

The Laplace Transform (Sect. 4.1).
The definition of the Laplace Transform.
Review: Improper integrals.
Examples of Laplace Transforms.
A table of Laplace Transforms.
Properties of the Laplace Transform.

Laplace Transform and differential equations.

Examples of Laplace Transforms.

Example

Compute $\mathcal{L}[1]$.

Solution: We have to find the Laplace Transform of f(t) = 1. Following the definition we obtain,

$$\mathcal{L}[1] = \int_0^\infty e^{-st} \, 1 \, dt = \int_0^\infty e^{-st} \, dt$$

But $\int_0^\infty e^{-at} dt = \frac{1}{a}$ for a > 0, and diverges for $a \le 0$.

Therefore $\mathcal{L}[1] = \frac{1}{s}$, for s > 0, and $\mathcal{L}[1]$ does not exists for $s \leq 0$. In other words, $F(s) = \mathcal{L}[1]$ is the function $F : D_F \to \mathbb{R}$ given by

$$f(t)=1, \qquad F(s)=rac{1}{s}, \qquad D_F=(0,\infty).$$

Examples of Laplace Transforms.

Example

Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.

Solution: Following the definition of Laplace Transform,

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt.$$

We have seen that the improper integral is given by

$$\int_0^\infty e^{-(s-a)t}\,dt=rac{1}{(s-a)}\quad ext{for}\quad (s-a)>0.$$

We conclude that $\mathcal{L}[e^{at}] = rac{1}{s-a}$ for s>a. In other words,

$$f(t) = e^{at}, \qquad F(s) = rac{1}{(s-a)}, \qquad s > a.$$

Examples of Laplace Transforms.
Example
Compute
$$\mathcal{L}[\sin(at)]$$
, where $a \in \mathbb{R}$.
Solution: In this case we need to compute

$$\mathcal{L}[\sin(at)] = \lim_{N \to \infty} \int_0^N e^{-st} \sin(at) dt.$$
Integrating by parts twice it is not difficult to obtain:

$$\int_0^N e^{-st} \sin(at) dt =$$

$$-\frac{1}{s} \left[e^{-st} \sin(at) \right] \Big|_0^N - \frac{a}{s^2} \left[e^{-st} \cos(at) \right] \Big|_0^N - \frac{a^2}{s^2} \int_0^N e^{-st} \sin(at) dt.$$
This identity implies
 $\left(1 + \frac{a^2}{s^2} \right) \int_0^N e^{-st} \sin(at) dt = -\frac{1}{s} \left[e^{-st} \sin(at) \right] \Big|_0^N - \frac{a}{s^2} \left[e^{-st} \cos(at) \right] \Big|_0^N.$

Examples of Laplace Transforms.

Example

Compute $\mathcal{L}[\sin(at)]$, where $a \in \mathbb{R}$.

Solution: Recall the identity:

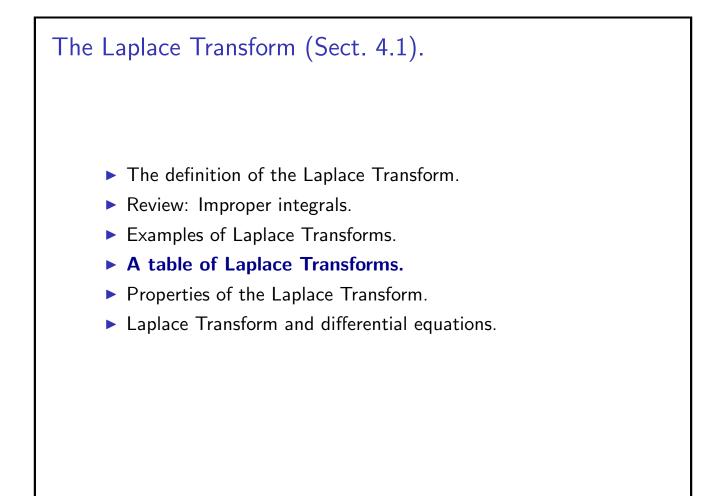
$$\left(1+\frac{a^2}{s^2}\right)\int_0^N e^{-st}\sin(at)\,dt = -\frac{1}{s}\left[e^{-st}\sin(at)\right]\Big|_0^N - \frac{a}{s^2}\left[e^{-st}\cos(at)\right]\Big|_0^N.$$

Hence, it is not difficult to see that

$$\left(\frac{s^2+a^2}{s^2}\right)\int_0^\infty e^{-st}\sin(at)\,dt=\frac{a}{s^2},\qquad s>0,$$

