Equations with regular-singular points (Sect. 3.3).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- Method to find solutions.
- Example: Method to find solutions.

Recall:
The point \( x_0 \in \mathbb{R} \) is a singular point of the equation
\[
P(x) y'' + Q(x) y' + R(x) y = 0
\]
iff holds that \( P(x_0) = 0 \).

Equations with regular-singular points.

Definition
A singular point \( x_0 \in \mathbb{R} \) of the equation
\[
P(x) y'' + Q(x) y' + R(x) y = 0
\]
is called a regular-singular point iff the following limits are finite,
\[
\lim_{x \to x_0} \frac{(x - x_0) Q(x)}{P(x)}, \quad \lim_{x \to x_0} \frac{(x - x_0)^2 R(x)}{P(x)},
\]
and both functions
\[
\frac{(x - x_0) Q(x)}{P(x)}, \quad \frac{(x - x_0)^2 R(x)}{P(x)}
\]
admit convergent Taylor series expansions around \( x_0 \).
Equations with regular-singular points.

Remark:
- If $x_0$ is a regular-singular point of
  \[ P(x) y'' + Q(x) y' + R(x) y = 0 \]
  and $P(x) \simeq (x - x_0)^n$ near $x_0$, then near $x_0$ holds
  \[ Q(x) \simeq (x - x_0)^{n-1}, \quad R(x) \simeq (x - x_0)^{n-2}. \]
- The main example is an Euler equation, case $n = 2$,
  \[ (x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0. \]

Equations with regular-singular points.

Example
Show that the singular point of every Euler equation is a regular-singular point.

Solution: Consider the general Euler equation
  \[ (x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0, \]
where $p_0$, $q_0$, $x_0$, are real constants. This is an equation
  \[ Py'' + Qy' + Ry = 0 \]
with
  \[ P(x) = (x - x_0)^2, \quad Q(x) = p_0(x - x_0), \quad R(x) = q_0. \]
Therefore, we obtain,
  \[ \lim_{x \to x_0} \frac{(x - x_0)Q(x)}{P(x)} = p_0, \quad \lim_{x \to x_0} \frac{(x - x_0)^2 R(x)}{P(x)} = q_0. \]
We conclude that $x_0$ is a regular-singular point.
Equations with regular-singular points.

Remark: Every equation \( Py'' + Qy' + Ry = 0 \) with a regular-singular point at \( x_0 \) is close to an Euler equation.

Proof:
For \( x \neq x_0 \) divide the equation by \( P(x) \),
\[
y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0,
\]
and multiply it by \((x - x_0)^2\),
\[
(x - x_0)^2 y'' + (x - x_0) \left[ \frac{(x - x_0) Q(x)}{P(x)} \right] y' + \left[ \frac{(x - x_0)^2 R(x)}{P(x)} \right] y = 0.
\]
The factors between \([\ ]\) approach constants, say \( p_0, q_0 \), as \( x \to x_0 \),
\[
(x - x_0)^2 y'' + (x - x_0) p_0 y' + q_0 y = 0.
\]

Equations with regular-singular points (Sect. 3.3).

- Equations with regular-singular points.
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Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation
\[(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,\]
where \(\alpha\) is a real constant.

Solution: Find the singular points of this equation,
\[0 = P(x) = (1 - x^2) = (1 - x)(1 + x) \quad \Rightarrow \quad \begin{cases} x_0 = 1, \\ x_1 = -1. \end{cases} \]

Case \(x_0 = 1\): We then have
\[
\frac{(x - 1) Q(x)}{P(x)} = \frac{(x - 1)(-2x)}{(1 - x)(1 + x)} = \frac{2x}{1 + x},
\]

\[
\frac{(x - 1)^2 R(x)}{P(x)} = \frac{(x - 1)^2[\alpha(\alpha + 1)]}{(1 - x)(1 + x)} = \frac{(x - 1)[\alpha(\alpha + 1)]}{1 + x};
\]
both functions above have Taylor series around \(x_0 = 1\).

Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation
\[(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,\]
where \(\alpha\) is a real constant.

