Second order linear homogeneous ODE (Sect. 2.3).

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - Review of Complex numbers.
  - A real-valued fundamental and general solutions.
- Application: The RLC circuit.

Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Theorem (Constant coefficients)

*Given real constants \( a_1, a_0 \), consider the homogeneous, linear differential equation on the unknown \( y: \mathbb{R} \to \mathbb{R} \) given by

\[
y'' + a_1 y' + a_0 y = 0.
\]

Let \( r_+, r_- \) be the roots of the characteristic polynomial \( p(r) = r^2 + a_1 r + a_0 \), and let \( c_0, c_1 \) be arbitrary constants. Then, the general solution \( y \) of the differential equation is given by

- (a) If \( r_+ \neq r_- \), real or complex, then \( y(t) = c_1 e^{r_+ t} + c_2 e^{r_- t} \).
- (b) If \( r_+ = r_- = \hat{r} \in \mathbb{R} \), then \( y(t) = c_1 e^{\hat{r} t} + c_2 t e^{\hat{r} t} \).

Furthermore, given real constants \( t_0, y_1 \) and \( y_2 \), there is a unique solution to the initial value problem

\[
y'' + a_1 y' + a_0 y = 0, \quad y(t_0) = y_1, \quad y'(t_0) = y_2.
\]
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Example
Find the general solution of the equation \( y'' - y' - 6y = 0 \).

Solution: Since solutions have the form \( e^{rt} \), we need to find the roots of the characteristic polynomial \( p(r) = r^2 - r - 6 \), that is,
\[
r_\pm = \frac{1}{2} \left( 1 \pm \sqrt{1 + 24} \right) = \frac{1}{2} (1 \pm 5) \quad \Rightarrow \quad r_+ = 3, \quad r_- = -2.
\]
So, \( r_\pm \) are real-valued. A fundamental solution set is formed by\[
y_1(t) = e^{3t}, \quad y_2(t) = e^{-2t}.
\]
The general solution of the differential equations is an arbitrary linear combination of the fundamental solutions, that is,
\[
y(t) = c_1 e^{3t} + c_2 e^{-2t}, \quad c_1, c_2 \in \mathbb{R}.
\]
Remark: Since \( c_1, c_2 \in \mathbb{R} \), then \( y \) is real-valued.

Second order linear homogeneous ODE (Sect. 2.3).

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- **Characteristic polynomial with complex roots.**
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- Application: The RLC circuit.
Two main sets of fundamental solutions.

Example
Find the general solution of the equation \( y'' - 2y' + 6y = 0 \).

Solution: We first find the roots of the characteristic polynomial,

\[
    r^2 - 2r + 6 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} (2 \pm \sqrt{4 - 24}) \quad \Rightarrow \quad r_{\pm} = 1 \pm i\sqrt{5}.
\]

A fundamental solution set is

\[
    \tilde{y}_1(t) = e^{(1+i\sqrt{5})t}, \quad \tilde{y}_2(t) = e^{(1-i\sqrt{5})t}.
\]

These are complex-valued functions. The general solution is

\[
    y(t) = \tilde{c}_1 e^{(1+i\sqrt{5})t} + \tilde{c}_2 e^{(1-i\sqrt{5})t}, \quad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}.
\]

Remark:
- The solutions found above include real-valued and complex-valued solutions.
- Since the differential equation is real-valued, it is usually important in applications to obtain the most general real-valued solution. (See RLC circuit below.)
- In the expression above it is difficult to take apart real-valued solutions from complex-valued solutions.
- In other words: It is not simple to see what values of \( \tilde{c}_1 \) and \( \tilde{c}_2 \) make the general solution above to be real-valued.
- One way to find the real-valued general solution is to find real-valued fundamental solutions.
Second order linear homogeneous ODE (Sect. 2.3).

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- **Characteristic polynomial with complex roots.**
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    - A real-valued fundamental and general solutions.
  - Application: The RLC circuit.

