The Laplace Transform (Sect. 4.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
- Properties of the Laplace Transform.
- Laplace Transform and differential equations.
The definition of the Laplace Transform.

**Definition**
The function $F : D_F \rightarrow \mathbb{R}$ is the *Laplace transform* of a function $f : [0, \infty) \rightarrow \mathbb{R}$ iff for all $s \in D_F$ holds,

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt,$$

where $D_F \subset \mathbb{R}$ is the set where the integral converges.

**Remark:** The domain $D_F$ of $F$ depends on the function $f$.

**Notation:** We often denote: $F(s) = \mathcal{L}[f(t)]$.

- This notation $\mathcal{L}[]$ emphasizes that the Laplace transform defines a map from a set of functions into a set of functions.
- Functions are denoted as $t \mapsto f(t)$.
- The Laplace transform is also a function: $f \mapsto \mathcal{L}[f]$.

The Laplace Transform (Sect. 4.1).

- The definition of the Laplace Transform.
- **Review: Improper integrals.**
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
- Properties of the Laplace Transform.
- Laplace Transform and differential equations.
Review: Improper integrals.

Recall: Improper integral are defined as a limit.

\[
\int_{t_0}^{\infty} g(t) \, dt = \lim_{N \to \infty} \int_{t_0}^{N} g(t) \, dt.
\]

- The integral **converges** iff the limit exists.
- The integral **diverges** iff the limit does not exist.

Example

Compute the improper integral \( \int_{0}^{\infty} e^{-at} \, dt \), with \( a > 0 \).

Solution:

\[
\int_{0}^{\infty} e^{-at} \, dt = \lim_{N \to \infty} \int_{0}^{N} e^{-at} \, dt = \lim_{N \to \infty} \frac{-1}{a} \left( e^{-aN} - 1 \right).
\]

Since \( \lim_{N \to \infty} e^{-aN} = 0 \) for \( a > 0 \), we conclude \( \int_{0}^{\infty} e^{-at} \, dt = \frac{1}{a}. \)
Examples of Laplace Transforms.

Example
Compute $\mathcal{L}[1]$.

Solution: We have to find the Laplace Transform of $f(t) = 1$. Following the definition we obtain,

$$\mathcal{L}[1] = \int_0^\infty e^{-st} \, dt = \int_0^\infty e^{-st} \, dt$$

But $\int_0^\infty e^{-at} \, dt = \frac{1}{a}$ for $a > 0$, and diverges for $a \leq 0$.

Therefore $\mathcal{L}[1] = \frac{1}{s}$, for $s > 0$, and $\mathcal{L}[1]$ does not exists for $s \leq 0$.

In other words, $F(s) = \mathcal{L}[1]$ is the function $F : D_F \rightarrow \mathbb{R}$ given by

$$f(t) = 1, \quad F(s) = \frac{1}{s}, \quad D_F = (0, \infty).$$

◁

Examples of Laplace Transforms.

Example
Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.

Solution: Following the definition of Laplace Transform,

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{-(s-a)t} \, dt.$$ 

We have seen that the improper integral is given by

$$\int_0^\infty e^{-(s-a)t} \, dt = \frac{1}{(s-a)} \quad \text{for} \quad (s-a) > 0.$$ 

We conclude that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s > a$. In other words,

$$f(t) = e^{at}, \quad F(s) = \frac{1}{s-a}, \quad s > a.$$ 

◁
Examples of Laplace Transforms.

Example
Compute $L[\sin(at)]$, where $a \in \mathbb{R}$.

Solution: In this case we need to compute

$$L[\sin(at)] = \lim_{N \to \infty} \int_0^N e^{-st} \sin(at) \, dt.$$ 

Integrating by parts twice it is not difficult to obtain:

$$\int_0^N e^{-st} \sin(at) \, dt =$$

$$-\frac{1}{s} [e^{-st} \sin(at)]_0^N - \frac{a}{s^2} [e^{-st} \cos(at)]_0^N - \frac{a^2}{s^2} \int_0^N e^{-st} \sin(at) \, dt.$$ 

This identity implies

$$\left(1 + \frac{a^2}{s^2}\right) \int_0^N e^{-st} \sin(at) \, dt = -\frac{1}{s} [e^{-st} \sin(at)]_0^N - \frac{a}{s^2} [e^{-st} \cos(at)]_0^N.$$ 

Hence, it is not difficult to see that

$$\left(\frac{s^2 + a^2}{s^2}\right) \int_0^\infty e^{-st} \sin(at) \, dt = \frac{a}{s^2}, \quad s > 0,$$

which is equivalent to

$$L[\sin(at)] = \frac{a}{s^2 + a^2}, \quad s > 0. \quad \triangledown$$
The Laplace Transform (Sect. 4.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- **A table of Laplace Transforms.**
- Properties of the Laplace Transform.
- Laplace Transform and differential equations.

---

**A table of Laplace Transforms.**

\[
\begin{align*}
    f(t) &= 1 & F(s) &= \frac{1}{s} & s > 0, \\
    f(t) &= e^{at} & F(s) &= \frac{1}{s - a} & s > \max\{a, 0\}, \\
    f(t) &= t^n & F(s) &= \frac{n!}{s^{n+1}} & s > 0, \\
    f(t) &= \sin(at) & F(s) &= \frac{a}{s^2 + a^2} & s > 0, \\
    f(t) &= \cos(at) & F(s) &= \frac{s}{s^2 + a^2} & s > 0, \\
    f(t) &= \sinh(at) & F(s) &= \frac{a}{s^2 - a^2} & s > |a|, \\
    f(t) &= \cosh(at) & F(s) &= \frac{s}{s^2 - a^2} & s > |a|, \\
    f(t) &= t^n e^{at} & F(s) &= \frac{n!}{(s - a)^{(n+1)}} & s > \max\{a, 0\}, \\
    f(t) &= e^{at} \sin(bt) & F(s) &= \frac{b}{(s - a)^2 + b^2} & s > \max\{a, 0\}.
\end{align*}
\]
The Laplace Transform (Sect. 4.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
- Properties of the Laplace Transform.
- Laplace Transform and differential equations.

