Power series solutions near regular points (Sect. 3.1).

- We study: $P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$.
- Review of power series.
- Regular point equations.
- Solutions using power series.
- Examples of the power series method.

Review of power series.

Definition

The *power series* of a function $y : \mathbb{R} \to \mathbb{R}$ centered at $x_0 \in \mathbb{R}$ is

$$y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n.$$

Example

- $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$. Here $x_0 = 0$ and $|x| < 1$.
- $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$. Here $x_0 = 0$ and $x \in \mathbb{R}$.
- The Taylor series of $y : \mathbb{R} \to \mathbb{R}$ centered at $x_0 \in \mathbb{R}$ is
  $$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}\,(x - x_0)^n = y(x_0) + y'(x_0)\,(x - x_0) + \cdots.$$

Review of power series.

Example

Find the Taylor series of $y(x) = \sin(x)$ centered at $x_0 = 0$.

Solution: $y(x) = \sin(x),\ y(0) = 0. \quad y'(x) = \cos(x),\ y'(0) = 1.$

$$y''(x) = -\sin(x),\ y''(0) = 0. \quad y'''(x) = -\cos(x),\ y'''(0) = -1.$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \ \Rightarrow\ \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{(2n+1)}. \ \triangleleft$$

Remark: The Taylor series of $y(x) = \cos(x)$ centered at $x_0 = 0$ is

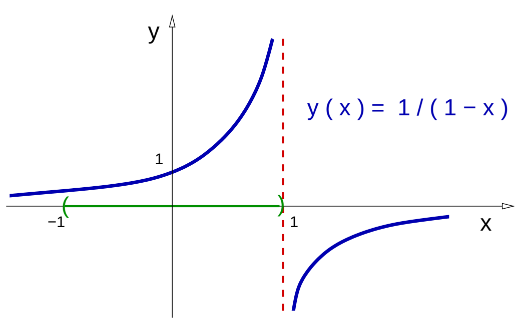
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{(2n)}.$$

Review of power series.

Remark: The power series of a function may not be defined on the whole domain of the function.

Example

The function $y(x) = \dfrac{1}{1-x}$ is defined for $x \in \mathbb{R} - \{1\}$.



The power series

$$y(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

converges only for $|x| < 1$.

$\triangleleft$

Review of power series.

Definition

The power series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ *converges absolutely* iff the series $\sum_{n=0}^{\infty} |a_n| \, |x - x_0|^n$ converges.

Example

The series $s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, but it does not converge absolutely, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Review of power series.

Definition

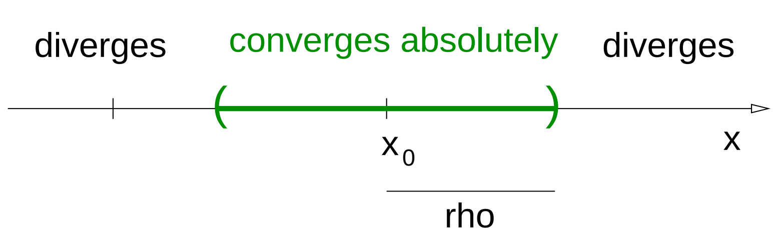
The *radius of convergence* of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is the number $\rho \geqslant 0$ that satisfies both

(a) the series converges absolutely for $|x - x_0| < \rho$;

(b) the series diverges for $|x - x_0| > \rho$.



Review of power series.

Example

(1) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ has radius of convergence $\rho = 1$.

(2) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence $\rho = \infty$.

(3) $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{(2n+1)}$ has radius $\rho = \infty$.

(4) $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{(2n)}$ has radius of convergence $\rho = \infty$.

Review of power series.

Theorem (Ratio test)

Given the power series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, *introduce the number* $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. *Then, the following statements hold:*

(1) *The power series converges in the domain* $|x - x_0|L < 1$.

(2) *The power series diverges in the domain* $|x - x_0|L > 1$.

