Overview of Fourier Series (Sect. 6.2).

- Origins of the Fourier Series.
- Periodic functions.
- Orthogonality of Sines and Cosines.
- Main result on Fourier Series.

Origins of the Fourier Series.

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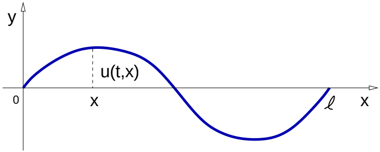

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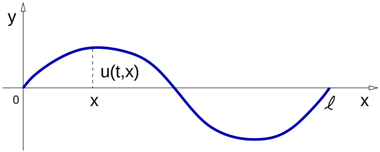

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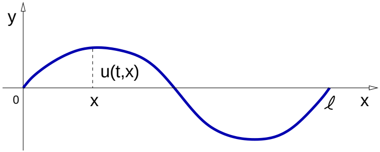

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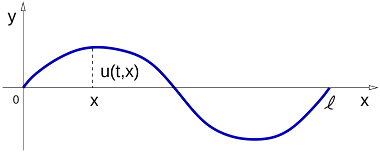

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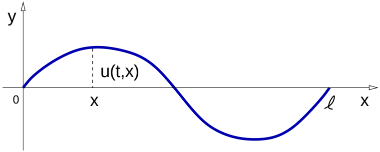

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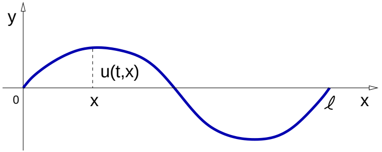

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Remark: The wave equation and its solutions provide a mathematical description of music.

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- More precisely: Every continuous, τ-periodic function F, there exist constants a_0, a_n, b_n, for $n = 1, 2, \cdots$ such that

$$F_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \Big[a_n \cos\Big(\frac{2n\pi x}{\tau}\Big) + b_n \sin\Big(\frac{2n\pi x}{\tau}\Big) \Big],$$

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Notation: $F(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{2n\pi x}{\tau}\Big) + b_n \sin\Big(\frac{2n\pi x}{\tau}\Big) \Big].$

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Given a continuous, τ-periodic function f, find the formulas for a_n and b_n such that

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Remarks: We need to review two main concepts:

- The notion of periodic functions.
- The notion of orthogonal functions, in particular the orthogonality of Sines and Cosines.

Fourier Series (Sect. 6.2).

- Origins of the Fourier Series.
- **Periodic functions.**
- Orthogonality of Sines and Cosines.
- Main result on Fourier Series.

Periodic functions.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* iff there exists $\tau > 0$ such that for all $x \in \mathbb{R}$ holds

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Notation:

A periodic function with period T is also called T-periodic.

Periodic functions.

Example

The following functions are periodic, with period T,

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$$f(x) = \cos(x), \qquad T = 2\pi.$$

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The proof of the latter statement is the following:

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The proof of the latter statement is the following:

$$f\left(x + \frac{2\pi}{a}\right)$$

Periodic functions.

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$\triangleleft$

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Show that the function below is periodic, and find its period,

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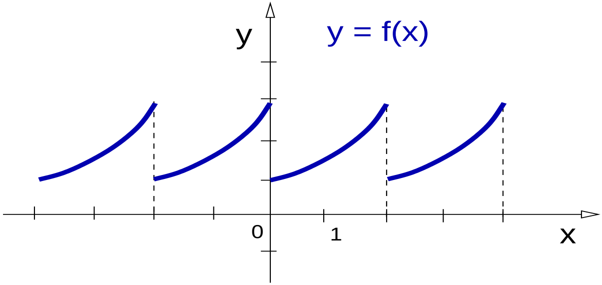

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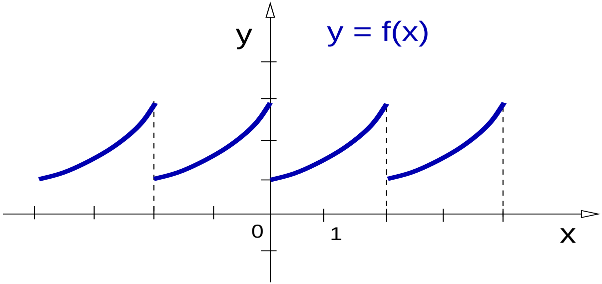

So the function is periodic with period $T = 2$. ◁

Periodic functions.

