Review of Linear Algebra (Sect. 5.2, 5.3)

This Class:

- $n \times n$ systems of linear algebraic equations.
- The matrix-vector product.
- A matrix is a function.
- The inverse of a square matrix.
- The determinant of a square matrix.

Next Class:

- Eigenvalues, eigenvectors of a matrix.
- Computing eigenvalues and eigenvectors.
- Diagonalizable matrices.

$n \times n$ systems of linear algebraic equations.

Definition

An *$n \times n$ algebraic system of linear equations* is the following: Given constants a_{ij} and b_i, where indices $i, j = 1 \cdots, n \geqslant 1$, find the constants x_j solutions of the system

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1,$$
$$\vdots$$
$$a_{n1}x_1 + \cdots + a_{nn}x_n = b_n.$$

The system is called homogeneous iff the sources vanish, that is, $b_1 = \cdots = b_n = 0$.

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Example

$$2 \times 2: \quad \begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3. \end{aligned}$$

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2×2: $\quad \begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3. \end{aligned} \qquad 3 \times 3$: $\quad \begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ -3x_1 + x_2 + 3x_3 &= 24, \\ x_2 - 4x_3 &= -1. \end{aligned}$ ◁

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Introduce the coefficient matrix, the source vector, and the unknown vector, respectively,

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The solution is: $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $\lhd$

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- A matrix is a function, and matrix multiplication is equivalent to function composition.

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Example

Show that $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a rotation in $\mathbb{R}^2$ by $\pi/2$ counterclockwise.

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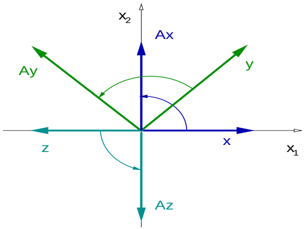
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \quad \triangleleft$$



A matrix is a function.

Definition

An $n \times n$ matrix I_n is called the identity matrix iff holds

$$I_n \mathbf{x} = \mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

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Example

Write down the identity matrices I_2, I_3, and I_n.

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Write down the identity matrices I_2, I_3, and I_n.

Solution:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad I_n = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}. \qquad \lhd$$

Review of Linear Algebra (Sect. 5.2, 5.3)

- $n \times n$ systems of linear algebraic equations.
- The matrix-vector product.
- A matrix is a function.
- **The inverse of a square matrix.**
- The determinant of a square matrix.

The inverse of a square matrix.

Definition

An $n \times n$ matrix A is called *invertible* iff there exists a matrix, denoted as A^{-1}, such

$$(A^{-1})A = I_n, \qquad A(A^{-1}) = I_n.$$

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Example

Show that $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ has the inverse $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$.

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$$A(A^{-1})$$

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Check that $(A^{-1})A = I_2$ also holds. $\triangleleft$

The inverse of a square matrix.

Remark: Not every $n \times n$ matrix is invertible.

The inverse of a square matrix.

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Theorem (2×2 case)

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is invertible iff holds that* $\Delta = ad - bc \neq 0$. *Furthermore, if* A *is invertible, then*

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Verify:

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Verify:

$A\left(A^{-1}\right)$

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$$A(A^{-1}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \Delta & -ab + ba \\ cd - dc & \Delta \end{bmatrix}$$

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It is not difficult to see that: $(A^{-1})A = I_2$ also holds.

The inverse of a square matrix.

Example

Find A^{-1} for $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

The inverse of a square matrix.

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Find A^{-1} for $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

Solution:

We use the formula in the previous Theorem.

The inverse of a square matrix.

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Find A^{-1} for $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

Solution:

We use the formula in the previous Theorem.

In this case: $\Delta = 6 - 2 = 4$,

The inverse of a square matrix.

Example

Find A^{-1} for $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

Solution:

We use the formula in the previous Theorem.

In this case: $\Delta = 6 - 2 = 4$, and

$$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse of a square matrix.

Example

Find A^{-1} for $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

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We use the formula in the previous Theorem.

In this case: $\Delta = 6 - 2 = 4$, and

$$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

$\triangleleft$

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Find A^{-1} for $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

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In this case: $\Delta = 6 - 2 = 4$, and

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$\triangleleft$

Remark: The formula for the inverse matrix can be generalized to $n \times n$ matrices having non-zero determinant.