Review of complex numbers.

- Complex numbers have the form \( z = a + ib \), where \( i^2 = -1 \).
- The complex conjugate of \( z \) is the number \( \bar{z} = a - ib \).
- \( \text{Re}(z) = a, \quad \text{Im}(z) = b \) are the real and imaginary parts of \( z \).
- Hence: \( \text{Re}(z) = \frac{z + \bar{z}}{2} \) and \( \text{Im}(z) = \frac{z - \bar{z}}{2i} \).
- \( e^{a+ib} = \sum_{n=0}^{\infty} \frac{(a + ib)^n}{n!} \). In particular holds \( e^{a+ib} = e^a e^{ib} \).
- Euler’s formula: \( e^{ib} = \cos(b) + i \sin(b) \).
- Hence, a complex number of the form \( e^{a+ib} \) can be written as
  \[
  e^{a+ib} = e^a [\cos(b) + i \sin(b)], \quad e^{a-ib} = e^a [\cos(b) - i \sin(b)].
  \]
- From \( e^{a+ib} \) and \( e^{a-ib} \) we get the real numbers
  \[
  \frac{1}{2} (e^{a+ib} + e^{a-ib}) = e^a \cos(b), \quad \frac{1}{2i} (e^{a+ib} - e^{a-ib}) = e^a \sin(b).
  \]
Two main sets of fundamental solutions.

Theorem (Complex roots)

If the constants $a_1, a_0 \in \mathbb{R}$ satisfy that $a_1^2 - 4a_0 < 0$, then the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$ of the equation

$$y'' + a_1 y' + a_0 y = 0 \quad (1)$$

has complex roots $r_+ = \alpha + i\beta$ and $r_- = \alpha - i\beta$, where

$$\alpha = -\frac{a_1}{2}, \quad \beta = \frac{1}{2} \sqrt{4a_0 - a_1^2}.$$ 

Furthermore, a fundamental set of solutions to Eq. (1) is

$$\tilde{y}_1(t) = e^{(\alpha+i\beta)t}, \quad \tilde{y}_2(t) = e^{(\alpha-i\beta)t},$$

while another fundamental set of solutions to Eq. (1) is

$$y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).$$

Review of complex numbers.

Idea of the Proof: Recall that the functions

$$\tilde{y}_1(t) = e^{(\alpha+i\beta)t}, \quad \tilde{y}_2(t) = e^{(\alpha-i\beta)t},$$

are solutions to $y'' + a_1 y' + a_0 y = 0$. Also recall that

$$\tilde{y}_1(t) = e^{\alpha t} [\cos(\beta t) + i \sin(\beta t)], \quad \tilde{y}_2(t) = e^{\alpha t} [\cos(\beta t) - i \sin(\beta t)].$$

Then the functions

$$y_1(t) = \frac{1}{2} (\tilde{y}_1(t) + \tilde{y}_2(t)) \quad y_2(t) = \frac{1}{2i} (\tilde{y}_1(t) - \tilde{y}_2(t))$$

are also solutions to the same differential equation. We conclude that $y_1$ and $y_2$ are real valued and

$$y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).$$
Second order linear homogeneous ODE (Sect. 2.3).

▶ Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.
▶ **Characteristic polynomial with complex roots.**
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  ▶ Review of Complex numbers.
  ▶ **A real-valued fundamental and general solutions.**
▶ Application: The RLC circuit.

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A real-valued fundamental and general solutions.

