Exact equations (Sect. 1.4).

- Exact differential equations.
- The Poincaré Lemma.
- Implicit solutions and the potential function.
- Generalization: The integrating factor method.

Exact differential equations.

**Definition**

The differential equation in the unknown function \( y : (t_1, t_2) \to \mathbb{R} \)

\[
N(t, y(t)) y'(t) + M(t, y(t)) = 0
\]

is called **exact** in an open rectangle \( R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2 \) iff for every point \( (t, u) \in R \) the functions \( M, N : R \to \mathbb{R} \) are continuously differentiable and satisfy the equation

\[
\partial_t N(t, u) = \partial_u M(t, u)
\]

**Remark:** We use the notation: \( \partial_t N = \frac{\partial N}{\partial t} \), and \( \partial_u M = \frac{\partial M}{\partial u} \).
Exact differential equations.

Example
Show whether the differential equation below is exact,

\[ 2ty(t) y'(t) + 2t + y^2(t) = 0. \]

Solution: We first identify the functions \( N \) and \( M \),

\[ [2ty(t)] y'(t) + [2t + y^2(t)] = 0 \quad \Rightarrow \quad \begin{cases} N(t, u) = 2tu, \\ M(t, u) = 2t + u^2. \end{cases} \]

The equation is exact iff \( \partial_t N = \partial_u M \). Since

\[ N(t, u) = 2tu \quad \Rightarrow \quad \partial_t N(t, u) = 2u, \]
\[ M(t, u) = 2t + u^2 \quad \Rightarrow \quad \partial_u M(t, u) = 2u. \]

We conclude: \( \partial_t N(t, u) = \partial_u M(t, u) \).

Remark: The ODE above is not separable and non-linear.

Exact differential equations.

Example
Show whether the differential equation below is exact,

\[ \sin(t)y'(t) + t^2e^{y(t)}y'(t) - y'(t) = -y(t) \cos(t) - 2te^{y(t)}. \]

Solution: We first identify the functions \( N \) and \( M \), if we write

\[ [\sin(t) + t^2e^{y(t)} - 1] y'(t) + [y(t) \cos(t) + 2te^{y(t)}] = 0, \]

we can see that

\[ N(t, u) = \sin(t) + t^2e^u - 1 \quad \Rightarrow \quad \partial_t N(t, u) = \cos(t) + 2te^u, \]
\[ M(t, u) = u \cos(t) + 2te^u \quad \Rightarrow \quad \partial_u M(t, u) = \cos(t) + 2te^u. \]

The equation is exact, since \( \partial_t N(t, u) = \partial_u M(t, u) \).
**Exact differential equations.**

**Example**
Show whether the linear differential equation below is exact,

\[ y'(t) = -a(t)y(t) + b(t), \quad a(t) \neq 0. \]

**Solution:** We first find the functions \( N \) and \( M \),

\[ y' + a(t)y - b(t) = 0 \quad \Rightarrow \quad \begin{cases} N(t, u) = 1, \\ M(t, u) = a(t)u - b(t). \end{cases} \]

The differential equation is not exact, since

\[ N(t, u) = 1 \quad \Rightarrow \quad \partial_t N(t, u) = 0, \]
\[ M(t, u) = a(t)u - b(t) \quad \Rightarrow \quad \partial_u M(t, u) = a(t). \]

This implies that \( \partial_t N(t, u) \neq \partial_u M(t, u) \).

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**Exact equations (Sect. 1.4).**

- **Exact differential equations.**
- **The Poincaré Lemma.**
- Implicit solutions and the potential function.
- Generalization: The integrating factor method.
The Poincaré Lemma.

Remark: The coefficients \( N \) and \( M \) of an exact equations are the derivatives of a potential function \( \psi \).

Lemma (Poincaré)

Continuously differentiable functions \( M, N : \mathbb{R} \rightarrow \mathbb{R} \), on an open rectangle \( R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2 \), satisfy the equation

\[
\partial_t N(t, u) = \partial_u M(t, u)
\]

iff there exists a twice continuously differentiable function \( \psi : \mathbb{R} \rightarrow \mathbb{R} \), called potential function, such that for all \( (t, u) \in R \) holds

\[
\begin{align*}
\partial_u \psi(t, u) &= N(t, u), \\
\partial_t \psi(t, u) &= M(t, u).
\end{align*}
\]

Proof: (\( \Leftarrow \)) Simple: \( \begin{cases} 
\partial_t N = \partial_t \partial_u \psi, \\
\partial_u M = \partial_u \partial_t \psi,
\end{cases} \Rightarrow \partial_t N = \partial_u M. \)

(\( \Rightarrow \)) Difficult: Poincaré, 1880.

