Equations with regular-singular points (Sect. 3.3).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- Method to find solutions.
- Example: Method to find solutions.

Recall:
The point $x_0 \in \mathbb{R}$ is a singular point of the equation

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$

iff holds that $P(x_0) = 0$.

Equations with regular-singular points.

Definition
A singular point $x_0 \in \mathbb{R}$ of the equation

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$

is called a *regular-singular* point iff the following limits are finite,

$$\lim_{x\to x_0} \frac{(x-x_0)\,Q(x)}{P(x)}, \qquad \lim_{x\to x_0} \frac{(x-x_0)^2\,R(x)}{P(x)},$$

and both functions

$$\frac{(x-x_0)\,Q(x)}{P(x)}, \qquad \frac{(x-x_0)^2\,R(x)}{P(x)},$$

admit convergent Taylor series expansions around x_0.

Equations with regular-singular points.

Remark:

- If x_0 is a regular-singular point of

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$

  and $P(x) \simeq (x - x_0)^n$ near x_0, then near x_0 holds

$$Q(x) \simeq (x - x_0)^{n-1}, \quad R(x) \simeq (x - x_0)^{n-2}.$$

- The main example is an Euler equation, case $n = 2$,

$$(x - x_0)^2\,y'' + p_0(x - x_0)\,y' + q_0\,y = 0.$$

Equations with regular-singular points.

Example

Show that the singular point of every Euler equation is a regular-singular point.

Solution: Consider the general Euler equation

$$(x - x_0)^2\,y'' + p_0(x - x_0)\,y' + q_0\,y = 0,$$

where p_0, q_0, x_0, are real constants. This is an equation $Py'' + Qy' + Ry = 0$ with

$$P(x) = (x - x_0)^2, \qquad Q(x) = p_0(x - x_0), \qquad R(x) = q_0.$$

Therefore, we obtain,

$$\lim_{x\to x_0} \frac{(x - x_0)\,Q(x)}{P(x)} = p_0, \qquad \lim_{x\to x_0} \frac{(x - x_0)^2\,R(x)}{P(x)} = q_0.$$

We conclude that x_0 is a regular-singular point. $\triangleleft$

Equations with regular-singular points.

Remark: Every equation $Py'' + Qy' + Ry = 0$ with a regular-singular point at x_0 is close to an Euler equation.

Proof:
For $x \neq x_0$ divide the equation by $P(x)$,

$$y'' + \frac{Q(x)}{P(x)}\,y' + \frac{R(x)}{P(x)}\,y = 0,$$

and multiply it by $(x - x_0)^2$,

$$(x - x_0)^2\,y'' + (x - x_0)\Big[\frac{(x - x_0)Q(x)}{P(x)}\Big]\,y' + \Big[\frac{(x - x_0)^2 R(x)}{P(x)}\Big]\,y = 0.$$

The factors between [ ] approach constants, say p_0, q_0, as $x \to x_0$,

$$(x - x_0)^2\,y'' + (x - x_0)p_0\,y' + q_0\,y = 0.$$

$\square$

Equations with regular-singular points (Sect. 3.3).

- Equations with regular-singular points.
- **Examples: Equations with regular-singular points.**
- Method to find solutions.
- Example: Method to find solutions.

Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$(1 - x^2)\, y'' - 2x\, y' + \alpha(\alpha + 1)\, y = 0,$$

where α is a real constant.

Solution: Find the singular points of this equation,

$$0 = P(x) = (1 - x^2) = (1 - x)(1 + x) \quad \Rightarrow \quad \begin{cases} x_0 = 1, \\ x_1 = -1. \end{cases}$$

Case $x_0 = 1$: We then have

$$\frac{(x-1)\, Q(x)}{P(x)} = \frac{(x-1)(-2x)}{(1-x)(1+x)} = \frac{2x}{1+x},$$

$$\frac{(x-1)^2\, R(x)}{P(x)} = \frac{(x-1)^2 \big[\alpha(\alpha+1)\big]}{(1-x)(1+x)} = \frac{(x-1)\big[\alpha(\alpha+1)\big]}{1+x};$$

both functions above have Taylor series around $x_0 = 1$.

Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$(1 - x^2)\, y'' - 2x\, y' + \alpha(\alpha + 1)\, y = 0,$$

where α is a real constant.

