq 0.
\end{cases}$$

**Example**
Graph the step function values $u(t)$ above, and the translations $u(t - c)$ and $u(t + c)$ with $c > 0$.

**Solution:**

![Graphs of step functions and their translations](image)
The definition of a step function.

Remark: Given any function values \( f(t) \) and \( c > 0 \), then \( f(t - c) \) is a right translation of \( f \) and \( f(t + c) \) is a left translation of \( f \).

Example

\[ f(t) = e^{at} \]
\[ f(t) = e^{a(t-c)} \]
\[ f(t) = u(t) e^{at} \]
\[ f(t) = u(t-c) e^{a(t-c)} \]

The Laplace Transform of step functions (Sect. 4.3).

- Overview and notation.
- The definition of a step function.
- **Piecewise discontinuous functions.**
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.
Piecewise discontinuous functions.

Example
Graph of the function $b(t) = u(t - a) - u(t - b)$, with $0 < a < b$.

Solution: The bump function $b$ can be graphed as follows:

![Graph of the function $b(t) = u(t - a) - u(t - b)$](image1)

Solution:

Graph of the function $f(t) = e^{at} [u(t - 1) - u(t - 2)]$.

Solution:

![Graph of the function $f(t) = e^{at} [u(t - 1) - u(t - 2)]$](image2)

Notation: It is common in the literature to denote the function values $u(t - c)$ as $u_c(t)$. 
The Laplace Transform of step functions (Sect. 4.3).

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- **The Laplace Transform of discontinuous functions.**
- Properties of the Laplace Transform.

The Laplace Transform of discontinuous functions.

**Theorem**

*Given any real number* \( c \geq 0 \), *the following equation holds,*

\[
\mathcal{L}[u(t - c)] = \frac{e^{-cs}}{s}, \quad s > 0.
\]

**Proof:**

\[
\mathcal{L}[u(t - c)] = \int_{0}^{\infty} e^{-st} u(t - c) \, dt = \int_{c}^{\infty} e^{-st} \, dt,
\]

\[
\mathcal{L}[u(t - c)] = \lim_{N \to \infty} \frac{1}{s} (e^{-Ns} - e^{-cs}) = \frac{e^{-cs}}{s}, \quad s > 0.
\]

We conclude that \( \mathcal{L}[u(t - c)] = \frac{e^{-cs}}{s} \). \( \square \)
The Laplace Transform of discontinuous functions.

Example
Compute $\mathcal{L}[3u(t - 2)]$.

Solution: $\mathcal{L}[3u(t - 2)] = 3\mathcal{L}[u(t - 2)] = 3 \frac{e^{-2s}}{s}$.

We conclude: $\mathcal{L}[3u(t - 2)] = \frac{3e^{-2s}}{s}$.

Example
Compute $\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s}\right]$.

Solution: Since $\mathcal{L}[u(t - c)] = \frac{e^{-cs}}{s}$, for $c = 3$ we get

$\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right] = u(t - 3)$. Therefore, $\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s}\right] = 2u(t - 3)$.

The Laplace Transform of step functions (Sect. 4.3).

- Overview and notation.
- The definition of a step function.
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- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.
Properties of the Laplace Transform.

Theorem (Translations)

If \( F(s) = \mathcal{L}[f(t)] \) exists for \( s > a \geq 0 \) and \( c \geq 0 \), then holds

\[
\mathcal{L}[u(t - c)f(t - c)] = e^{-cs} F(s), \quad s > a.
\]

Furthermore,

\[
\mathcal{L}[e^{ct}f(t)] = F(s - c), \quad s > a + c.
\]

Remark:

\begin{itemize}
  \item \( \mathcal{L}\left[\text{translation } (uf)\right] = (\exp) \left( \mathcal{L}[f] \right) \).
  \item \( \mathcal{L}\left[\text{(exp)}(f)\right] = \text{translation}(\mathcal{L}[f]) \).
\end{itemize}

Equivalent notation:

\begin{itemize}
  \item \( \mathcal{L}[u(t - c)f(t - c)] = e^{-cs} \mathcal{L}[f(t)], \)
  \item \( \mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f](s - c). \)
\end{itemize}

Properties of the Laplace Transform.

Example

Compute \( \mathcal{L}[u(t - 2) \sin(a(t - 2))] \).

Solution: \( \mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}, \mathcal{L}[u(t - c)f(t - c)] = e^{-cs} \mathcal{L}[f(t)]. \)

\[
\mathcal{L}[u(t - 2) \sin(a(t - 2))] = e^{-2s} \mathcal{L}[\sin(at)] = e^{-2s} \frac{a}{s^2 + a^2}.
\]

We conclude: \( \mathcal{L}[u(t - 2) \sin(a(t - 2))] = e^{-2s} \frac{a}{s^2 + a^2}. \) \( \triangleleft \)

Example

Compute \( \mathcal{L}[e^{3t} \sin(at)] \).

Solution: Recall: \( \mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f](s - c), \mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2} \).

We conclude: \( \mathcal{L}[e^{3t} \sin(at)] = \frac{a}{(s - 3)^2 + a^2}, \text{ with } s > 3. \) \( \triangleleft \)
Properties of the Laplace Transform.

Example

Find the Laplace transform of \( f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geq 1. \end{cases} \)

Solution: Using step function notation,

\[
f(t) = u(t - 1)(t^2 - 2t + 2).
\]

Completing the square we obtain,

\[
t^2 - 2t + 2 = (t^2 - 2t + 1) - 1 + 2 = (t - 1)^2 + 1.
\]

This is a parabola \( t^2 \) translated to the right by 1 and up by one. Because of the step function, this is a discontinuous function at \( t = 1 \).