Theorem

A linear combination of T-periodic functions is also T-periodic.

Periodic functions.

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Proof: If $f(x+T) = f(x)$ and $g(x+T) = g(x)$, then

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Since f and g are invariant under translations by τ/n, they are also invariant under translations by τ.

Periodic functions.

Corollary

Any function f given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi x}{\tau}\right) + b_n \sin\left(\frac{2n\pi x}{\tau}\right) \right]$$

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Remark: We will show that the converse statement is true.

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Theorem

A function f is τ-periodic iff holds

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{2n\pi x}{\tau}\Big) + b_n \sin\Big(\frac{2n\pi x}{\tau}\Big) \Big].$$

Fourier Series (Sect. 6.2).

- Origins of the Fourier Series.
- Periodic functions.
- **Orthogonality of Sines and Cosines.**
- Main result on Fourier Series.

Orthogonality of Sines and Cosines.

Remark:
From now on we work on the following domain: $[-L, L]$.

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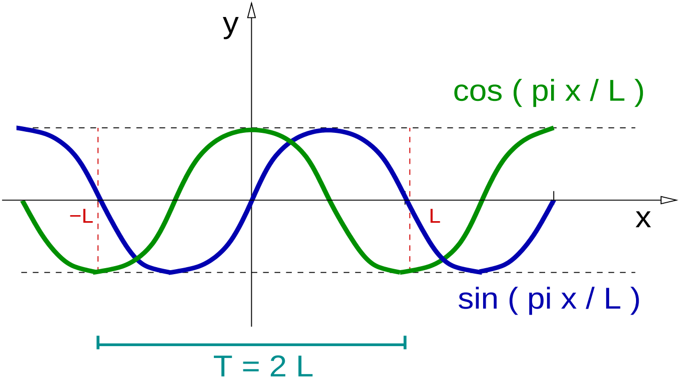

Orthogonality of Sines and Cosines.

Theorem (Orthogonality)

The following relations hold for all $n,\ m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\, \cos\Big(\frac{m\pi x}{L}\Big)\, dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big)\, \sin\Big(\frac{m\pi x}{L}\Big)\, dx = \begin{cases} 0 & n \neq m, \\ L & n = m, \end{cases}$$

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Remark:

- The operation $f \cdot g = \displaystyle\int_{-L}^{L} f(x)\,g(x)\,dx$ is an inner product in the vector space of functions.

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- The operation $f \cdot g = \displaystyle\int_{-L}^{L} f(x)\,g(x)\,dx$ is an inner product in the vector space of functions. Like the dot product is in $\mathbb{R}^2$.
- Two functions f, g, are orthogonal iff $f \cdot g = 0$.

Orthogonality of Sines and Cosines.

Recall:

$$\cos(\theta)\,\cos(\phi) = \frac{1}{2}\left[\cos(\theta + \phi) + \cos(\theta - \phi)\right];$$

$$\sin(\theta)\,\sin(\phi) = \frac{1}{2}\left[\cos(\theta - \phi) - \cos(\theta + \phi)\right];$$

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Orthogonality of Sines and Cosines.

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Proof: First formula: If $n = m = 0$, it is simple to see that

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \int_{-L}^{L} dx = 2L.$$

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In the case where one of n or m is non-zero, use the relation

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx$$
$$+\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

Orthogonality of Sines and Cosines.

Proof: Since one of n or m is non-zero,

Orthogonality of Sines and Cosines.

Proof: Since one of n or m is non-zero, holds

$$\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx = \frac{L}{2(n+m)\pi}\,\sin\Big[\frac{(n+m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

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We obtain that

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

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If we further restrict $n \neq m$, then

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If $n = m \neq 0$, we have that

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This establishes the first equation in the Theorem. The remaining equations are proven in a similar way. $\square$

Overview of Fourier Series (Sect. 6.2).