Solution: Recall:
\[
\frac{(x - 1) Q(x)}{P(x)} = \frac{2x}{1 + x}, \quad \frac{(x - 1)^2 R(x)}{P(x)} = \frac{(x - 1)[\alpha(\alpha + 1)]}{1 + x}.
\]

Furthermore, the following limits are finite,
\[
\lim_{x \to 1} \frac{(x - 1) Q(x)}{P(x)} = 1, \quad \lim_{x \to 1} \frac{(x - 1)^2 R(x)}{P(x)} = 0.
\]

We conclude that \(x_0 = 1\) is a regular-singular point.
Example

Find the regular-singular points of the differential equation

\[(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,\]

where \(\alpha\) is a real constant.

Solution:

Case \(x_1 = -1\):

\[
\frac{(x + 1) Q(x)}{P(x)} = \frac{2x}{(1-x)(1+x)} = -\frac{2x}{1-x},
\]

\[
\frac{(x + 1)^2 R(x)}{P(x)} = \frac{(x + 1)^2 \left[ \alpha(\alpha + 1) \right]}{(1-x)(1+x)} = \frac{(x + 1) \left[ \alpha(\alpha + 1) \right]}{1-x}.
\]

Both functions above have Taylor series \(x_1 = -1\).

Example: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

\[(1 - x^2) y'' - 2x y' + \alpha(\alpha + 1) y = 0,\]

where \(\alpha\) is a real constant.

Solution: Recall:

\[
\frac{(x + 1) Q(x)}{P(x)} = -\frac{2x}{1-x}, \quad \frac{(x + 1)^2 R(x)}{P(x)} = \frac{(x + 1) \left[ \alpha(\alpha + 1) \right]}{1-x}.
\]

Furthermore, the following limits are finite,

\[
\lim_{{x \to -1}} \frac{(x + 1) Q(x)}{P(x)} = 1, \quad \lim_{{x \to -1}} \frac{(x + 1)^2 R(x)}{P(x)} = 0.
\]

Therefore, the point \(x_1 = -1\) is a regular-singular point.
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[(x + 2)^2(x - 1)y'' + 3(x - 1)y' + 2y = 0.\]

Solution: Find the singular points: \(x_0 = -2\) and \(x_1 = 1\).

Case \(x_0 = -2\):

\[\lim_{x \to -2} \frac{(x + 2)Q(x)}{P(x)} = \lim_{x \to -2} \frac{(x + 2)3(x - 1)}{(x + 2)^2(x - 1)} = \lim_{x \to -2} \frac{3}{x + 2} = \pm \infty.\]

So \(x_0 = -2\) is not a regular-singular point. Case \(x_1 = 1\):

\[\frac{(x - 1) Q(x)}{P(x)} = \frac{(x - 1)[3(x - 1)]}{(x + 2)(x - 1)} = -\frac{3(x - 1)}{(x + 2)^2},\]

\[\frac{(x - 1)^2 R(x)}{P(x)} = \frac{2(x - 1)^2}{(x + 2)^2(x - 1)} = \frac{2(x - 1)}{(x + 2)^2};\]

Both functions have Taylor series around \(x_1 = 1\).

\[\triangleq\]

Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[(x + 2)^2(x - 1)y'' + 3(x - 1)y' + 2y = 0.\]

Solution: Recall:

\[\frac{(x - 1) Q(x)}{P(x)} = -\frac{3(x - 1)}{(x + 2)^2}, \quad \frac{(x - 1)^2 R(x)}{P(x)} = \frac{2(x - 1)}{(x + 2)^2}.\]

Furthermore, the following limits are finite,

\[\lim_{x \to 1} \frac{(x - 1) Q(x)}{P(x)} = 0; \quad \lim_{x \to 1} \frac{(x - 1)^2 R(x)}{P(x)} = 0.\]

Therefore, the point \(x_1 = -1\) is a regular-singular point. \(\triangleq\)
Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3xy = 0. \]

Solution: The singular point is \( x_0 = 0 \). We compute the limit

\[ \lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} \frac{x \left[ -x \ln(|x|) \right]}{x} = \lim_{x \to 0} -\ln(|x|) \frac{1}{x}. \]

Use L’Hôpital’s rule:

\[ \lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} -\frac{1}{x} = \lim_{x \to 0} x = 0. \]

The other limit is:

\[ \lim_{x \to 0} \frac{x^2R(x)}{P(x)} = \lim_{x \to 0} \frac{x^2(3x)}{x} = \lim_{x \to 0} 3x^2 = 0. \]

Examples: Equations with regular-singular points.