Solution: Recall:

$$\frac{(x-1)\, Q(x)}{P(x)} = \frac{2x}{1+x}, \qquad \frac{(x-1)^2\, R(x)}{P(x)} = \frac{(x-1)\big[\alpha(\alpha+1)\big]}{1+x}.$$

Furthermore, the following limits are finite,

$$\lim_{x \to 1} \frac{(x-1)\, Q(x)}{P(x)} = 1, \qquad \lim_{x \to 1} \frac{(x-1)^2\, R(x)}{P(x)} = 0.$$

We conclude that $x_0 = 1$ is a regular-singular point.

Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$(1-x^2)\,y'' - 2x\,y' + \alpha(\alpha+1)\,y = 0,$$

where α is a real constant.

Solution:

Case $x_1 = -1$:

$$\frac{(x+1)\,Q(x)}{P(x)} = \frac{(x+1)(-2x)}{(1-x)(1+x)} = -\frac{2x}{1-x},$$

$$\frac{(x+1)^2\,R(x)}{P(x)} = \frac{(x+1)^2\big[\alpha(\alpha+1)\big]}{(1-x)(1+x)} = \frac{(x+1)\big[\alpha(\alpha+1)\big]}{1-x}.$$

Both functions above have Taylor series $x_1 = -1$.

Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$(1-x^2)\,y'' - 2x\,y' + \alpha(\alpha+1)\,y = 0,$$

where α is a real constant.

Solution: Recall:

$$\frac{(x+1)\,Q(x)}{P(x)} = -\frac{2x}{1-x}, \qquad \frac{(x+1)^2\,R(x)}{P(x)} = \frac{(x+1)\big[\alpha(\alpha+1)\big]}{1-x}.$$

Furthermore, the following limits are finite,

$$\lim_{x\to -1} \frac{(x+1)\,Q(x)}{P(x)} = 1, \qquad \lim_{x\to -1} \frac{(x+1)^2\,R(x)}{P(x)} = 0.$$

Therefore, the point $x_1 = -1$ is a regular-singular point. $\triangleleft$

Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$(x+2)^2(x-1)\,y'' + 3(x-1)\,y' + 2\,y = 0.$$

Solution: Find the singular points: $x_0 = -2$ and $x_1 = 1$.
Case $x_0 = -2$:

$$\lim_{x\to -2} \frac{(x+2)Q(x)}{P(x)} = \lim_{x\to -2} \frac{(x+2)3(x-1)}{(x+2)^2(x-1)} = \lim_{x\to -2} \frac{3}{(x+2)} = \pm\infty.$$

So $x_0 = -2$ is not a regular-singular point. Case $x_1 = 1$:

$$\frac{(x-1)\,Q(x)}{P(x)} = \frac{(x-1)\big[3(x-1)\big]}{(x+2)(x-1)} = -\frac{3(x-1)}{(x+2)^2},$$

$$\frac{(x-1)^2\,R(x)}{P(x)} = \frac{2(x-1)^2}{(x+2)^2(x-1)} = \frac{2(x-1)}{(x+2)^2};$$

Both functions have Taylor series around $x_1 = 1$.

Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$(x+2)^2(x-1)\,y'' + 3(x-1)\,y' + 2\,y = 0.$$

Solution: Recall:

$$\frac{(x-1)\,Q(x)}{P(x)} = -\frac{3(x-1)}{(x+2)^2}, \qquad \frac{(x-1)^2\,R(x)}{P(x)} = \frac{2(x-1)}{(x+2)^2}.$$

Furthermore, the following limits are finite,

$$\lim_{x\to 1} \frac{(x-1)\,Q(x)}{P(x)} = 0; \qquad \lim_{x\to 1} \frac{(x-1)^2\,R(x)}{P(x)} = 0.$$

Therefore, the point $x_1 = -1$ is a regular-singular point. $\triangleleft$

Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$x\,y'' - x\ln(|x|)\,y' + 3x\,y = 0.$$

Solution: The singular point is $x_0 = 0$. We compute the limit

$$\lim_{x\to 0} \frac{xQ(x)}{P(x)} = \lim_{x\to 0} \frac{x\big[-x\ln(|x|)\big]}{x} = \lim_{x\to 0} -\frac{\ln(|x|)}{\frac{1}{x}}.$$

