Non-homogeneous equations (Sect. 2.5).

- We study: \( y'' + a_1 y' + a_0 y = b(t) \).
- Operator notation and preliminary results.
- Summary of the undetermined coefficients method.
- Using the method in few examples.
- The guessing solution table.

Operator notation and preliminary results.

**Notation:** Given functions \( p, q \), denote

\[
L(y) = y'' + p(t) y' + q(t) y.
\]

Therefore, the differential equation

\[
y'' + p(t) y' + q(t) y = f(t)
\]

can be written as

\[
L(y) = f.
\]

The homogeneous equation can be written as

\[
L(y) = 0.
\]

The function \( L \) acting on a function \( y \) is called an **operator**.
Operator notation and preliminary results.

Remark: The operator $L$ is a linear function of $y$.

Theorem
For every continuously differentiable functions $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ and every $c_1, c_2 \in \mathbb{R}$ holds that

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2).$$

Proof:

$$L(c_1y_1 + c_2y_2) = (c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2)$$

$$L(c_1y_1 + c_2y_2) = (c_1y_1'' + p(t)c_1y_1' + q(t)c_1y_1)$$
$$+ (c_2y_2'' + p(t)c_2y_2' + q(t)c_2y_2)$$

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2).$$  $\square$

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Operator notation and preliminary results.

Theorem
Given functions $p, q, f$, let $L(y) = y'' + p(t)y' + q(t)y$.
If the functions $y_1$ and $y_2$ are fundamental solutions of the homogeneous equation

$$L(y) = 0,$$

and $y_p$ is any solution of the non-homogeneous equation

$$L(y_p) = f,$$  \hspace{1cm} (1)

then any other solution $y$ of the non-homogeneous equation above is given by

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t),$$  \hspace{1cm} (2)

where $c_1, c_2 \in \mathbb{R}$.

Notation: The expression for $y$ in Eq. (2) is called the general solution of the non-homogeneous Eq. (1).
Operator notation and preliminary results.

Theorem

Given functions $p$, $q$, let $L(y) = y'' + p(t) y' + q(t) y$.

A particular solution to the differential equation

$$L(y_p) = f_1 + \cdots + f_n,$$

is a function $y_p = y_{p_1} + \cdots + y_{p_n}$, where each function $y_{p_i}$, with $i = 1, \cdots, n$ is a solution of the equation

$$L(y_{p_i}) = f_i.$$

Non-homogeneous equations (Sect. 2.5).

- We study: $y'' + a_1 y' + a_0 y = b(t)$.
- Operator notation and preliminary results.
- **Summary of the undetermined coefficients method.**
- Using the method in few examples.
- The guessing solution table.
Summary of the undetermined coefficients method.

Problem: Given a constant coefficients linear operator
\[ L(y) = y'' + a_1 y' + a_0 y, \] with \( a_1, a_2 \in \mathbb{R} \), find every solution of the non-homogeneous differential equation
\[ L(y) = f. \]

Remarks:
- The undetermined coefficients is a method to find solutions to linear, non-homogeneous, constant coefficients, differential equations.
- It consists in guessing the solution \( y_p \) of the non-homogeneous equation
\[ L(y_p) = f, \]
for particularly simple source functions \( f \).

Summary of the undetermined coefficients method.

Summary:
1. Find the general solution of the homogeneous equation \( L(y_h) = 0 \).
2. If \( f \) has the form \( f = f_1 + \cdots + f_n \), with \( n \geq 1 \), then look for solutions \( y_{p_i} \), with \( i = 1, \cdots, n \) to the equations
\[ L(y_{p_i}) = f_i. \]
   Once the functions \( y_{p_i} \) are found, then construct
\[ y_p = y_{p_1} + \cdots + y_{p_n}. \]
3. Given the source functions \( f_i \), guess the solutions functions \( y_{p_i} \) following the Table below.
Summary of the undetermined coefficients method.