Properties of the Laplace Transform.

**Theorem (Sufficient conditions)**

*If the function \( f : [0, \infty) \rightarrow \mathbb{R} \) is piecewise continuous and there exist positive constants \( k \) and \( a \) such that

\[
|f(t)| \leq k e^{at},
\]

then the Laplace Transform of \( f \) exists for all \( s > a \).*

**Theorem (Linear combination)**

*If the \( \mathcal{L}[f] \) and \( \mathcal{L}[g] \) are well-defined and \( a, b \) are constants, then

\[
\mathcal{L}[af + bg] = a \mathcal{L}[f] + b \mathcal{L}[g].
\]

**Proof:** Integration is a linear operation:

\[
\int [af(t) + bg(t)] \, dt = a \int f(t) \, dt + b \int g(t) \, dt.
\]
Properties of the Laplace Transform.

Theorem (Derivatives)
If the \( \mathcal{L}[f] \) and \( \mathcal{L}[f'] \) are well-defined, then holds,

\[
\mathcal{L}[f'] = s \mathcal{L}[f] - f(0). \tag{1}
\]

Furthermore, if \( \mathcal{L}[f''] \) is well-defined, then it also holds

\[
\mathcal{L}[f''] = s^2 \mathcal{L}[f] - sf(0) - f'(0). \tag{2}
\]

Proof of Eq (2): Use Eq. (1) twice:

\[
\mathcal{L}[f''] = \mathcal{L}[(f')'] = s\mathcal{L}[(f')] - f'(0) = s(s\mathcal{L}[f] - f(0)) - f'(0),
\]

that is,

\[
\mathcal{L}[f''] = s^2 \mathcal{L}[f] - sf(0) - f'(0).
\]

Proof of Eq (1): Recall the definition of the Laplace Transform,

\[
\mathcal{L}[f'] = \int_{0}^{\infty} e^{-st} f'(t) \, dt = \lim_{n \to \infty} \int_{0}^{n} e^{-st} f'(t) \, dt
\]

Integrating by parts,

\[
\lim_{n \to \infty} \int_{0}^{n} e^{-st} f'(t) \, dt = \lim_{n \to \infty} \left[ (e^{-st} f(t)) \bigg|_{0}^{n} - \int_{0}^{n} (-s) e^{-st} f(t) \, dt \right]
\]

\[
\mathcal{L}[f'] = \lim_{n \to \infty} \left[ e^{-sn} f(n) - f(0) \right] + s \int_{0}^{\infty} e^{-st} f(t) \, dt = -f(0) + s \mathcal{L}[f],
\]

where we used that \( \lim_{n \to \infty} e^{-sn} f(n) = 0 \) for \( s \) big enough, and we also used that \( \mathcal{L}[f] \) is well-defined.

We then conclude that \( \mathcal{L}[f'] = s \mathcal{L}[f] - f(0) \).
Laplace Transform and differential equations.

Remark: Laplace Transforms can be used to find solutions to differential equations with constant coefficients.

Idea of the method:

\[ \mathcal{L} \left[ \text{Differential Eq.}\right. \]

\[ \text{for } y(t) \] \[ \xrightarrow{(1)} \]

\[ \text{Algebraic Eq.}\]

\[ \text{for } \mathcal{L}[y(t)] \] \[ \xrightarrow{(2)} \]

\[ \text{Solve the } \]

\[ \text{Algebraic Eq.}\]

\[ \text{for } \mathcal{L}[y(t)] \] \[ \xrightarrow{(2)} \]

\[ \text{Transform back}\]

\[ \text{to obtain } y(t) \]

\[ \xrightarrow{(3)} \]

(Using the table.)
Laplace Transform and differential equations.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \quad y(0) = 3.$$ 

Solution: We know the solution: $y(t) = 3e^{-2t}$.

(1): Compute the Laplace transform of the differential equation,

$$\mathcal{L}[y' + 2y] = \mathcal{L}[0] \quad \Rightarrow \quad \mathcal{L}[y' + 2y] = 0.$$ 

Find an algebraic equation for $\mathcal{L}[y]$. Recall linearity:

$$\mathcal{L}[y'] + 2 \mathcal{L}[y] = 0.$$ 

Also recall the property: $\mathcal{L}[y'] = s \mathcal{L}[y] - y(0)$, that is,

$$\left[ s \mathcal{L}[y] - y(0) \right] + 2 \mathcal{L}[y] = 0 \quad \Rightarrow \quad (s + 2) \mathcal{L}[y] = y(0).$$

Laplace Transform and differential equations.

Example
Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \quad y(0) = 3.$$ 

Solution: Recall: $(s + 2) \mathcal{L}[y] = y(0)$.

(2): Solve the algebraic equation for $\mathcal{L}[y]$.

$$\mathcal{L}[y] = \frac{y(0)}{s + 2}, \quad y(0) = 3, \quad \Rightarrow \quad \mathcal{L}[y] = \frac{3}{s + 2}.$$ 

(3): Transform back to $y(t)$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s - a} \Rightarrow \frac{3}{s + 2} = 3 \mathcal{L}[e^{-2t}] \Rightarrow \frac{3}{s + 2} = \mathcal{L}[3e^{-2t}].$$ 

Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}] \quad \Rightarrow \quad y(t) = 3e^{-2t}$. ◀