Review of Linear Algebra (Sect. 5.2, 5.3)

- $n \times n$ systems of linear algebraic equations.
- The matrix-vector product.
- A matrix is a function.
- The inverse of a square matrix.
- **The determinant of a square matrix.**

The determinant of a square matrix.

Definition

The *determinant* of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the number

$$\Delta = ad - bc.$$

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Notation: The determinant can be denoted in different ways:

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$$\Delta = \det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

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Example

(a) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

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(b) $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$

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(a) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2.$

(b) $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3$

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(c) $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

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(c) $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0.$ $\lhd$

Remark: $\left| \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right|$ is the area of the parallelogram formed by the vectors

$$\begin{bmatrix} a \\ c \end{bmatrix} \text{ and } \begin{bmatrix} b \\ d \end{bmatrix}.$$

The determinant of a square matrix.

Definition

The *determinant* of a 3×3 matrix A is given by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

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Remark: The $|\det(A)|$ is the volume of the parallelepiped formed by the column vectors of A.

The determinant of a square matrix.

Example

Find the determinant of $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$.

The determinant of a square matrix.

Example

Find the determinant of $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$.

Solution: We use the definition above, that is,

$$\det(A) = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix},$$

The determinant of a square matrix.

Example

Find the determinant of $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$.

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$$\det(A) = (1 - 2) - 3(2 - 3) - (4 - 3)$$

The determinant of a square matrix.

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The determinant of a square matrix.

Example

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$$\det(A) = (1-2) - 3(2-3) - (4-3) = -1 + 3 - 1.$$

We conclude: $\det(A) = 1$. ◁

Linear Algebra and differential systems (Sect. 5.4, 5.5, 5.6)

- **Eigenvalues, eigenvectors of a matrix (5.5).**
- Computing eigenvalues and eigenvectors (5.5).
- Diagonalizable matrices (5.5).
- $n \times n$ linear differential systems (5.4).
- Constant coefficients homogenoues systems (5.6).
- Examples: 2×2 linear systems (5.6).

Eigenvalues, eigenvectors of a matrix

Definition

A number λ and a non-zero n-vector $\mathbf{v}$ are respectively called an *eigenvalue* and *eigenvector* of an $n \times n$ matrix A iff the following equation holds,

$$A\mathbf{v} = \lambda\mathbf{v}\,.$$

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Example

Verify that the pair $\lambda_1 = 4$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = -2$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvalue and eigenvector pairs of matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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Solution: $A\mathbf{v}_1$

Eigenvalues, eigenvectors of a matrix

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Example

Verify that the pair $\lambda_1 = 4$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = -2$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvalue and eigenvector pairs of matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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Eigenvalues, eigenvectors of a matrix

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Remarks:

- If we interpret an $n \times n$ matrix A as a function $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, then the eigenvector $\mathbf{v}$ determines a particular *direction* on $\mathbb{R}^n$ where the action of A is *simple*:

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Example

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

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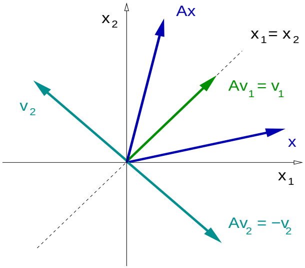

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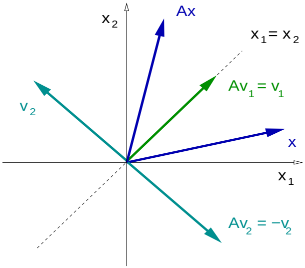

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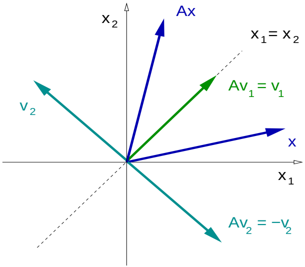

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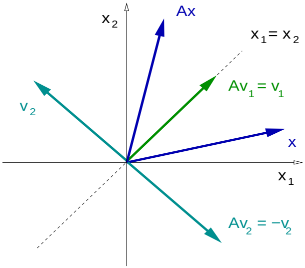

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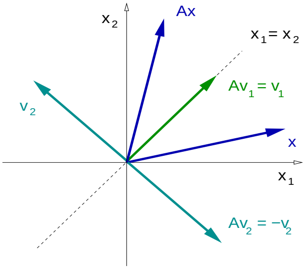