**Example**
Find the real-valued general solution of the equation

$$y'' - 2y' + 6y = 0.$$ 

**Solution:** Recall: Complex valued solutions are

$$\tilde{y}_1(t) = e^{(1+i\sqrt{5})t}, \quad \tilde{y}_2(t) = e^{(1-i\sqrt{5})t}.$$

Any linear combination of these functions is solution of the differential equation. In particular,

$$y_1(t) = \frac{1}{2} [\tilde{y}_1(t) + \tilde{y}_2(t)], \quad y_2(t) = \frac{1}{2i} [\tilde{y}_1(t) - \tilde{y}_2(t)].$$

Now, recalling $e^{(1\pm i\sqrt{5})t} = e^t e^{\pm i\sqrt{5}t}$

$$y_1(t) = \frac{1}{2} [e^t e^{i\sqrt{5}t} + e^t e^{-i\sqrt{5}t}], \quad y_2(t) = \frac{1}{2i} [e^t e^{i\sqrt{5}t} - e^t e^{-i\sqrt{5}t}],$$
A real-valued fundamental and general solutions.

Example
Find the real-valued general solution of the equation
\[ y'' - 2y' + 6y = 0. \]

Solution: \[ y_1 = \frac{e^t}{2} [e^{i\sqrt{5}t} + e^{-i\sqrt{5}t}], \quad y_2 = \frac{e^t}{2i} [e^{i\sqrt{5}t} - e^{-i\sqrt{5}t}]. \]

The Euler formula and its complex-conjugate formula
\[ e^{i\sqrt{5}t} = [\cos(\sqrt{5}t) + i\sin(\sqrt{5}t)], \]
\[ e^{-i\sqrt{5}t} = [\cos(\sqrt{5}t) - i\sin(\sqrt{5}t)], \]

imply the inverse relations
\[ e^{i\sqrt{5}t} + e^{-i\sqrt{5}t} = 2\cos(\sqrt{5}t), \quad e^{i\sqrt{5}t} - e^{-i\sqrt{5}t} = 2i\sin(\sqrt{5}t). \]

So functions \( y_1 \) and \( y_2 \) can be written as
\[ y_1(t) = e^t \cos(\sqrt{5}t), \quad y_2(t) = e^t \sin(\sqrt{5}t). \]

A real-valued fundamental and general solutions.

Example
Find the real-valued general solution of the equation
\[ y'' - 2y' + 6y = 0. \]

Solution: Recall: \( y(t) = \bar{c}_1 e^{(1+i\sqrt{5})t} + \bar{c}_2 e^{(1-i\sqrt{5})t}, \quad \bar{c}_1, \bar{c}_2 \in \mathbb{C}. \)

The calculation above says that a real-valued fundamental set is
\[ y_1(t) = e^t \cos(\sqrt{5}t), \quad y_2(t) = e^t \sin(\sqrt{5}t). \]

Hence, the complex-valued general solution can also be written as
\[ y(t) = [c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)] e^t, \quad c_1, c_2 \in \mathbb{C}. \]

The real-valued general solution is simple to obtain:
\[ y(t) = [c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)] e^t, \quad c_1, c_2 \in \mathbb{R}. \]

We just restricted the coefficients \( c_1, c_2 \) to be real-valued. ◄
A real-valued fundamental and general solutions.

Example

Show that \( y_1(t) = e^t \cos(\sqrt{5} t) \) and \( y_2(t) = e^t \sin(\sqrt{5} t) \) are fundamental solutions to the equation \( y'' - 2y' + 6y = 0 \).

Solution: \( y_1(t) = e^t \cos(\sqrt{5} t), \ y_2(t) = e^t \sin(\sqrt{5} t) \).

Summary:

- These functions are solutions of the differential equation.
- They are not proportional to each other, Hence li.
- Therefore, \( y_1, y_2 \) form a fundamental set.
- The general solution of the equation is
  \[
  y(t) = [c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)] e^t.
  \]
- \( y \) is real-valued for \( c_1, c_2 \in \mathbb{R} \).
- \( y \) is complex-valued for \( c_1, c_2 \in \mathbb{C} \).

A real-valued fundamental and general solutions.