Use L'Hôpital's rule: $\displaystyle\lim_{x\to 0} \frac{xQ(x)}{P(x)} = \lim_{x\to 0} \frac{-\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} x = 0.$

The other limit is: $\displaystyle\lim_{x\to 0} \frac{x^2R(x)}{P(x)} = \lim_{x\to 0} \frac{x^2(3x)}{x} = \lim_{x\to 0} 3x^2 = 0.$

Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$x\,y'' - x\ln(|x|)\,y' + 3x\,y = 0.$$

Solution: Recall: $\displaystyle\lim_{x\to 0} \frac{xQ(x)}{P(x)} = 0$ and $\displaystyle\lim_{x\to 0} \frac{x^2R(x)}{P(x)} = 0.$

However, at the point $x_0 = 0$ the function xQ/P does not have a power series expansion around zero, since

$$\frac{xQ(x)}{P(x)} = -x\ln(|x|),$$

and the log function does not have a Taylor series at $x_0 = 0$.

We conclude that $x_0 = 0$ is not a regular-singular point. $\triangleleft$

Equations with regular-singular points (Sect. 3.3).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- **Method to find solutions.**
- Example: Method to find solutions.

Method to find solutions.

Recall: If x_0 is a regular-singular point of

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0,$$

with limits $\displaystyle\lim_{x\to x_0} \frac{(x-x_0)Q(x)}{P(x)} = p_0$ and $\displaystyle\lim_{x\to x_0} \frac{(x-x_0)^2 R(x)}{P(x)} = q_0$,

then the coefficients of the differential equation above near x_0 are close to the coefficients of the Euler equation

$$(x-x_0)^2\,y'' + p_0(x-x_0)\,y' + q_0\,y = 0.$$

Idea: If the differential equation is close to an Euler equation, then the solutions of the differential equation might be close to the solutions of an Euler equation.

Recall: One solution of an Euler equation is $y(x) = (x-x_0)^r$.

Method to find solutions.

Summary: Solutions for equations with regular-singular points:

(1) Look for a solution y of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)};$$

(2) Introduce this power series expansion into the differential equation and find both a the exponent r and a recurrence relation for the coefficients a_n;

(3) First find the solutions for the constant r. Then, introduce this result for r into the recurrence relation for the coefficients a_n. Only then, solve this latter recurrence relation for the coefficients a_n.

Equations with regular-singular points (Sect. 3.3).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- Method to find solutions.
- **Example: Method to find solutions.**

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We look for a solution $y(x) = \sum_{n=0}^{\infty} a_n\, x^{(n+r)}$.

The first and second derivatives are given by

$$y' = \sum_{n=0}^{\infty} (n+r) a_n\, x^{(n+r-1)}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n\, x^{(n+r-2)}.$$

In the case $r = 0$ we had the relation

$$\sum_{n=0}^{\infty} n a_n\, x^{(n-1)} = \sum_{n=1}^{\infty} n a_n\, x^{(n-1)},$$

but for $r \neq 0$ this relation is not true.

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We now compute the term $(x+3)y$,

$$(x+3)\, y = (x+3) \sum_{n=0}^{\infty} a_n\, x^{(n+r)}$$

$$(x+3)\, y = \sum_{n=0}^{\infty} a_n\, x^{(n+r+1)} + \sum_{n=0}^{\infty} 3a_n\, x^{(n+r)}$$

$$(x+3)\, y = \sum_{n=1}^{\infty} a_{(n-1)}\, x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n\, x^{(n+r)}.$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We now compute the term $-x(x+3)\, y'$,

$$-x(x+3)\, y' = -(x^2 + 3x) \sum_{n=0}^{\infty} (n+r) a_n\, x^{(n+r-1)}$$

$$-x(x+3)\, y' = -\sum_{n=0}^{\infty} (n+r) a_n\, x^{(n+r+1)} - \sum_{n=0}^{\infty} 3(n+r) a_n\, x^{(n+r)},$$

$$-x(x+3)\, y' = -\sum_{n=1}^{\infty} (n+r-1) a_{(n-1)}\, x^{(n+r)} - \sum_{n=0}^{\infty} 3(n+r) a_n\, x^{(n+r)}.$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We compute the term $x^2\, y''$,

$$x^2\, y'' = x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n\, x^{(n+r-2)}$$

$$x^2\, y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n\, x^{(n+r)}.$$

The guiding principle to rewrite each term is to have the power function $x^{(n+r)}$ labeled in the same way on every term.

