Convolution solutions (Sect. 4.5).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.
Convolution solutions (Sect. 4.5).

- **Convolution of two functions.**
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.
Convolution of two functions.

**Definition**

The *convolution* of piecewise continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ is the function $f \ast g : \mathbb{R} \to \mathbb{R}$ given by

$$(f \ast g)(t) = \int_{0}^{t} f(\tau)g(t - \tau) \, d\tau.$$
Definition
The \textit{convolution} of piecewise continuous functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ is the function $f \ast g : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$(f \ast g)(t) = \int_{0}^{t} f(\tau)g(t - \tau) \, d\tau.$$ 

Remarks:
- $f \ast g$ is also called the generalized product of $f$ and $g$. 
Convolution of two functions.

Definition
The *convolution* of piecewise continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ is the function $f \ast g : \mathbb{R} \to \mathbb{R}$ given by

$$(f \ast g)(t) = \int_0^t f(\tau)g(t - \tau) \, d\tau.$$  

Remarks:

▶ $f \ast g$ is also called the generalized product of $f$ and $g$.

▶ The definition of convolution of two functions also holds in the case that one of the functions is a generalized function, like Dirac’s delta.
Convolution of two functions.

Example

Find the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$.
Convolution of two functions.

Example
Find the convolution of \( f(t) = e^{-t} \) and \( g(t) = \sin(t) \).

Solution: By definition: 
\[
(f * g)(t) = \int_{0}^{t} e^{-\tau} \sin(t - \tau) d\tau.
\]
Convolution of two functions.

Example
Find the convolution of \( f(t) = e^{-t} \) and \( g(t) = \sin(t) \).

Solution: By definition: \( (f \ast g)(t) = \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau \).

Integrate by parts twice:
Convolution of two functions.

Example
Find the convolution of \( f(t) = e^{-t} \) and \( g(t) = \sin(t) \).

Solution: By definition: \( (f \ast g)(t) = \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau \).

Integrate by parts twice:

\[
\int_0^t e^{-\tau} \sin(t - \tau) \, d\tau = \left[ e^{-\tau} \cos(t - \tau) \right]_0^t - \left[ e^{-\tau} \sin(t - \tau) \right]_0^t - \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau,
\]

We conclude:

\( (f \ast g)(t) = \frac{1}{2} \left( e^{-t} + \sin(t) - \cos(t) \right) \).
Convolution of two functions.

Example
Find the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Solution: By definition: $(f \ast g)(t) = \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau$.

Integrate by parts twice:

\[
\int_0^t e^{-\tau} \sin(t - \tau) \, d\tau = \left[ e^{-\tau} \cos(t - \tau) \right]_0^t - \left[ e^{-\tau} \sin(t - \tau) \right]_0^t - \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau,
\]

\[
2 \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau = \left[ e^{-\tau} \cos(t - \tau) \right]_0^t - \left[ e^{-\tau} \sin(t - \tau) \right]_0^t,
\]
Convolution of two functions.

Example
Find the convolution of \( f(t) = e^{-t} \) and \( g(t) = \sin(t) \).

Solution: By definition: \((f \ast g)(t) = \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau \).

Integrate by parts twice:

\[
\int_0^t e^{-\tau} \sin(t - \tau) \, d\tau =
\left[ e^{-\tau} \cos(t - \tau) \right]_0^t - \left[ e^{-\tau} \sin(t - \tau) \right]_0^t - \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau,
\]

\[
2 \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau = \left[ e^{-\tau} \cos(t - \tau) \right]_0^t - \left[ e^{-\tau} \sin(t - \tau) \right]_0^t,
\]

\[
2(f \ast g)(t) = e^{-t} - \cos(t) - 0 + \sin(t).
\]
Convolution of two functions.

Example

Find the convolution of \( f(t) = e^{-t} \) and \( g(t) = \sin(t) \).

Solution: By definition: \((f * g)(t) = \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau\).