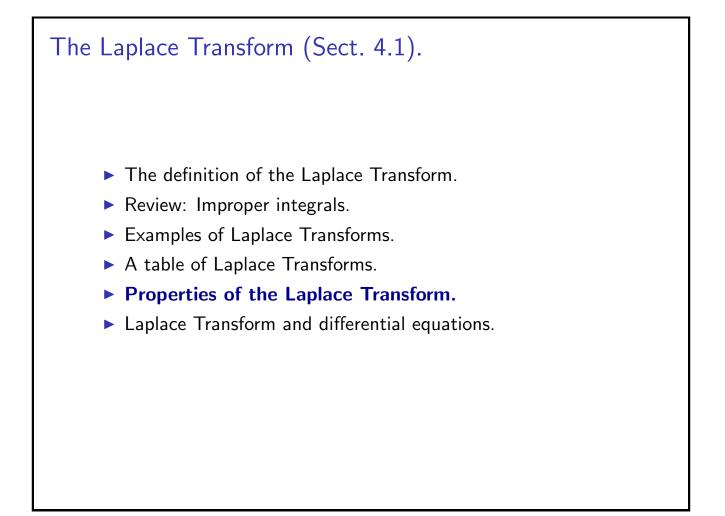
which is equivalent to

$$\mathcal{L}[\sin(at)] = rac{a}{s^2 + a^2}, \qquad s > 0.$$



A table of Laplace Transforms.

$$\begin{split} f(t) &= 1 & F(s) = \frac{1}{s} & s > 0, \\ f(t) &= e^{at} & F(s) = \frac{1}{s-a} & s > \max\{a,0\}, \\ f(t) &= t^n & F(s) = \frac{n!}{s^{(n+1)}} & s > 0, \\ f(t) &= \sin(at) & F(s) = \frac{a}{s^2 + a^2} & s > 0, \\ f(t) &= \cos(at) & F(s) = \frac{s}{s^2 + a^2} & s > 0, \\ f(t) &= \cosh(at) & F(s) = \frac{a}{s^2 - a^2} & s > |a|, \\ f(t) &= \cosh(at) & F(s) = \frac{s}{s^2 - a^2} & s > |a|, \\ f(t) &= t^n e^{at} & F(s) = \frac{n!}{(s-a)^{(n+1)}} & s > \max\{a,0\}, \\ f(t) &= e^{at} \sin(bt) & F(s) = \frac{b}{(s-a)^2 + b^2} & s > \max\{a,0\}. \end{split}$$



Properties of the Laplace Transform.

Theorem (Sufficient conditions)

If the function $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous and there exist positive constants k and a such that

 $|f(t)|\leqslant k\,e^{at},$

then the Laplace Transform of f exists for all s > a.

Theorem (Linear combination) If the $\mathcal{L}[f]$ and $\mathcal{L}[g]$ are well-defined and a, b are constants, then

$$\mathcal{L}[af + bg] = a \mathcal{L}[f] + b \mathcal{L}[g].$$

Proof: Integration is a linear operation:

$$\int [af(t) + bg(t)] dt = a \int f(t) dt + b \int g(t) dt$$

Properties of the Laplace Transform.

Theorem (Derivatives) If the $\mathcal{L}[f]$ and $\mathcal{L}[f']$ are well-defined, then holds,

$$\mathcal{L}[f'] = s \, \mathcal{L}[f] - f(0). \tag{1}$$

Furthermore, if $\mathcal{L}[f'']$ is well-defined, then it also holds

$$\mathcal{L}[f''] = s^2 \mathcal{L}[f] - s f(0) - f'(0).$$
(2)

Proof of Eq (??): Use Eq. (??) twice:

$$\mathcal{L}[f''] = \mathcal{L}[(f')'] = s\mathcal{L}[(f')] - f'(0) = s(s\mathcal{L}[f] - f(0)) - f'(0),$$

that is,

$$\mathcal{L}[f''] = s^2 \mathcal{L}[f] - s f(0) - f'(0).$$

Properties of the Laplace Transform.