- Origins of the Fourier Series.
- Periodic functions.
- Orthogonality of Sines and Cosines.
- **Main result on Fourier Series.**

Main result on Fourier Series.

Theorem (Fourier Series)

If the function $f : [-L, L] \subset \mathbb{R} \to \mathbb{R}$ *is continuous, then* f *can be expressed as an infinite series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big] \qquad (1)$$

with the constants a_n *and* b_n *given by*

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

Furthermore, the Fourier series in Eq. (1) provides a $2L$*-periodic extension of* f *from the domain* $[-L, L] \subset \mathbb{R}$ *to* $\mathbb{R}$*.*

Examples of the Fourier Theorem (Sect. 6.2).

- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.

The Fourier Theorem: Continuous case.

Theorem (Fourier Series)

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with the constants a_n and b_n given by

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Furthermore, the Fourier series in Eq. (2) provides a $2L$-periodic extension of function f from the domain $[-L, L] \subset \mathbb{R}$ to $\mathbb{R}$.

The Fourier Theorem: Continuous case.

Sketch of the Proof:

- Define the partial sum functions

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big]$$

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- Express f_N as a convolution of Sine, Cosine, functions and the original function f.

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- Express f_N as a convolution of Sine, Cosine, functions and the original function f.
- Use the convolution properties to show that

$$\lim_{N\to\infty} f_N(x) = f(x), \qquad x \in [-L, L].$$

□

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- The Fourier Theorem: Continuous case.
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Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Example: Using the Fourier Theorem.

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x) \big],$$

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$$a_0 = \int_{-1}^{1} f(x)\, dx$$

Example: Using the Fourier Theorem.

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$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (1 + x)\,dx + \int_{0}^{1} (1 - x)\,dx.$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: In this case $L = 1$. The Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x)\big],$$

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$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (1 + x)\,dx + \int_{0}^{1} (1 - x)\,dx.$$

$$a_0 = \Big(x + \frac{x^2}{2}\Big)\Big|_{-1}^{0} + \Big(x - \frac{x^2}{2}\Big)\Big|_{0}^{1}$$

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We obtain: $a_0 = 1$.

Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$.

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Solution: Recall: $a_0 = 1$. Similarly, the rest of the a_n are given by,

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx$$

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Recall the integrals $\displaystyle\int \cos(n\pi x)\, dx = \frac{1}{n\pi} \sin(n\pi x),$

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Recall the integrals $\displaystyle\int \cos(n\pi x)\, dx = \frac{1}{n\pi} \sin(n\pi x),$ and

$$\int x \cos(n\pi x)\, dx = \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x).$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: It is not difficult to see that

$$a_n = \frac{1}{n\pi} \sin(n\pi x)\Big|_{-1}^{0} + \left[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x)\right]\Big|_{-1}^{0}$$
$$+ \frac{1}{n\pi} \sin(n\pi x)\Big|_{0}^{1} - \left[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x)\right]\Big|_{0}^{1}$$

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We then conclude that $a_n = \frac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]$.

Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

Solution: Recall: $a_0 = 1$, and $a_n = \dfrac{2}{n^2\pi^2}\big[1 - \cos(n\pi)\big]$.

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Finally, we must find the coefficients b_n.

Example: Using the Fourier Theorem.

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A similar calculation shows that $b_n = 0$.

Example: Using the Fourier Theorem.

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Finally, we must find the coefficients b_n.

A similar calculation shows that $b_n = 0$.

Then, the Fourier series of f is given by

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1-\cos(n\pi)\big]\,\cos(n\pi x).$$

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Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

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We can obtain a simpler expression for the Fourier coefficients a_n.

Example: Using the Fourier Theorem.

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Recall the relations $\cos(n\pi) = (-1)^n$,

Example: Using the Fourier Theorem.

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$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \big[1 + (-1)^{n+1}\big] \cos(n\pi x).$$

Example: Using the Fourier Theorem.

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If $n = 2k$,

Example: Using the Fourier Theorem.

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If $n = 2k$, so n is even,

Example: Using the Fourier Theorem.