(3) *The power series may or may not converge at* $|x - x_0|L = 1$.

Therefore, if $L \neq 0$, *then* $\rho = \frac{1}{L}$ *is the series radius of convergence; if* $L = 0$, *then the radius of convergence is* $\rho = \infty$.

Review of power series.

Remarks: On summation indices:

$$y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

$$y(x) = \sum_{k=0}^{\infty} a_k\,(x - x_0)^k = \sum_{m=-3}^{\infty} a_{m+3}\,(x - x_0)^{m+3}.$$

$$y'(x) = \sum_{n=0}^{\infty} n\,a_n\,(x - x_0)^{n-1} = a_1 + 2a_2(x - x_0) + \cdots$$

$$y'(x) = \sum_{n=1}^{\infty} n\,a_n\,(x - x_0)^{n-1} = \sum_{m=0}^{\infty} (m+1)\,a_{m+1}\,(x - x_0)^m$$

where $m = n - 1$, that is, $n = m + 1$.

Power series solutions near regular points (Sect. 3.1).

- We study: $P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$.
- Review of power series.
- **Regular point equations.**
- Solutions using power series.
- Examples of the power series method.

Regular point equations.

Problem: We look for solutions y of the variable coefficients equation

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0.$$

around $x_0 \in \mathbb{R}$ where $P(x_0) \neq 0$ using a power series representation of the solution centered at x_0, that is,

$$y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n.$$

Definition

Given continuous functions P, Q, $R : (x_1, x_2) \to \mathbb{R}$, a point $x_0 \in (x_1, x_2)$ is called a *regular point* of the equation

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0.$$

iff $P(x_0) \neq 0$. The point x_0 is called a *singular point* iff $P(x_0) = 0$.

Remark: The equation order does not change near regular points.

Power series solutions near regular points (Sect. 3.1).

- We study: $P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0.$
- Review of power series.
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- **Solutions using power series.**
- Examples of the power series method.

Solutions using power series.

Summary for regular points:

(1) Propose a power series representation of the solution centered at x_0, given by

$$y(x) = \sum_{n=0}^{\infty} a_n \, (x - x_0)^n; \tag{1}$$

(2) Introduce Eq. (1) into the differential equation $P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0$.

(3) Find a recurrence relation among the coefficients a_n;

(4) Solve the recurrence relation in terms of free coefficients;

(5) If possible, add up the resulting power series for the solution y.

Power series solutions near regular points (Sect. 3.1).

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- Review of power series.
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Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c\,y = 0, \qquad c \in \mathbb{R}.$$

Solution: Recall: The solution is $y(x) = a_0\, e^{-c\,x}$.

We now use the power series method. We propose a power series centered at $x_0 = 0$:

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^n \quad \Rightarrow \quad y'(x) = \sum_{n=0}^{\infty} n a_n\, x^{(n-1)} = \sum_{n=1}^{\infty} n a_n\, x^{(n-1)}.$$

Change the summation index: $m = n - 1$, so $n = m + 1$.

$$y'(x) = \sum_{m=0}^{\infty} (m+1) a_{m+1}\, x^m = \sum_{n=0}^{\infty} (n+1) a_{n+1}\, x^n.$$

Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c\,y = 0, \qquad c \in \mathbb{R}.$$

Solution: $y(x) = \sum_{n=0}^{\infty} a_n\, x^n$, and $y'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1}\, x^n$.

Introduce y and y' into the differential equation,

$$\sum_{n=0}^{\infty} (n+1) a_{(n+1)}\, x^n + \sum_{n=0}^{\infty} c\, a_n\, x^n = 0$$

$$\sum_{n=0}^{\infty} \big[(n+1) a_{(n+1)} + c\, a_n\big]\, x^n = 0$$

The recurrence relation is $(n+1) a_{(n+1)} + c\, a_n = 0$ for all $n \geqslant 0$.

Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c\,y = 0, \qquad c \in \mathbb{R}.$$

Solution: Recurrence relation: $(n+1)a_{(n+1)} + c\,a_n = 0, \quad n \geqslant 0.$

Equivalently: $a_{n+1} = -\dfrac{c}{n+1}\,a_n$. That is,

$$n = 0, \quad a_1 = -c\,a_0 \quad \Rightarrow \quad a_1 = -c\,a_0,$$

$$n = 1, \quad 2a_2 = -c\,a_1 \quad \Rightarrow \quad a_2 = \frac{c^2}{2!}\,a_0,$$

$$n = 2, \quad 3a_3 = -c\,a_2 \quad \Rightarrow \quad a_3 = -\frac{c^3}{3!}\,a_0,$$

$$n = 3, \quad 4a_4 = -c\,a_3 \quad \Rightarrow \quad a_4 = \frac{c^4}{4!}\,a_0.$$

Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y' + c\,y = 0, \qquad c \in \mathbb{R}.$$

Solution: Solved recurrence relation: $a_n = \dfrac{(-c)^n}{n!}\,a_0$.

The solution y of the differential equation is given by

$$y(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!}\,a_0\,x^n \quad \Rightarrow \quad y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-c\,x)^n}{n!}.$$

If we recall the power series $e^{ax} = \displaystyle\sum_{n=0}^{\infty} \frac{(ax)^n}{n!}$,

then, we conclude that the solution is $y(x) = a_0\,e^{-c\,x}$. $\triangleleft$

Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation
$$y'' + y = 0.$$

Solution: Recall: The characteristic polynomial is $r^2 + 1 = 0$, hence the general solution is $y(x) = a_0 \cos(x) + a_1 \sin(x)$.

We re-obtain this solution using the power series method:
$$y = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{(n-1)} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m,$$
where $m = n - 1$, so $n = m + 1$;
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{(n-2)} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m.$$
where $m = n - 2$, so $n = m + 2$.

Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation
$$y'' + y = 0.$$

Solution: Introduce y and y'' into the differential equation,
$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$
$$\sum_{n=0}^{\infty} \big[(n+2)(n+1) a_{(n+2)} + a_n\big] x^n = 0.$$

The recurrence relation is $(n+2)(n+1) a_{(n+2)} + a_n = 0, \quad n \geqslant 0.$

Equivalently: $(n+2)(n+1)\, a_{(n+2)} = -a_n,$

Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$

Solution: Recall: $(n+2)(n+1)\,a_{(n+2)} = -a_n, \quad n \geqslant 0.$

For n even: $n = 0, \quad (2)(1)a_2 = -a_0 \quad \Rightarrow \quad a_2 = -\frac{1}{2!}\,a_0,$

$$n = 2, \quad (4)(3)a_4 = -a_2 \quad \Rightarrow \quad a_4 = \frac{1}{4!}\,a_0,$$

$$n = 4, \quad (6)(5)a_6 = -a_4 \quad \Rightarrow \quad a_6 = -\frac{1}{6!}\,a_0.$$

We obtain: $a_{2k} = \dfrac{(-1)^k}{(2k)!}\,a_0$, for $k \geqslant 0$.

Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$

Solution: Recall: $a_{2k} = \dfrac{(-1)^k}{(2k)!}\,a_0$ and $(n+2)(n+1)\,a_{(n+2)} = -a_n$.

For n odd: $n = 1, \quad (3)(2)a_3 = -a_1 \quad \Rightarrow \quad a_3 = -\dfrac{1}{3!}\,a_1,$

$$n = 3, \quad (5)(4)a_5 = -a_3 \quad \Rightarrow \quad a_5 = \frac{1}{5!}\,a_1,$$

$$n = 5, \quad (7)(6)a_7 = -a_5 \quad \Rightarrow \quad a_7 = -\frac{1}{7!}\,a_1.$$

We obtain $a_{2k+1} = \dfrac{(-1)^k}{(2k+1)!}\,a_1$ for $k \geqslant 0$.

Examples of the power series method.