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: The differential equation is given by

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n\, x^{(n+r)} - \sum_{n=1}^{\infty}(n+r-1)a_{(n-1)}\, x^{(n+r)}$$

$$-\sum_{n=0}^{\infty}3(n+r)a_n\, x^{(n+r)} + \sum_{n=1}^{\infty}a_{(n-1)}\, x^{(n+r)} + \sum_{n=0}^{\infty}3a_n\, x^{(n+r)} = 0.$$

We split the sums into the term $n = 0$ and a sum containing the terms with $n \geqslant 1$, that is,

$$0 = \big[r(r-1) - 3r + 3\big]\, a_0 x^r +$$

$$\sum_{n=1}^{\infty}\Big[(n+r)(n+r-1)a_n - (n+r-1)a_{(n-1)} - 3(n+r)a_n + a_{(n-1)} + 3a_n\Big]\, x^{(n+r)}$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: Therefore, $\big[r(r-1) - 3r + 3\big] = 0$ and

$$\Big[(n+r)(n+r-1)a_n - (n+r-1)a_{(n-1)} - 3(n+r)a_n + a_{(n-1)} + 3a_n\Big] = 0.$$

The last expression can be rewritten as follows,

$$\Big[\big[(n+r)(n+r-1) - 3(n+r) + 3\big]\, a_n - (n+r-1-1)a_{(n-1)}\Big] = 0,$$

$$\Big[\big[(n+r)(n+r-1) - 3(n+r-1)\big]\, a_n - (n+r-2)a_{(n-1)}\Big] = 0.$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: Hence, the recurrence relation is given by the equations

$$r(r-1) - 3r + 3 = 0,$$
$$(n+r-1)(n+r-3)a_n - (n+r-2)a_{(n-1)} = 0.$$

First: solve the first equation for $r_\pm$.

Second: Introduce the first solution r_+ into the second equation above and solve for the a_n; the result is a solution y_+ of the original differential equation;

Third: Introduce the second solution r_- into into the second equation above and solve for the a_n; the result is a solution y_- of the original differential equation;

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We first solve $r(r-1) - 3r + 3 = 0$.

$$r^2 - 4r + 3 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big[4 \pm \sqrt{16-12}\big] \quad \Rightarrow \quad \begin{cases} r_+ = 3, \\ r_- = 1. \end{cases}$$

Introduce $r_+ = 3$ into the equation for a_n:

$$(n+2)n\, a_n - (n+1)a_{n-1} = 0.$$

One can check that the solution y_+ is

$$y_+ = a_0\, x^3 \Big[1 + \frac{2}{3}\, x + \frac{1}{4}\, x^2 + \frac{1}{15}\, x^3 + \cdots\Big].$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\,y'' - x(x+3)\,y' + (x+3)\,y = 0.$$

Solution: Introduce $r_- = 1$ into the equation for a_n:

$$n(n-2)a_n - (n-1)a_{n-1} = 0.$$

One can also check that the solution y_- is

$$y_- = a_2\,x\Big[x^2 + \frac{2}{3}\,x^3 + \frac{1}{4}\,x^4 + \frac{1}{15}\,x^5 + \cdots\Big].$$

Notice:

$$y_- = a_2\,x^3\Big[1 + \frac{2}{3}\,x + \frac{1}{4}\,x^2 + \frac{1}{15}\,x^3 + \cdots\Big] \;\Rightarrow\; y_- = \frac{a_2}{a_1}\,y_+.$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\,y'' - x(x+3)\,y' + (x+3)\,y = 0.$$

Solution: The solutions y_+ and y_- are not linearly independent.

This Example shows that the method does not provide all solutions of a differential equation near a regular-singular point, it only provides at least one solution near a regular-singular point.

Remark: It can be shown the following result:
If the roots of the Euler characteristic polynomial r_+, r_- differ by an integer, then the second solution y_-, the solution corresponding to the smaller root, is not given by the method above.
This solution involves logarithmic terms.
We do not study this type of solutions in these notes. $\triangleleft$