The Poincaré Lemma.

Example

Show that the function \( \psi(t, u) = t^2 + tu^2 \) is the potential function for the exact differential equation

\[
2ty(t)y'(t) + 2t + y^2(t) = 0.
\]

Solution: We already saw that the differential equation above is exact, since the functions \( M \) and \( N \),

\[
\begin{align*}
N(t, u) &= 2tu, \\
M(t, u) &= 2t + u^2
\end{align*}
\]

\( \Rightarrow \) \( \partial_t N = 2u = \partial_u M. \)

The potential function is \( \psi(t, u) = t^2 + tu^2 \), since

\[
\begin{align*}
\partial_t \psi &= 2t + u^2 = M, \\
\partial_u \psi &= 2tu = N.
\end{align*}
\]

Remark: The Poincaré Lemma only states necessary and sufficient conditions on \( N \) and \( M \) for the existence of \( \psi \).
Exact equations (Sect. 1.4).

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Theorem (Exact differential equations)

If the differential equation

\[ N(t, y(t)) y'(t) + M(t, y(t)) = 0 \quad (1) \]

is exact on \( R = (t_1, t_2) \times (u_1, u_2) \), then every solution \( y \) must satisfy the algebraic equation

\[ \psi(t, y(t)) = c, \]

where \( c \in \mathbb{R} \) and \( \psi : R \rightarrow \mathbb{R} \) is a potential function for Eq. (1).

Proof: \( 0 = N(t, y) y' + M(t, y) = \partial_y \psi(t, y) \frac{dy}{dt} + \partial_t \psi(t, y). \)

\[ 0 = \frac{d}{dt} \psi(t, y(t)) \quad \Leftrightarrow \quad \psi(t, y(t)) = c. \]
Implicit solutions and the potential function.

Example
Find all solutions $y$ to the equation

$$[\sin(t) + t^2 e^{y(t)} - 1] y'(t) + y(t) \cos(t) + 2te^{y(t)} = 0.$$ 

Solution: Recall: The equation is exact,

$$N(t, u) = \sin(t) + t^2 e^u - 1 \quad \Rightarrow \quad \partial_t N(t, u) = \cos(t) + 2te^u,$$

$$M(t, u) = u \cos(t) + 2te^u \quad \Rightarrow \quad \partial_u M(t, u) = \cos(t) + 2te^u,$$

hence, $\partial_t N = \partial_u M$. Poincaré Lemma says the exists $\psi$,

$$\partial_u \psi(t, u) = N(t, u), \quad \partial_t \psi(t, u) = M(t, u).$$

These are actually equations for $\psi$. From the first one,

$$\psi(t, u) = \int [\sin(t) + t^2 e^u - 1] \ du + g(t).$$

Integrating,

$$\psi(t, u) = u \sin(t) + t^2 e^u - u + g(t).$$

Introduce this expression into $\partial_t \psi(t, u) = M(t, u)$, that is,

$$\partial_t \psi(t, u) = u \cos(t) + 2te^u + g'(t) = M(t, u) = u \cos(t) + 2te^u,$$

Therefore, $g'(t) = 0$, so we choose $g(t) = 0$. We obtain,

$$\psi(t, u) = u \sin(t) + t^2 e^u - u.$$

So the solution $y$ satisfies $y(t) \sin(t) + t^2 e^{y(t)} - y(t) = c. \quad \triangle$
Exact equations (Sect. 1.4).

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- **Generalization: The integrating factor method.**

**Remark:**
Sometimes a non-exact equation can we transformed into an exact equation multiplying the equation by an integrating factor. Just like in the case of linear differential equations.

### Generalization: The integrating factor method.

**Theorem (Integrating factor)**

Assume that the differential equation

\[ N(t, y(t)) y'(t) + M(t, y(t)) = 0 \]

is not exact in the sense that the continuously differentiable functions \(M, N : \mathbb{R} \to \mathbb{R}\) satisfy \(\partial_t N(t, u) \neq \partial_u M(t, u)\) on \(R = (t_1, t_2) \times (u_1, u_2)\). If \(N \neq 0\) and the function

\[
\frac{1}{N(t, u)} \left[ \partial_u M(t, u) - \partial_t N(t, u) \right]
\]

does not depend on the variable \(u\), then the equation

\[
\mu(t) N(t, y(t)) y'(t) + \mu(t) M(t, y(t)) = 0
\]

is exact, where \(\frac{\mu'(t)}{\mu(t)} = \frac{1}{N(t, u)} \left[ \partial_u M(t, u) - \partial_t N(t, u) \right]\).
Generalization: The integrating factor method.

