# Review for Final Exam.

- Exam is cumulative.
- Heat equation not included.
- 15 problems.
- Two and half hours.

- Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

# Fourier Series: Even/Odd-periodic extensions.

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We conclude: $f(x) = -\dfrac{4}{\pi}\displaystyle\sum_{k=1}^{\infty}\dfrac{1}{(2k-1)}\sin[(2k-1)\pi x].$ $\triangleleft$

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$$b_n = \frac{-4}{n\pi}\big[\cos(n\pi) - 1\big] + \Big[\frac{4}{n\pi} \cos(n\pi) - 0\Big] \quad \Rightarrow \quad b_n = \frac{4}{n\pi}.$$

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Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \dfrac{-2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) - \displaystyle\int \Big(\dfrac{-2}{n\pi}\Big) \cos\Big(\dfrac{n\pi x}{2}\Big)\, dx.$

$I = -\dfrac{2x}{n\pi} \cos\Big(\dfrac{n\pi x}{2}\Big) + \Big(\dfrac{2}{n\pi}\Big)^2 \sin\Big(\dfrac{n\pi x}{2}\Big).$ So, we get

$$b_n = 2\Big[\frac{-2}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 + \Big[\frac{2x}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big(\frac{2}{n\pi}\Big)^2 \sin\Big(\frac{n\pi x}{2}\Big)\Big|_0^2$$

$$b_n = \frac{-4}{n\pi}\big[\cos(n\pi) - 1\big] + \Big[\frac{4}{n\pi} \cos(n\pi) - 0\Big] \quad \Rightarrow \quad b_n = \frac{4}{n\pi}.$$

We conclude: $f(x) = \dfrac{4}{\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n} \sin\Big(\dfrac{n\pi x}{2}\Big).$ $\triangleleft$

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We conclude: $f(x) = 1 + \dfrac{8}{\pi^2}\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos\Big(\frac{(2k-1)\pi x}{2}\Big).$ $\triangleleft$

# Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- **Eigenvalue-Eigenfunction BVP (Chptr. 6).**
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

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Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

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$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that $r$ is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

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Then, for $n = 1, 2, \cdots$ holds

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$$y(x) = c_1 + c_2 x.$$

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Then, choosing $c_1 = 1$, holds,

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Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

$$0 = y'(8) = 0.$$

Then, choosing $c_1 = 1$, holds,

$$\lambda = 0, \qquad y_0(x) = 1.$$

$\triangleleft$

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# Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- **Systems of linear Equations (Chptr. 5).**
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

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## Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

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We conclude: $\mathbf{x}(t) = \frac{5}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$ $\triangleleft$

# Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- **Laplace transforms (Chptr. 4).**
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

# Laplace transforms.

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- Main Properties:

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- Partial fraction decompositions, completing the squares.

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Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

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$$\mathcal{L}[y] = \frac{(3s+2)}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

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Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad \mathcal{L}[y''] = s^2\,\mathcal{L}[y] - 3s - 2.$$

$$(s^2 + 9)\,\mathcal{L}[y] - 3s - 2 = \frac{e^{-5s}}{s}$$

$$\mathcal{L}[y] = \frac{(3s+2)}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

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Therefore, we conclude that,

$$y(t) = 3\cos(3t) + \frac{2}{3}\,\sin(3t) + \frac{u_5(t)}{9}\Big[1 - \cos(3(t-5))\Big].$$

$\triangleleft$

# Review for Final Exam.

- Fourier Series expansions (Chptr. 6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- **Second order linear equations (Chptr. 2).**
- First order differential equations (Chptr. 1).

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Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

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Non-homogeneous equations: $g \neq 0$.

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Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution $y_2$ not proportional to $y_1$.

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Choose $c_1 = 0$, $c_2 = 1$. Hence $y_2(x) = x^3$, and $y_1(x) = x^2$. $\triangleleft$

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## Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

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Then propose $y_p(t) = kt\,e^{-t}$.

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Since $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$, we obtain,

$$y(t) = \frac{1}{2}\left(e^{3t} + e^{-t}\right) - \frac{3}{4}\,t\,e^{-t}. \qquad \triangleleft$$

# Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- **First order differential equations (Chptr. 1).**

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6. Exact equation with integrating factor.
   (Very complicated to check.)

# First order differential equations.

## Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

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Find the solution $y$ to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

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Divide by $y^3$. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

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