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If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd,

Example: Using the Fourier Theorem.

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If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2\pi^2} (1 - 1)$$

Example: Using the Fourier Theorem.

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If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2\pi^2} (1 - 1) \quad \Rightarrow \quad a_{2k} = 0.$$

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If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

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If $n = 2k - 1$, so n is odd,

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If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

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If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

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If $n = 2k - 1$, so n is odd, so $n + 1 = 2k$ is even, then

$$a_{2k-1} = \frac{2}{(2k-1)^2\pi^2} (1 + 1)$$

Example: Using the Fourier Theorem.

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If $n = 2k$, so n is even, so $n + 1 = 2k + 1$ is odd, then

$$a_{2k} = \frac{2}{(2k)^2\pi^2}\,(1 - 1) \quad \Rightarrow \quad a_{2k} = 0.$$

If $n = 2k - 1$, so n is odd, so $n + 1 = 2k$ is even, then

$$a_{2k-1} = \frac{2}{(2k-1)^2\pi^2}\,(1 + 1) \quad \Rightarrow \quad a_{2k-1} = \frac{4}{(2k-1)^2\pi^2}.$$

Example: Using the Fourier Theorem.

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Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

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Recall: $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[1 + (-1)^{n+1}\right] \cos(n\pi x)$, and

$$a_{2k} = 0, \qquad a_{2k-1} = \frac{4}{(2k-1)^2\pi^2}.$$

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Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0), \\ 1 - x & x \in [0, 1]. \end{cases}$$

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Recall: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \dfrac{2}{n^2\pi^2}\left[1 + (-1)^{n+1}\right] \cos(n\pi x)$, and

$$a_{2k} = 0, \qquad a_{2k-1} = \frac{4}{(2k-1)^2\pi^2}.$$

We conclude: $f(x) = \dfrac{1}{2} + \displaystyle\sum_{k=1}^{\infty} \dfrac{4}{(2k-1)^2\pi^2} \cos((2k-1)\pi x)$. $\lhd$

Examples of the Fourier Theorem (Sect. 6.2).

- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- **The Fourier Theorem: Piecewise continuous case.**
- Example: Using the Fourier Theorem.

The Fourier Theorem: Piecewise continuous case.

Recall:

Definition

A function $f : [a, b] \to \mathbb{R}$ is called *piecewise continuous* iff holds,

(a) $[a, b]$ can be partitioned in a finite number of sub-intervals such that f is continuous on the interior of these sub-intervals.

(b) f has finite limits at the endpoints of all sub-intervals.

The Fourier Theorem: Piecewise continuous case.

Theorem (Fourier Series)

If $f : [-L, L] \subset \mathbb{R} \to \mathbb{R}$ is piecewise continuous, then the function

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big]$$

where a_n and b_n given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

satisfies that:

(a) $f_F(x) = f(x)$ *for all x where f is continuous;*

(b) $f_F(x_0) = \frac{1}{2}\Big[\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x) \Big]$ *for all x_0 where f is discontinuous.*

Examples of the Fourier Theorem (Sect. 6.2).

- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- **Example: Using the Fourier Theorem.**

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Example: Using the Fourier Theorem.

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Solution: We start computing the Fourier coefficients b_n;

Example: Using the Fourier Theorem.

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Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

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Solution: We start computing the Fourier coefficients b_n;

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx,$$

Example: Using the Fourier Theorem.

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$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \qquad L = 1,$$

$$b_n = \int_{-1}^{0} (-1) \sin\big(n\pi x\big) \, dx + \int_{0}^{1} (1) \sin\big(n\pi x\big) \, dx,$$

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$$b_n = \frac{(-1)}{n\pi} \Big[-\cos(n\pi x)\Big|_{-1}^{0} \Big] + \frac{1}{n\pi} \Big[-\cos(n\pi x)\Big|_{0}^{1} \Big],$$

Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: We start computing the Fourier coefficients b_n;

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad L = 1,$$

$$b_n = \int_{-1}^{0} (-1) \sin\big(n\pi x\big)\, dx + \int_{0}^{1} (1) \sin\big(n\pi x\big)\, dx,$$

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$$b_n = \frac{(-1)}{n\pi} \big[-1 + \cos(-n\pi) \big] + \frac{1}{n\pi} \big[-\cos(n\pi) + 1 \big].$$

Example: Using the Fourier Theorem.