Example
Find the regular-singular points of the differential equation

\[ x y'' - x \ln(|x|) y' + 3xy = 0. \]

Solution: Recall:

\[ \lim_{x \to 0} \frac{xQ(x)}{P(x)} = 0 \quad \text{and} \quad \lim_{x \to 0} \frac{x^2R(x)}{P(x)} = 0. \]

However, at the point \( x_0 = 0 \) the function \( xQ/P \) does not have a
power series expansion around zero, since

\[ \frac{xQ(x)}{P(x)} = -x \ln(|x|), \]

and the log function does not have a Taylor series at \( x_0 = 0 \).

We conclude that \( x_0 = 0 \) is not a regular-singular point. △
Equations with regular-singular points (Sect. 3.3).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- **Method to find solutions.**
- Example: Method to find solutions.

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**Method to find solutions.**

**Recall:** If $x_0$ is a regular-singular point of

$$P(x) y'' + Q(x) y' + R(x) y = 0,$$

with limits $\lim_{x \to x_0} \frac{(x - x_0)Q(x)}{P(x)} = p_0$ and $\lim_{x \to x_0} \frac{(x - x_0)^2 R(x)}{P(x)} = q_0$,

then the coefficients of the differential equation above near $x_0$ are close to the coefficients of the Euler equation

$$(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0.$$

**Idea:** If the differential equation is close to an Euler equation, then the solutions of the differential equation might be close to the solutions of an Euler equation.

**Recall:** One solution of an Euler equation is $y(x) = (x - x_0)^r$. 

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Method to find solutions.

**Summary:** Solutions for equations with regular-singular points:

1. Look for a solution $y$ of the form

   $$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(n+r)};$$

2. Introduce this power series expansion into the differential equation and find both a the exponent $r$ and a recurrence relation for the coefficients $a_n$;

3. First find the solutions for the constant $r$. Then, introduce this result for $r$ into the recurrence relation for the coefficients $a_n$. Only then, solve this latter recurrence relation for the coefficients $a_n$.

Equations with regular-singular points (Sect. 3.3).

- Equations with regular-singular points.
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- Method to find solutions.
- **Example:** Method to find solutions.
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: We look for a solution $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

The first and second derivatives are given by
\[ y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{n+r-2}. \]

In the case $r = 0$ we had the relation
\[ \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \]

but for $r \neq 0$ this relation is not true.

Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: We now compute the term $(x + 3)y$,
\[ (x + 3)y = (x + 3) \sum_{n=0}^{\infty} a_n x^{n+r}. \]

\[ (x + 3)y = \sum_{n=0}^{\infty} a_n x^{n+r+1} + \sum_{n=0}^{\infty} 3a_n x^{n+r}. \]

\[ (x + 3)y = \sum_{n=1}^{\infty} a_{n-1} x^{n+r} + \sum_{n=0}^{\infty} 3a_n x^{n+r}. \]
Example: Method to find solutions.

Example
Find the solution \( y \) near the regular-singular point \( x_0 = 0 \) of
\[
x^2 y'' - x(x + 3) y' + (x + 3) y = 0.
\]

Solution: We now compute the term \(-x(x + 3) y'\),
\[
-x(x + 3) y' = -(x^2 + 3x) \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}
\]
\[
-x(x + 3) y' = -\sum_{n=0}^{\infty} (n + r) a_n x^{n+r+1} - \sum_{n=0}^{\infty} 3(n + r) a_n x^{n+r},
\]
\[
-x(x + 3) y' = -\sum_{n=1}^{\infty} (n + r - 1) a_{n-1} x^{n+r} - \sum_{n=0}^{\infty} 3(n + r) a_n x^{n+r}.
\]