Example
Find all solutions \( y \) to the differential equation
\[
\left[ t^2 + t y(t) \right] y'(t) + \left[ 3t y(t) + y^2(t) \right] = 0.
\]

Solution: The equation is not exact:
\[
N(t, u) = t^2 + tu \quad \Rightarrow \quad \partial_t N(t, u) = 2t + u,
\]
\[
M(t, u) = 3tu + u^2 \quad \Rightarrow \quad \partial_u M(t, u) = 3t + 2u,
\]
hence \( \partial_t N \neq \partial_u M \). We now verify whether the extra condition in Theorem above holds:
\[
\frac{[\partial_u M(t, u) - \partial_t N(t, u)]}{N(t, u)} = \frac{1}{(t^2 + tu)} \left[ (3t + 2u) - (2t + u) \right]
\]
\[
= \frac{1}{t(t + u)} (t + u) = \frac{1}{t}.
\]

Generalization: The integrating factor method.

Example
Find all solutions \( y \) to the differential equation
\[
\left[ t^2 + t y(t) \right] y'(t) + \left[ 3t y(t) + y^2(t) \right] = 0.
\]

Solution: \[
\frac{[\partial_u M(t, u) - \partial_t N(t, u)]}{N(t, u)} = \frac{1}{t}.
\]

We find a function \( \mu \) solution of \( \frac{\mu'}{\mu} = \frac{[\partial_u M - \partial_t N]}{N} \), that is
\[
\frac{\mu'(t)}{\mu(t)} = \frac{1}{t} \quad \Rightarrow \quad \ln(\mu(t)) = \ln(t) \quad \Rightarrow \quad \mu(t) = t.
\]

Therefore, the equation below is exact:
\[
\left[ t^3 + t^2 y(t) \right] y'(t) + \left[ 3t^2 y(t) + t y^2(t) \right] = 0.
\]
Example
Find all solutions $y$ to the differential equation
\[
[t^2 + t \, y(t)] \, y'(t) + [3t \, y(t) + y^2(t)] = 0.
\]
Solution: $[t^3 + t^2 \, y(t)] \, y'(t) + [3t^2 \, y(t) + t \, y^2(t)] = 0$.

This equation is exact:
\[
\tilde{N}(t, u) = t^3 + t^2 \, u \quad \Rightarrow \quad \partial_t \tilde{N}(t, u) = 3t^2 + 2tu,
\]
\[
\tilde{M}(t, u) = 3t^2 \, u + tu^2 \quad \Rightarrow \quad \partial_u \tilde{M}(t, u) = 3t^2 + 2tu,
\]
that is, $\partial_t \tilde{N} = \partial_u \tilde{M}$. Therefore, there exists $\psi$ such that
\[
\partial_u \psi(t, u) = \tilde{N}(t, u), \quad \partial_t \psi(t, u) = \tilde{M}(t, u).
\]
From the first equation above we obtain
\[
\partial_u \psi = t^3 + t^2 \, u \quad \Rightarrow \quad \psi(t, u) = \int (t^3 + t^2 \, u) \, du + g(t).
\]

Integrating, $\psi(t, u) = t^3 \, u + \frac{1}{2} \, t^2 \, u^2 + g(t)$.

Introduce $\psi$ in $\partial_t \psi = \tilde{M}$, where $\tilde{M} = 3t^2 \, u + tu^2$. So,
\[
\partial_t \psi(t, u) = 3t^2 \, u + tu^2 + g'(t) = \tilde{M}(t, u) = 3t^2 \, u + tu^2,
\]
So $g'(t) = 0$ and we choose $g(t) = 0$. We conclude that a potential function is $\psi(t, u) = t^3 \, u + \frac{1}{2} \, t^2 \, u^2$.

And every solution $y$ satisfies
\[
t^3 \, y(t) + \frac{1}{2} \, t^2 \, [y(t)]^2 = c. \quad \triangleleft