Example

Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$

Solution: Recall: $a_{2k} = \frac{(-1)^k}{(2k)!}\, a_0$ and $a_{2k+1} = \frac{(-1)^k}{(2k+1)!}\, a_1$.

Therefore, the solution of the differential equation is given by

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1}.$$

One can check that these are precisely the power series representations of the cosine and sine functions, respectively,

$$y(x) = a_0 \cos(x) + a_1 \sin(x).$$

$\triangleleft$

Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x\, y = 0.$$

Solution: We propose: $y = \sum_{n=0}^{\infty} a_n (x-2)^n$.

It is convenient to rewrite the function xy as follows,

$$xy = \sum_{n=0}^{\infty} a_n x (x-2)^n = \sum_{n=0}^{\infty} a_n \big[(x-2) + 2\big](x-2)^n,$$

$$xy = \sum_{n=0}^{\infty} a_n (x-2)^{n+1} + \sum_{n=0}^{\infty} 2a_n (x-2)^n.$$

We relabel the first sum: $\sum_{n=0}^{\infty} a_n (x-2)^{n+1} = \sum_{n=1}^{\infty} a_{(n-1)} (x-2)^n$.

Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x\,y = 0.$$

Solution: We relabel the y'',

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{(n+2)}(x-2)^n.$$

Introduce y'' and xy in the differential equation

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{(n+2)}(x-2)^n - \sum_{n=0}^{\infty} 2a_n(x-2)^n - \sum_{n=1}^{\infty} a_{(n-1)}(x-2)^n = 0$$

$$(2)(1)a_2 - 2a_0 + \sum_{n=1}^{\infty}\Big[(n+2)(n+1)a_{(n+2)} - 2a_n - a_{(n-1)}\Big]\,(x-2)^n = 0.$$

The recurrence relation for the coefficients a_n is:

$$a_2 - a_0 = 0, \qquad (n+2)(n+1)a_{(n+2)} - 2a_n - a_{(n-1)} = 0, \qquad n \geqslant 1.$$

Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x\,y = 0.$$

Solution: The recurrence relation is:

$$a_2 - a_0 = 0, \quad (n+2)(n+1)a_{(n+2)} - 2a_n - a_{(n-1)} = 0, \quad n \geqslant 1.$$

We solve this recurrence relation for the first four coefficients,

$$n = 0 \quad a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = a_0,$$

$$n = 1 \quad (3)(2)a_3 - 2a_1 - a_0 = 0 \quad \Rightarrow \quad a_3 = \frac{a_0}{6} + \frac{a_1}{3},$$

$$n = 2 \quad (4)(3)a_4 - 2a_2 - a_1 = 0 \quad \Rightarrow \quad a_4 = \frac{a_0}{6} + \frac{a_1}{12}.$$

$$y \simeq a_0 + a_1(x-2) + a_0(x-2)^2 + \Big(\frac{a_0}{6} + \frac{a_1}{3}\Big)(x-2)^3 + \Big(\frac{a_0}{6} + \frac{a_1}{12}\Big)(x-2)^4.$$

Examples of the power series method.

Example

Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x\,y = 0.$$

Solution: The first terms in the power series expression for y are

$$y \simeq a_0 + a_1(x-2) + a_0(x-2)^2 + \Big(\frac{a_0}{6} + \frac{a_1}{3}\Big)(x-2)^3 + \Big(\frac{a_0}{6} + \frac{a_1}{12}\Big)(x-2)^4.$$

$$y = a_0\Big[1 + (x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{6}(x-2)^4 + \cdots\Big]$$
$$+ a_1\Big[(x-2) + \frac{1}{3}(x-2)^3 + \frac{1}{12}(x-2)^4 + \cdots\Big]$$

So the first three terms on each fundamental solution are given by

$$y_1 \simeq 1+(x-2)^2+\frac{1}{6}(x-2)^3, \quad y_2 \simeq (x-2)+\frac{1}{3}(x-2)^3+\frac{1}{12}(x-2)^4.$$