The guiding principle to rewrite each term is to have the power function \( x^{(n+r)} \) labeled in the same way on every term.
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x + 3) y' + (x + 3) y = 0.$$ 

Solution: The differential equation is given by

$$
\sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{(n+r)} - \sum_{n=1}^{\infty} (n + r - 1)a_{(n-1)} x^{(n+r)}

\quad - \sum_{n=0}^{\infty} 3(n + r)a_n x^{(n+r)} + \sum_{n=1}^{\infty} a_{(n-1)} x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n x^{(n+r)} = 0.
$$

We split the sums into the term $n = 0$ and a sum containing the terms with $n \geq 1$, that is,

$$
0 = [r(r - 1) - 3r + 3] a_0 x' + \sum_{n=1}^{\infty} \left[ (n+r)(n+r-1)a_n - (n+r-1)a_{(n-1)} - 3(n+r)a_n + a_{(n-1)} + 3a_n \right] x^{(n+r)}.
$$

Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x + 3) y' + (x + 3) y = 0.$$ 

Solution: Therefore, \([r(r - 1) - 3r + 3] = 0\) and

$$
[(n+r)(n+r-1)a_n - (n+r-1)a_{(n-1)} - 3(n+r)a_n + a_{(n-1)} + 3a_n] = 0.
$$

The last expression can be rewritten as follows,

$$
\left[ [(n + r)(n + r - 1) - 3(n + r) + 3] a_n - (n + r - 1 - 1)a_{(n-1)} \right] = 0,
$$

$$
\left[ [(n + r)(n + r - 1) - 3(n + r - 1)] a_n - (n + r - 2)a_{(n-1)} \right] = 0.
$$
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: Hence, the recurrence relation is given by the equations

\[ r(r - 1) - 3r + 3 = 0, \]
\[ (n + r - 1)(n + r - 3)a_n - (n + r - 2)a_{n-1} = 0. \]

First: solve the first equation for $r_{\pm}$.

Second: Introduce the first solution $r_+$ into the second equation above and solve for the $a_n$; the result is a solution $y_+$ of the original differential equation;

Third: Introduce the second solution $r_-$ into into the second equation above and solve for the $a_n$; the result is a solution $y_-$ of the original differential equation;

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Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: We first solve $r(r - 1) - 3r + 3 = 0$.

\[ r^2 - 4r + 3 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 12}] \quad \Rightarrow \quad \begin{cases} r_+ = 3, \\
                     r_- = 1. \end{cases} \]

Introduce $r_+ = 3$ into the equation for $a_n$:
\[ (n + 2) n a_n - (n + 1) a_{n-1} = 0. \]

One can check that the solution $y_+$ is
\[ y_+ = a_0 x^3 \left[ 1 + \frac{2}{3} x + \frac{1}{4} x^2 + \frac{1}{15} x^3 + \cdots \right]. \]
Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: Introduce $r_- = 1$ into the equation for $a_n$:
\[ n(n - 2)a_n - (n - 1)a_{n-1} = 0. \]

One can also check that the solution $y_-$ is
\[ y_- = a_2 x \left[ x^2 + \frac{2}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{15} x^5 + \cdots \right]. \]

Notice:
\[ y_- = a_2 x^3 \left[ 1 + \frac{2}{3} x + \frac{1}{4} x^2 + \frac{1}{15} x^3 + \cdots \right] \Rightarrow y_- = \frac{a_2}{a_1} y_+. \]

Example: Method to find solutions.

Example
Find the solution $y$ near the regular-singular point $x_0 = 0$ of
\[ x^2 y'' - x(x + 3) y' + (x + 3) y = 0. \]

Solution: The solutions $y_+$ and $y_-$ are not linearly independent.

This Example shows that the method does not provide all solutions of a differential equation near a regular-singular point, it only provides at least one solution near a regular-singular point.

Remark: It can be shown the following result:
If the roots of the Euler characteristic polynomial $r_+, r_-$ differ by an integer, then the second solution $y_-$, the solution corresponding to the smaller root, is not given by the method above.
This solution involves logarithmic terms.
We do not study this type of solutions in these notes.