Summary (cont.):

<table>
<thead>
<tr>
<th>$f_i(t)$ ($K$, $m$, $a$, $b$, given.)</th>
<th>$y_{pi}(t)$ (Guess) ($k$ not given.)</th>
</tr>
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<tbody>
<tr>
<td>$Ke^{at}$</td>
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</table>

(4) If any guessed function $y_{pi}$ satisfies the homogeneous equation $L(y_{pi}) = 0$, then change the guess to the function

\[ t^s y_{pi}, \quad \text{with} \quad s \geq 1, \]

and $s$ sufficiently large such that $L(t^s y_{pi}) \neq 0$.

(5) Impose the equation $L(y_{pi}) = f_i$ to find the undetermined constants $k_1, \cdots, k_m$, for the appropriate $m$, given in the table above.

(6) The general solution to the original differential equation $L(y) = f$ is then given by

\[ y(t) = y_h(t) + y_{p1} + \cdots + y_{pn}. \]
Non-homogeneous equations (Sect. 2.5).

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Using the method in few examples.

**Example**

Find all solutions to the non-homogeneous equation

\[
y'' - 3y' - 4y = 3e^{2t}.
\]

**Solution:** Notice: \( L(y) = y'' - 3y' - 4y \) and \( f(t) = 3e^{2t} \).

1. Find all solutions \( y_h \) to the homogeneous equation \( L(y_h) = 0 \).

   The characteristic equation is

   \[
r^2 - 3r - 4 = 0 \implies \begin{cases} r_1 = 4, \\ r_2 = -1. \end{cases}
   \]

   \[
y_h(t) = c_1 e^{4t} + c_2 e^{-t}.
   \]

2. Trivial in our case. The source function \( f(t) = 3e^{2t} \) cannot be simplified into a sum of simpler functions.

3. Table says: For \( f(t) = 3e^{2t} \) guess \( y_p(t) = k e^{2t} \)
Using the method in few examples.

Example
Find all solutions to the non-homogeneous equation

\[ y'' - 3y' - 4y = 3e^{2t}. \]

Solution: Recall: \( y_p(t) = ke^{2t} \). We need to find \( k \).

(4) Trivial here, since \( L(y_p) \neq 0 \), we do not modify our guess.
(Recall: \( L(y_h) = 0 \) iff \( y_h(t) = c_1 e^{4t} + c_2 e^{-t} \).)

(5) Introduce \( y_p \) into \( L(y_p) = f \) and find \( k \).

\[ (2^2 - 6 - 4)ke^{2t} = 3e^{2t} \quad \Rightarrow \quad -6k = 3 \quad \Rightarrow \quad k = -\frac{1}{2}. \]

We have obtained that \( y_p(t) = -\frac{1}{2} e^{2t} \).

(6) The general solution to the inhomogeneous equation is

\[ y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2} e^{2t}. \]

\[ \triangleq \]

Using the method in few examples.

Example
Find all solutions to the non-homogeneous equation

\[ y'' - 3y' - 4y = 3e^{4t}. \]

Solution: We know that the general solution to homogeneous equation is \( y_h(t) = c_1 e^{4t} + c_2 e^{-t} \).

Following the table we guess \( y_p \) as \( y_p = ke^{4t} \).

However, this guess satisfies \( L(y_p) = 0 \).

So we modify the guess to \( y_p = kt e^{4t} \).

Introduce the guess into \( L(y_p) = f \). We need to compute

\[ y'_p = ke^{4t} + 4kt e^{4t}, \quad y''_p = 8ke^{4t} + 16kt e^{4t}. \]
Using the method in few examples.

Example
Find all solutions to the non-homogeneous equation
\[ y'' - 3y' - 4y = 3e^{4t}. \]

Solution: Recall:
\[ y_p = kt e^{4t}, \quad y'_p = k e^{4t} + 4kt e^{4t}, \quad y''_p = 8k e^{4t} + 16kt e^{4t}. \]
\[ [(8k + 16kt) - 3(k + 4kt) - 4kt] e^{4t} = 3e^{4t}. \]
\[ [(8 + 16t) - 3(1 + 4t) - 4t] \quad \Rightarrow \quad [5 + (16 - 12 - 4) t] \quad k = 3 \]
We obtain that \( k = \frac{3}{5}. \) Therefore, \( y_p(t) = \frac{3}{5} t e^{4t}, \) and
\[ y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{3}{5} t e^{4t}. \]

\[ \triangle \]

Using the method in few examples.