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Solution: $b_n = \frac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \frac{1}{n\pi}\big[-\cos(n\pi) + 1\big].$

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Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

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$$b_n = \frac{1}{n\pi}\big[1 - \cos(-n\pi) - \cos(n\pi) + 1\big] = \frac{2}{n\pi}\big[1 - \cos(n\pi)\big],$$

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If $n = 2k$,

Example: Using the Fourier Theorem.

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If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[1 - (-1)^{2k}\big]$,

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If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[1 - (-1)^{2k}\big]$, hence $b_{2k} = 0$.

Example: Using the Fourier Theorem.

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If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[1 - (-1)^{2k}\big]$, hence $b_{2k} = 0$.

If $n = 2k - 1$,

Example: Using the Fourier Theorem.

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Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

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Solution: $b_n = \dfrac{(-1)}{n\pi}\big[-1 + \cos(-n\pi)\big] + \dfrac{1}{n\pi}\big[-\cos(n\pi) + 1\big]$.

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If $n = 2k$, then $b_{2k} = \dfrac{2}{2k\pi}\big[1 - (-1)^{2k}\big]$, hence $b_{2k} = 0$.

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hence $b_{2k} = \dfrac{4}{(2k-1)\pi}$.

Example: Using the Fourier Theorem.

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Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, and $b_{2k} = \dfrac{4}{(2k-1)\pi}$.

Example: Using the Fourier Theorem.

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Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

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$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx,$$

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Example: Using the Fourier Theorem.

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Example: Using the Fourier Theorem.

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1). \end{cases}$

and periodic with period $T = 2$.

Solution: Recall: $b_{2k} = 0$, $b_{2k} = \dfrac{4}{(2k-1)\pi}$, and $a_n = 0$.

Therefore, we conclude that

$$f_F(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin\big((2k-1)\pi\, x\big).$$

$\lhd$

Sine and Cosine Series (Sect. 6.2).

- Even, odd functions.
- Main properties of even, odd functions.
- Sine and cosine series.
- Even-periodic, odd-periodic extensions of functions.

Even, odd functions.

Definition

A function $f : [-L, L] \to \mathbb{R}$ is *even* iff for all $x \in [-L, L]$ holds

$$f(-x) = f(x).$$

A function $f : [-L, L] \to \mathbb{R}$ is *odd* iff for all $x \in [-L, L]$ holds

$$f(-x) = -f(x).$$

Even, odd functions.

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Remarks:

- The only function that is both odd and even is $f = 0$.

Even, odd functions.

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Remarks:

- The only function that is both odd and even is $f = 0$.
- Most functions are neither odd nor even.

Even, odd functions.

Example

Show that the function $f(x) = x^2$ is even on $[-L, L]$.

Even, odd functions.

Example

Show that the function $f(x) = x^2$ is even on $[-L, L]$.

Solution: The function is even, since

$$f(-x) = (-x)^2$$

Even, odd functions.

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Even, odd functions.

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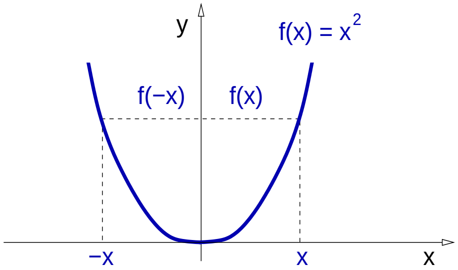

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Even, odd functions.

Example

Show that the function $f(x) = x^3$ is odd on $[-L, L]$.

Even, odd functions.

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Show that the function $f(x) = x^3$ is odd on $[-L, L]$.

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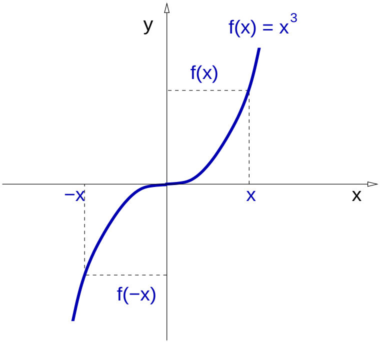

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Even, odd functions.