Properties of the Laplace Transform.

Example

Find the Laplace transform of \( f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geq 1. \end{cases} \)

Solution: Recall: \( f(t) = u(t - 1) [(t - 1)^2 + 1] \).

This is equivalent to

\[
f(t) = u(t - 1) (t - 1)^2 + u(t - 1).
\]

Since \( \mathcal{L}[t^2] = 2/s^3 \), and \( \mathcal{L}[u(t - c)g(t - c)] = e^{-cs} \mathcal{L}[g(t)] \), then

\[
\mathcal{L}[f(t)] = \mathcal{L}[u(t - 1) (t - 1)^2] + \mathcal{L}[u(t - 1)] = e^{-s} \frac{2}{s^3} + e^{-s} \frac{1}{s}.
\]

We conclude: \( \mathcal{L}[f(t)] = \frac{e^{-s}}{s^3} (2 + s^2) \). ❆
Properties of the Laplace Transform.

Remark: The inverse of the formulas in the Theorem above are:

\[ \mathcal{L}^{-1}[e^{-cs} F(s)] = u(t - c) f(t - c), \]
\[ \mathcal{L}^{-1}[F(s - c)] = e^{ct} f(t). \]

Example
Find \( \mathcal{L}^{-1}\left[ \frac{e^{-4s}}{s^2 + 9} \right] \).

Solution: \( \mathcal{L}^{-1}\left[ \frac{e^{-4s}}{s^2 + 9} \right] = \frac{1}{3} \mathcal{L}^{-1}\left[ \frac{3}{s^2 + 9} \right] \).

Recall: \( \mathcal{L}^{-1}\left[ \frac{a}{s^2 + a^2} \right] = \sin(at) \). Then, we conclude that
\[ \mathcal{L}^{-1}\left[ \frac{e^{-4s}}{s^2 + 9} \right] = \frac{1}{3} u(t - 4) \sin(3(t - 4)). \]

Properties of the Laplace Transform.

Example
Find \( \mathcal{L}^{-1}\left[ \frac{(s - 2)}{(s - 2)^2 + 9} \right] \).

Solution: \( \mathcal{L}^{-1}\left[ \frac{s}{s^2 + 9} \right] = \cos(at), \mathcal{L}^{-1}[F(s - c)] = e^{ct} f(t) \).

We conclude: \( \mathcal{L}^{-1}\left[ \frac{(s - 2)}{(s - 2)^2 + 9} \right] = e^{2t} \cos(3t). \)

Example
Find \( \mathcal{L}^{-1}\left[ \frac{2e^{-3s}}{s^2 - 4} \right] \).

Solution: Recall: \( \mathcal{L}^{-1}\left[ \frac{a}{s^2 - a^2} \right] = \sinh(at) \) and \( \mathcal{L}^{-1}[e^{-cs} F(s)] = u(t - c) f(t - c). \)
Properties of the Laplace Transform.

Example
Find $\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right]$.

Solution: Recall:

$$\mathcal{L}^{-1}\left[\frac{a}{s^2 - a^2}\right] = \sinh(at), \quad \mathcal{L}^{-1}[e^{-cs} F(s)] = u(t - c) f(t - c).$$

$$\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right] = \mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right].$$

We conclude: $\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right] = u(t - 3) \sinh(2(t - 3)).$

Properties of the Laplace Transform.

Example
Find $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right]$.

Solution: Find the roots of the denominator:

$$s_{\pm} = \frac{1}{2} \left[-1 \pm \sqrt{1 + 8}\right] \Rightarrow \begin{cases} s_+ = 1, \\ s_- = -2. \end{cases}$$

Therefore, $s^2 + s - 2 = (s - 1)(s + 2)$.

Use partial fractions to simplify the rational function:

$$\frac{1}{s^2 + s - 2} = \frac{1}{(s - 1)(s + 2)} = \frac{a}{s - 1} + \frac{b}{s + 2},$$

$$\frac{1}{s^2 + s - 2} = a(s + 2) + b(s - 1) = \frac{(a + b) s + (2a - b)}{(s - 1)(s + 2)}. $$
Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right]$.

Solution: Recall: 

\[ \frac{1}{s^2 + s - 2} = \frac{(a + b) s + (2a - b)}{(s - 1)(s + 2)} \]

\[ a + b = 0, \quad 2a - b = 1, \quad \Rightarrow \quad a = \frac{1}{3}, \quad b = -\frac{1}{3}. \]

\[ \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right] = \frac{1}{3} \mathcal{L}^{-1}\left[e^{-2s} \frac{1}{s - 1}\right] - \frac{1}{3} \mathcal{L}^{-1}\left[e^{-2s} \frac{1}{s + 2}\right]. \]

Recall: $\mathcal{L}^{-1}\left[\frac{1}{s - a}\right] = e^{at}$, $\mathcal{L}^{-1}\left[e^{-cs} F(s)\right] = u(t - c) f(t - c)$,

\[ \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right] = \frac{1}{3} u(t - 2) e^{(t-2)} - \frac{1}{3} u(t - 2) e^{-2(t-2)}. \]

Hence: $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right] = \frac{1}{3} u(t - 2) \left[e^{(t-2)} - e^{-2(t-2)}\right]. \quad \triangle$