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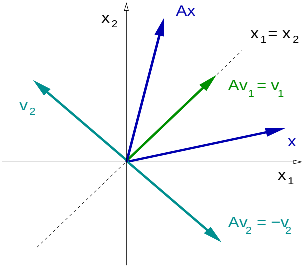

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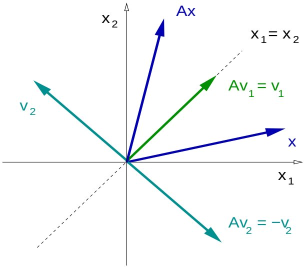

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An eigenvalue eigenvector pair is: $\lambda_1 = 1, \ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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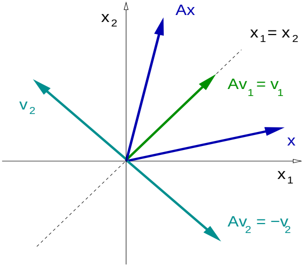

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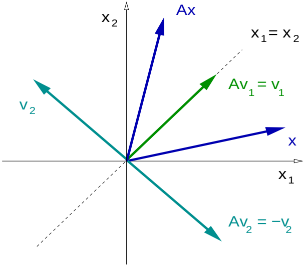

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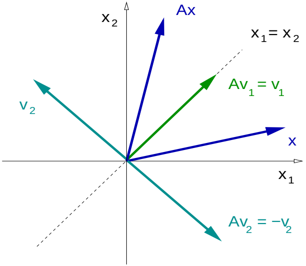

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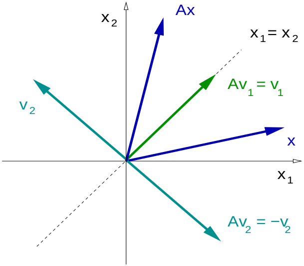

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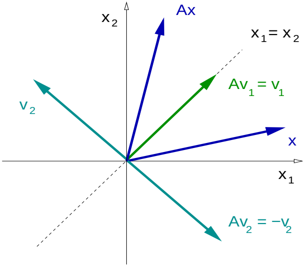

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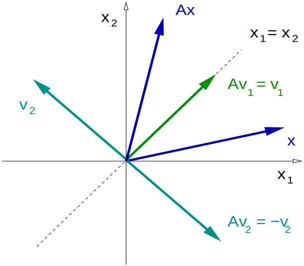

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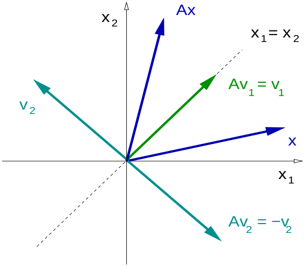

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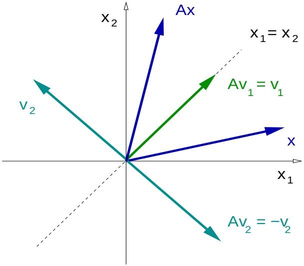

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A second eigenvector eigenvalue pair is:

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \lambda_2 = -1.$$

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Fix $\theta \in (0, \pi)$ and define $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.
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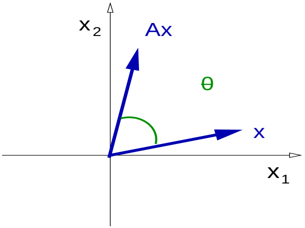

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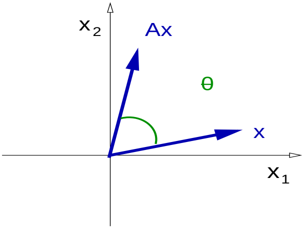

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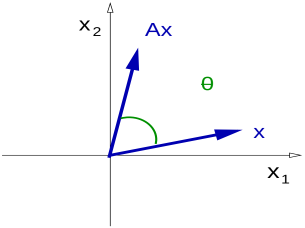

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Remark:

Matrix A has complex-values eigenvalues and eigenvectors.

Linear Algebra and differential systems (Sect. 5.4, 5.5, 5.6)

- Eigenvalues, eigenvectors of a matrix (5.5).
- **Computing eigenvalues and eigenvectors (5.5).**
- Diagonalizable matrices (5.5).
- $n \times n$ linear differential systems (5.4).
- Constant coefficients homogenoues systems (5.6).
- Examples: 2×2 linear systems (5.6).