Example

(1) The function $f(x) = \cos(ax)$ is even on $[-L, L]$;

Even, odd functions.

Example

(1) The function $f(x) = \cos(ax)$ is even on $[-L, L]$;

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Even, odd functions.

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(3) The functions $f(x) = e^x$ and $f(x) = (x-2)^2$ are neither even nor odd.

Even, odd functions.

Example

(1) The function $f(x) = \cos(ax)$ is even on $[-L, L]$;

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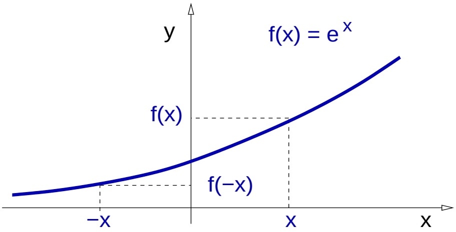

Even, odd functions.

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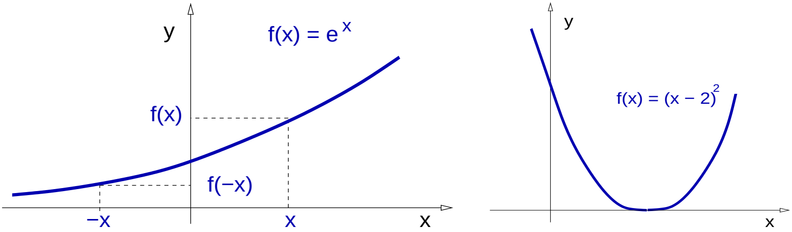

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Sine and Cosine Series (Sect. 6.2).

- Even, odd functions.
- **Main properties of even, odd functions.**
- Sine and cosine series.
- Even-periodic, odd-periodic extensions of functions.

Main properties of even, odd functions.

Theorem

(1) *A linear combination of even (odd) functions is even (odd).*

(2) *The product of two odd functions is even.*

(3) *The product of two even functions is even.*

(4) *The product of an even function by an odd function is odd.*

Main properties of even, odd functions.

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Proof:

(1) Let f and g be even,

Main properties of even, odd functions.

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Proof:

(1) Let f and g be even, that is, $f(-x) = f(x)$, $g(-x) = g(x)$.
Then, for all $a, b \in \mathbb{R}$ holds,

$(af + bg)(-x)$

Main properties of even, odd functions.

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Case "odd" is similar.

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Proof:

(2) Let f and g be odd, that is, $f(-x) = -f(x)$, $g(-x) = -g(x)$.

Main properties of even, odd functions.

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Proof:

(2) Let f and g be odd, that is, $f(-x) = -f(x)$, $g(-x) = -g(x)$. Then, for all $a, b \in \mathbb{R}$ holds,

$$(fg)(-x)$$

Main properties of even, odd functions.

Theorem

(1) *A linear combination of even (odd) functions is even (odd).*

(2) *The product of two odd functions is even.*

(3) *The product of two even functions is even.*

(4) *The product of an even function by an odd function is odd.*

Proof:

(2) Let f and g be odd, that is, $f(-x) = -f(x)$, $g(-x) = -g(x)$. Then, for all $a, b \in \mathbb{R}$ holds,

$$(fg)(-x) = f(-x)g(-x)$$

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Cases (3), (4) are similar. □

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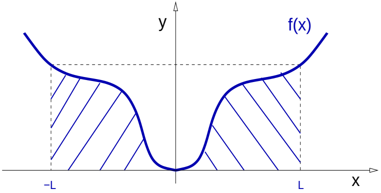

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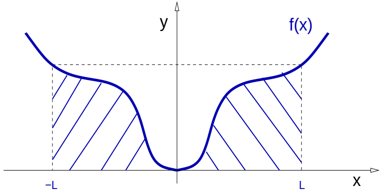

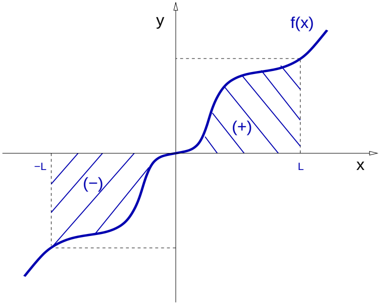

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Sine and Cosine Series (Sect. 6.2).