Proof of Eq (??): Recall the definition of the Laplace Transform,

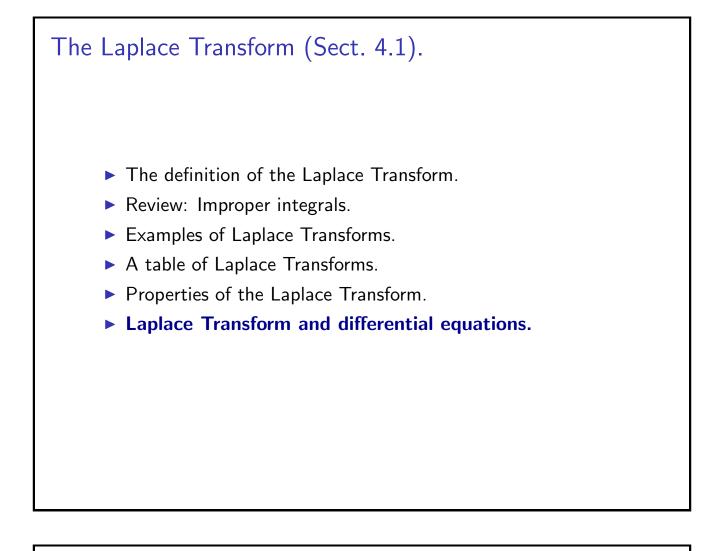
$$\mathcal{L}[f'] = \int_0^\infty e^{-st} f'(t) \, dt = \lim_{n \to \infty} \int_0^n e^{-st} f'(t) \, dt$$

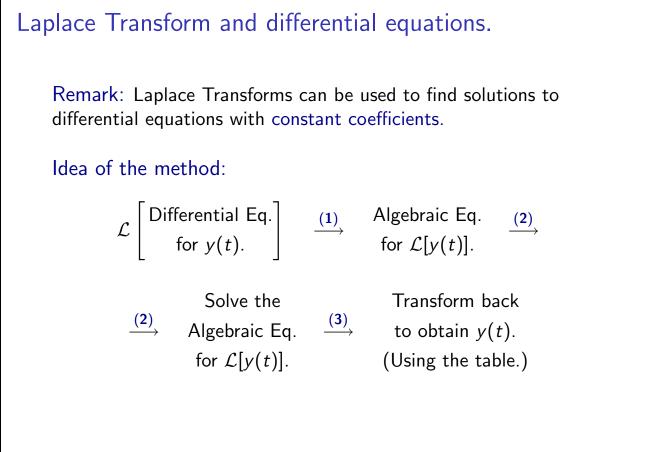
Integrating by parts,

$$\lim_{n \to \infty} \int_0^n e^{-st} f'(t) dt = \lim_{n \to \infty} \left[\left(e^{-st} f(t) \right) \Big|_0^n - \int_0^n (-s) e^{-st} f(t) dt \right]$$
$$\mathcal{L}[f'] = \lim_{n \to \infty} \left[e^{-sn} f(n) - f(0) \right] + s \int_0^\infty e^{-st} f(t) dt = -f(0) + s \mathcal{L}[f],$$

where we used that $\lim_{n\to\infty} e^{-sn}f(n) = 0$ for s big enough, and we also used that $\mathcal{L}[f]$ is well-defined.

We then conclude that $\mathcal{L}[f'] = s \mathcal{L}[f] - f(0)$.





Laplace Transform and differential equations. Example Use the Laplace transform to find the solution y(t) to the IVP y' + 2y = 0, y(0) = 3. Solution: We know the solution: $y(t) = 3e^{-2t}$. (1): Compute the Laplace transform of the differential equation, $\mathcal{L}[y' + 2y] = \mathcal{L}[0] \Rightarrow \mathcal{L}[y' + 2y] = 0$. Find an algebraic equation for $\mathcal{L}[y]$. Recall linearity: $\mathcal{L}[y'] + 2\mathcal{L}[y] = 0$. Also recall the property: $\mathcal{L}[y'] = s \mathcal{L}[y] - y(0)$, that is, $\left[s \mathcal{L}[y] - y(0)\right] + 2\mathcal{L}[y] = 0 \Rightarrow (s + 2)\mathcal{L}[y] = y(0)$.

Laplace Transform and differential equations.

Example

Use the Laplace transform to find the solution y(t) to the IVP

$$y' + 2y = 0,$$
 $y(0) = 3.$

Solution: Recall: $(s+2)\mathcal{L}[y] = y(0)$.

(2): Solve the algebraic equation for $\mathcal{L}[y]$.

$$\mathcal{L}[y] = \frac{y(0)}{s+2}, \quad y(0) = 3, \quad \Rightarrow \quad \mathcal{L}[y] = \frac{3}{s+2}$$

(3): Transform back to y(t). From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \Rightarrow \frac{3}{s+2} = 3\mathcal{L}[e^{-2t}] \Rightarrow \frac{3}{s+2} = \mathcal{L}[3e^{-2t}].$$

Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}] \Rightarrow y(t) = 3e^{-2t}.$