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Given an $n \times n$ matrix A, find, if possible, λ and $\mathbf{v} \neq \mathbf{0}$ solution of

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Theorem (Eigenvalues-eigenvectors)

(a) *The number λ is an eigenvalue of an $n \times n$ matrix A iff*

$$\det(A - \lambda I) = 0.$$

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Find λ such that for a non-zero vector $\mathbf{v}$ holds,

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Since $(A - \lambda I)$ is not invertible, then $\det(A - \lambda I) = 0$.

Once λ is known, the original eigenvalue-eigenvector equation $A\mathbf{v} = \lambda \mathbf{v}$ is equivalent to $(A - \lambda I)\mathbf{v} = \mathbf{0}$. □

Computing eigenvalues and eigenvectors.

Example

Find the eigenvalues λ and eigenvectors $\mathbf{v}$ of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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The eigenvalues are the roots of the characteristic polynomial.

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Computing eigenvalues and eigenvectors.

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Find the eigenvalues λ and eigenvectors $\mathbf{v}$ of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^+ = v_2^+, \\ v_2^+ \quad \text{free.} \end{cases}$$

Al solutions to the equation above are then given by

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Computing eigenvalues and eigenvectors.

Example

Find the eigenvalues λ and eigenvectors $\mathbf{v}$ of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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The first eigenvalue eigenvector pair is $\lambda_+ = 4, \ \mathbf{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

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Solve $(A + 2I)\mathbf{v}_- = \mathbf{0}$, using Gauss operations on $A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$.

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The second eigenvalue eigenvector pair: $\lambda_- = -2$, $\mathbf{v}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $\lhd$

Linear Algebra and differential systems (Sect. 5.4, 5.5, 5.6)

- Eigenvalues, eigenvectors of a matrix (5.5).
- Computing eigenvalues and eigenvectors (5.5).
- **Diagonalizable matrices (5.5).**
- $n \times n$ linear differential systems (5.4).
- Constant coefficients homogenoues systems (5.6).
- Examples: 2×2 linear systems (5.6).

Diagonalizable matrices.

Definition

An $n \times n$ matrix D is called *diagonal* iff $D = \begin{bmatrix} d_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_{nn} \end{bmatrix}$.

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$$A = PDP^{-1}.$$

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- Systems of linear *differential* equations are simple to solve in the case that the coefficient matrix A is diagonalizable.

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- Systems of linear *differential* equations are simple to solve in the case that the coefficient matrix A is diagonalizable.
- In such case, it is simple to *decouple* the differential equations.
- One solves the decoupled equations, and then transforms back to the original unknowns.

Diagonalizable matrices.

Theorem (Diagonalizability and eigenvectors)

An $n \times n$ matrix A is diagonalizable iff matrix A has a linearly independent set of n eigenvectors. Furthermore,

$$A = PDP^{-1}, \quad P = [\mathbf{v}_1, \cdots, \mathbf{v}_n], \quad D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix},$$

where $\lambda_i, \mathbf{v}_i$, for $i = 1, \cdots, n$, are eigenvalue-eigenvector pairs of A.

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Remark: It is not simple to know whether an $n \times n$ matrix A has a linearly independent set of n eigenvectors. One simple case is given in the following result.

Theorem (n different eigenvalues)

If an $n \times n$ matrix A has n different eigenvalues, then A is diagonalizable.

Diagonalizable matrices.

Example

Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Diagonalizable matrices.

Example

Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Solution: We known that the eigenvalue eigenvector pairs are

$$\lambda_1 = 4, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -2, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

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Introduce P and D as follows,

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$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

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Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

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$$PDP^{-1} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Diagonalizable matrices.

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Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

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$$PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$PDP^{-1} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

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We conclude,

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We conclude,

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Diagonalizable matrices.

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We conclude,

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that is, A is diagonalizable. $\triangleleft$

Linear Algebra and differential systems (Sect. 5.4, 5.5, 5.6)

- Eigenvalues, eigenvectors of a matrix (5.5).
- Computing eigenvalues and eigenvectors (5.5).
- Diagonalizable matrices (5.5).
- **$n \times n$ linear differential systems (5.4).**
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- Examples: 2×2 linear systems (5.6).

$n \times n$ linear differential systems (5.4).

Definition

An *$n \times n$ linear differential system* is a the following: Given an $n \times n$ matrix-valued function A, and an n-vector-valued function $\mathbf{b}$, find an n-vector-valued function $\mathbf{x}$ solution of

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t).$$

The system above is called *homogeneous* iff holds $\mathbf{b} = 0$.