Example

Find real-valued fundamental solutions to the equation

\[
y'' + 2y' + 6y = 0.
\]

Solution:

The roots of the characteristic polynomial \( p(r) = r^2 + 2r + 6 \) are

\[
r_{\pm} = \frac{1}{2} [-2 \pm \sqrt{4 - 24}] = \frac{1}{2} [-2 \pm \sqrt{-20}] \Rightarrow r_{\pm} = -1 \pm i\sqrt{5}.
\]

These are complex-valued roots, with

\[
\alpha = -1, \quad \beta = \sqrt{5}.
\]

Real-valued fundamental solutions are

\[
y_1(t) = e^{-t} \cos(\sqrt{5} t), \quad y_2(t) = e^{-t} \sin(\sqrt{5} t) \quad \triangleleft
\]
A real-valued fundamental and general solutions.

Example
Find real-valued fundamental solutions to the equation
\[ y'' + 2y' + 6y = 0. \]

Solution: \( y_1(t) = e^{-t} \cos(\sqrt{5} \, t), \quad y_2(t) = e^{-t} \sin(\sqrt{5} \, t). \)

Differential equations like the one in this example describe physical processes related to damped oscillations. For example, pendulums with friction.

A real-valued fundamental and general solutions.

Example
Find the real-valued general solution of \( y'' + 5y = 0. \)

Solution: The characteristic polynomial is \( p(r) = r^2 + 5. \)

Its roots are \( r_{\pm} = \pm \sqrt{5} \, i. \) This is the case \( \alpha = 0, \) and \( \beta = \sqrt{5}. \)

Real-valued fundamental solutions are
\[ y_1(t) = \cos(\sqrt{5} \, t), \quad y_2(t) = \sin(\sqrt{5} \, t). \]

The real-valued general solution is
\[ y(t) = c_1 \cos(\sqrt{5} \, t) + c_2 \sin(\sqrt{5} \, t), \quad c_1, \, c_2 \in \mathbb{R}. \]

Remark: Equations like the one in this example describe oscillatory physical processes without dissipation, \( \alpha = 0. \)
Second order linear homogeneous ODE (Sect. 2.3).

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- **Application: The RLC circuit.**

Application: The RLC circuit.

Consider an electric circuit with resistance \( R \), non-zero capacitor \( C \), and non-zero inductance \( L \), as in the figure.

The electric current flowing in such circuit satisfies:

\[
L I''(t) + R I'(t) + \frac{1}{C} \int_{t_0}^{t} I(s) \, ds = 0.
\]

Derivate both sides above: \( L I''(t) + R I'(t) + \frac{1}{C} I(t) = 0 \).

Divide by \( L \): \( I''(t) + 2 \left( \frac{R}{2L} \right) I'(t) + \frac{1}{LC} I(t) = 0 \).

Introduce \( \alpha = \frac{R}{2L} \) and \( \omega = \frac{1}{\sqrt{LC}} \), then \( I'' + 2\alpha I' + \omega^2 I = 0 \).
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$. The roots are:

$$r_\pm = \frac{1}{2} [-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}] \Rightarrow r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.$$  

Case (a) $R = 0$. This implies $\alpha = 0$, so $r_\pm = \pm i\omega$. Therefore,

$$I_1(t) = \cos(\omega t), \quad I_2(t) = \sin(\omega t).$$

Remark: When the circuit has no resistance, the current oscillates without dissipation.

Application: The RLC circuit.

Example
Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: Recall: $r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$.

Case (b) $R < \sqrt{4L/C}$. This implies

$$R^2 < \frac{4L}{C} \iff \frac{R^2}{4L^2} < \frac{1}{LC} \iff \alpha^2 < \omega^2.$$  

Therefore, $r_\pm = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$. The fundamental solutions are

$$I_1(t) = e^{-\alpha t} \cos(\sqrt{\omega^2 - \alpha^2} t), \quad I_2(t) = e^{-\alpha t} \sin(\sqrt{\omega^2 - \alpha^2} t).$$

The resistance $R$ damps the current oscillations.