Example
Find all the solutions to the inhomogeneous equation
\[ y'' - 3y' - 4y = 2 \sin(t). \]

Solution: We know that the general solution to homogeneous equation is \( y(t) = c_1 e^{4t} + c_2 e^{-t}. \)

Following the table: Since \( f = 2 \sin(t), \) then we guess
\[ y_p = k_1 \sin(t) + k_2 \cos(t). \]

This guess satisfies \( L(y_p) \neq 0. \)
Compute: \( y'_p = k_1 \cos(t) - k_2 \sin(t), \quad y''_p = -k_1 \sin(t) - k_2 \cos(t). \)
\[ L(y_p) = \left[ -k_1 \sin(t) - k_2 \cos(t) \right] - 3[k_1 \cos(t) - k_2 \sin(t)] \]
\[ -4[k_1 \sin(t) + k_2 \cos(t)] = 2 \sin(t), \]
Using the method in few examples.

Example
Find all the solutions to the inhomogeneous equation
\[ y'' - 3y' - 4y = 2\sin(t). \]

Solution: Recall:
\[
L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]
- 4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),
\]
\[-5k_1 + 3k_2 = 2,\]
\[-3k_1 - 5k_2 = 0,\]
\[\Rightarrow \left\{ \begin{array}{l}
k_1 = -\frac{5}{17}, \\
k_2 = \frac{3}{17}.
\end{array} \right.\]

Using the method in few examples.

Example
Find all the solutions to the inhomogeneous equation
\[ y'' - 3y' - 4y = 2\sin(t). \]

Solution: Recall: \( k_1 = -\frac{5}{17} \) and \( k_2 = \frac{3}{17} \).

So the particular solution to the inhomogeneous equation is
\[ y_p(t) = \frac{1}{17} [-5\sin(t) + 3\cos(t)]. \]

The general solution is
\[ y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{1}{17} [-5\sin(t) + 3\cos(t)]. \]
Using the method in few examples.

Example
Find all the solutions to the inhomogeneous equation
\[ y'' - 3y' - 4y = 3e^{2t} + 2\sin(t). \]

Solution: We know that the general solution \( y \) is given by
\[ y(t) = y_h(t) + y_{p1}(t) + y_{p2}(t), \]
where \( y_h(t) = c_1 e^{4t} + c_2 e^{2t}, \) \( L(y_{p1}) = 3e^{2t}, \) and \( L(y_{p2}) = 2\sin(t). \)
We have just found out that
\[ y_p(t) = -\frac{1}{2} e^{2t}, \quad y_{p2}(t) = \frac{1}{17} \left[-5\sin(t) + 3\cos(t)\right]. \]
We conclude that
\[ y(t) = c_1 e^{4t} + c_2 e^{2t} - \frac{1}{2} e^{2t} + \frac{1}{17} \left[-5\sin(t) + 3\cos(t)\right]. \]

Using the method in few examples.

Example
- For \( y'' - 3y' - 4y = 3e^{2t}\sin(t), \) guess
\[ y_p(t) = [k_1 \sin(t) + k_2 \cos(t)] e^{2t}. \]
- For \( y'' - 3y' - 4y = 2t^2 e^{3t}, \) guess
\[ y_p(t) = (k_0 + k_1 t + k_2 t^2) e^{3t}. \]
- For \( y'' - 3y' - 4y = 3t \sin(t), \) guess
\[ y_p(t) = (1 + k_1 t) [k_2 \sin(t) + k_3 \cos(t)]. \]
Non-homogeneous equations (Sect. 2.5).

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