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$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t)$$

$n \times n$ linear differential systems (5.4).

Definition

An *$n \times n$ linear differential system* is a the following: Given an $n \times n$ matrix-valued function A, and an n-vector-valued function $\mathbf{b}$, find an n-vector-valued function $\mathbf{x}$ solution of

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t).$$

The system above is called *homogeneous* iff holds $\mathbf{b} = 0$.

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$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t) \;\Leftrightarrow\; \begin{aligned} x_1' &= a_{11}(t)\,x_1 + \cdots + a_{1n}(t)\,x_n + b_1(t) \\ &\vdots \\ x_n' &= a_{n1}(t)\,x_1 + \cdots + a_{nn}(t)\,x_n + b_n(t). \end{aligned}$$

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Example

Find the explicit expression for the linear system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ in the case that

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad \mathbf{b}(t) = \begin{bmatrix} e^{t} \\ 2e^{3t} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

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Solution: The 2×2 linear system is given by

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^t \\ 2e^{3t} \end{bmatrix}.$$

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That is,

$$x_1'(t) = x_1(t) + 3x_2(t) + e^t,$$
$$x_2'(t) = 3x_1(t) + x_2(t) + 2e^{3t}.$$

$\lhd$

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◁

Linear Algebra and differential systems (Sect. 5.4, 5.5, 5.6)

- Eigenvalues, eigenvectors of a matrix (5.5).
- Computing eigenvalues and eigenvectors (5.5).
- Diagonalizable matrices (5.5).
- $n \times n$ linear differential systems (5.4).
- **Constant coefficients homogenoues systems (5.6).**
- Examples: 2×2 linear systems (5.6).

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Summary:

- Given an $n \times n$ matrix $A(t)$, n-vector $\mathbf{b}(t)$, find $\mathbf{x}(t)$ solution
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Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to the homogeneous, constant coefficients, linear system

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is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

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Linear Algebra and differential systems (Sect. 5.4, 5.5, 5.6)

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Find the general solution to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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Solution: Find eigenvalues and eigenvectors of A.

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Solution: Find eigenvalues and eigenvectors of A. We found that:

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Fundamental solutions are

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$\triangleleft$

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Example

Verify that $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$, and $\mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$ are solutions to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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Solution: We compute $\mathbf{x}^{(1)\prime}$

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Solution: We compute $\mathbf{x}^{(1)\prime}$ and then we compare it with $A\mathbf{x}^{(1)}$,

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$$A\mathbf{x}^{(1)} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} e^{4t}$$

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Examples: 2×2 linear systems (5.6).

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Verify that $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$, and $\mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$ are solutions to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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We conclude that $\mathbf{x}^{(1)\prime} = A\mathbf{x}^{(1)}$.

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$$A\mathbf{x}^{(2)} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-2t}$$

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So, $A\mathbf{x}^{(2)} = -2\mathbf{x}^{(2)}$. Hence, $\mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)}$. ◁

Examples: 2×2 linear systems (5.6).

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Solve the IVP $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

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Solution: The general solution: $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$.

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Therefore, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, hence $\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$. $\lhd$

Constant coefficients homogenoues systems (5.6).

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Proof: Recall: $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, and $\mathbf{y}(t) = \begin{bmatrix} c_1\, e^{\lambda_1 t} \\ \vdots \\ c_n\, e^{\lambda_n t} \end{bmatrix}$.

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Transform back to $\mathbf{x}(t)$,

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Transform back to $\mathbf{x}(t)$, that is,

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Transform back to $\mathbf{x}(t)$, that is,

$$\mathbf{x}(t) = P \, \mathbf{y}(t) = \big[\mathbf{v}_1, \cdots, \mathbf{v}_n\big] \begin{bmatrix} c_1 \, e^{\lambda_1 t} \\ \vdots \\ c_n \, e^{\lambda_n t} \end{bmatrix}$$

Constant coefficients homogenoues systems (5.6).

Proof: Recall: $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, and $\mathbf{y}(t) = \begin{bmatrix} c_1\, e^{\lambda_1 t} \\ \vdots \\ c_n\, e^{\lambda_n t} \end{bmatrix}$.

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- The eigenvalues and eigenvectors of A are crucial to solve the differential linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$.