- Even, odd functions.
- Main properties of even, odd functions.
- **Sine and cosine series.**
- Even-periodic, odd-periodic extensions of functions.

Sine and cosine series.

Theorem (Cosine and Sine Series)

Consider the function $f : [-L, L] \to \mathbb{R}$ *with Fourier expansion*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right].$$

(1) *If* f *is even, then* $b_n = 0$ *for* $n = 1, 2, \cdots$ *, and the Fourier series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

is called a Cosine Series.

(2) *If* f *is odd, then* $a_n = 0$ *for* $n = 0, 1, \cdots$ *, and the Fourier series*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

is called a Sine Series.

Sine and cosine series.

Proof:

If f is even, and since the Sine function is odd,

Sine and cosine series.

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If f is even, and since the Sine function is odd, then

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0,$$

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Proof:

If f is even, and since the Sine function is odd, then

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\left(\frac{n\pi x}{L}\right)dx = 0,$$

since we are integrating an odd function on $[-L, L]$.

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Sine and Cosine Series (Sect. 6.2).

- Even, odd functions.
- Main properties of even, odd functions.
- Sine and cosine series.
- **Even-periodic, odd-periodic extensions of functions.**

Even-periodic, odd-periodic extensions of functions.

(1) Even-periodic case:
A function $f : [0, L] \to \mathbb{R}$ can be extended as an even function $f : [-L, L] \to \mathbb{R}$ requiring for $x \in [0, L]$ that

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Example

Sketch the graph of the even-periodic extension of $f(x) = x^5$, with $x \in [0, 1]$.

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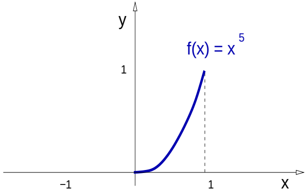

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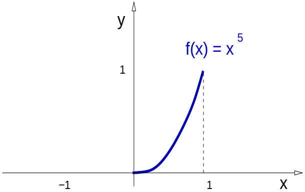

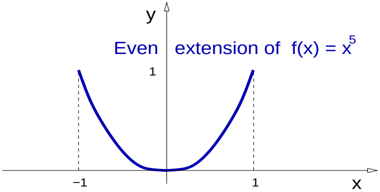

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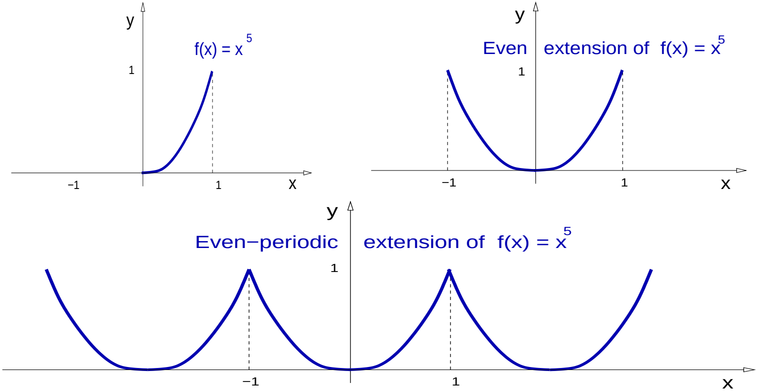

Even-periodic, odd-periodic extensions of functions.

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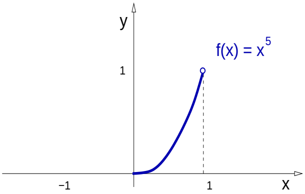

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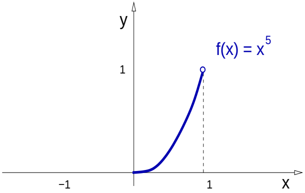

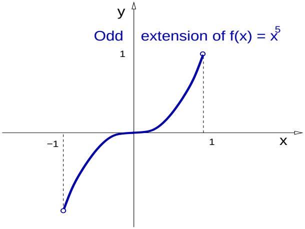

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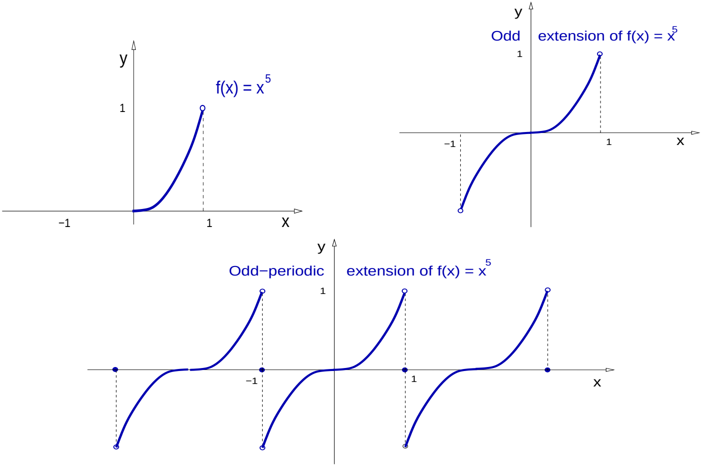

Even-periodic, odd-periodic extensions of functions.

Example

Sketch the graph of the even-periodic extension of $f(x) = x$, with $x \in [0, 1]$, and then find its Fourier Series.

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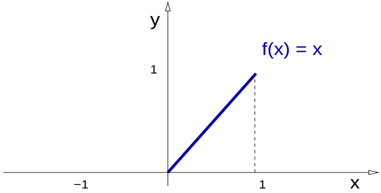

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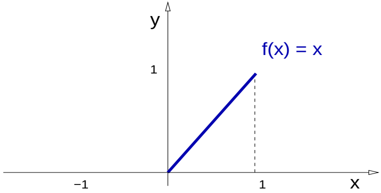

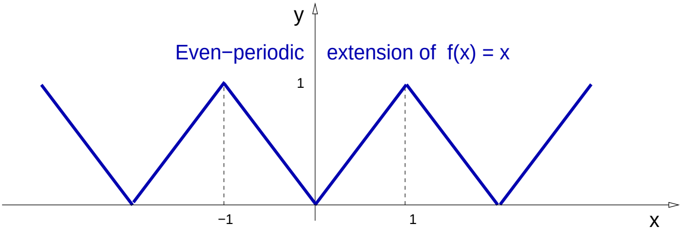

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Solution: Since f is even and periodic,

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Sketch the graph of the even-periodic extension of $f(x) = x$, with $x \in [0, 1]$, and then find its Fourier Series.

Solution: Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$. From the graph: $a_0 = 1$.

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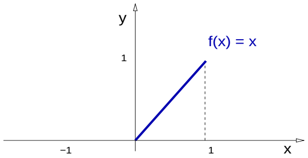

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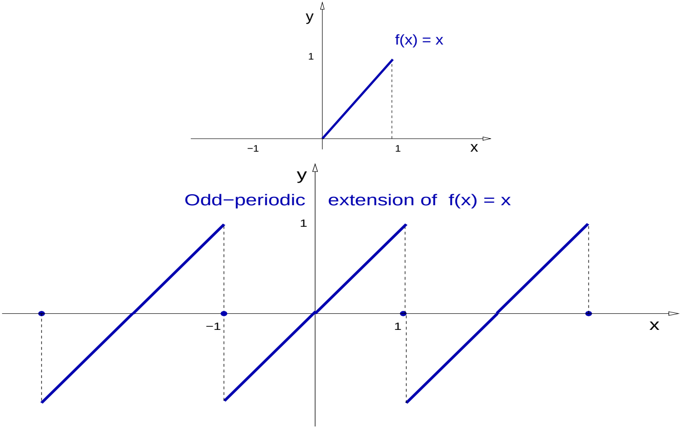

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$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n} \sin(n\pi x). \qquad \triangleleft$$