Integrate by parts twice:

\[
\int_0^t e^{-\tau} \sin(t - \tau) \, d\tau = \left[ e^{-\tau} \cos(t - \tau) \right]_0^t - \left[ e^{-\tau} \sin(t - \tau) \right]_0^t - \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau,
\]

\[
2 \int_0^t e^{-\tau} \sin(t - \tau) \, d\tau = \left[ e^{-\tau} \cos(t - \tau) \right]_0^t - \left[ e^{-\tau} \sin(t - \tau) \right]_0^t,
\]

\[
2(f * g)(t) = e^{-t} - \cos(t) - 0 + \sin(t).
\]

We conclude: \((f * g)(t) = \frac{1}{2} [e^{-t} + \sin(t) - \cos(t)]\). \(\triangleleft\)
Convolution solutions (Sect. 4.5).

- Convolution of two functions.
- **Properties of convolutions.**
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.
Properties of convolutions.

Theorem (Properties)

For every piecewise continuous functions $f$, $g$, and $h$, hold:

(i) Commutativity: $f * g = g * f$;

(ii) Associativity: $f * (g * h) = (f * g) * h$;

(iii) Distributivity: $f * (g + h) = f * g + f * h$;

(iv) Neutral element: $f * 0 = 0$;

(v) Identity element: $f * \delta = f$. 

Properties of convolutions.

**Theorem (Properties)**

For every piecewise continuous functions $f$, $g$, and $h$, hold:

(i) **Commutativity**: $f * g = g * f$;

(ii) **Associativity**: $f * (g * h) = (f * g) * h$;

(iii) **Distributivity**: $f * (g + h) = f * g + f * h$;

(iv) **Neutral element**: $f * 0 = 0$;

(v) **Identity element**: $f * \delta = f$.

**Proof:**

(v):

$$(f * \delta)(t) = \int_0^t f(\tau) \delta(t - \tau) \, d\tau$$
Properties of convolutions.

Theorem (Properties)
For every piecewise continuous functions $f$, $g$, and $h$, hold:

(i) Commutativity: $f * g = g * f$;

(ii) Associativity: $f * (g * h) = (f * g) * h$;

(iii) Distributivity: $f * (g + h) = f * g + f * h$;

(iv) Neutral element: $f * 0 = 0$;

(v) Identity element: $f * \delta = f$.

Proof:
(v):

$$(f * \delta)(t) = \int_{0}^{t} f(\tau) \delta(t - \tau) \, d\tau = \int_{0}^{t} f(\tau) \delta(\tau - t) \, d\tau$$
Properties of convolutions.

Theorem (Properties)

For every piecewise continuous functions \( f \), \( g \), and \( h \), hold:

(i) **Commutativity:** \( f \ast g = g \ast f \);

(ii) **Associativity:** \( f \ast (g \ast h) = (f \ast g) \ast h \);

(iii) **Distributivity:** \( f \ast (g + h) = f \ast g + f \ast h \);

(iv) **Neutral element:** \( f \ast 0 = 0 \);

(v) **Identity element:** \( f \ast \delta = f \).

Proof:

(v):

\[
(f \ast \delta)(t) = \int_0^t f(\tau) \delta(t - \tau) \, d\tau = \int_0^t f(\tau) \delta(\tau - t) \, d\tau = f(t).
\]
Properties of convolutions.

Proof:
(1): Commutativity: \( f \ast g = g \ast f \).
Properties of convolutions.

Proof:
(1): Commutativity: $f * g = g * f$.

The definition of convolution is,

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau.$$
Properties of convolutions.

Proof:
(1): Commutativity: \( f \ast g = g \ast f \).

The definition of convolution is,

\[
(f \ast g)(t) = \int_0^t f(\tau) g(t - \tau) \, d\tau.
\]

Change the integration variable: \( \hat{\tau} = t - \tau \),

\[
(f \ast g)(t) = \int_0^t g(\hat{\tau}) f(t - \hat{\tau}) \, d\hat{\tau}.
\]
Properties of convolutions.

Proof:
(1): Commutativity: \( f \ast g = g \ast f \).

The definition of convolution is,

\[
(f \ast g)(t) = \int_0^t f(\tau) g(t - \tau) \, d\tau.
\]

Change the integration variable: \( \hat{\tau} = t - \tau \), hence \( d\hat{\tau} = -d\tau \),
Properties of convolutions.

Proof:

(1): Commutativity: \( f \ast g = g \ast f \).

The definition of convolution is,

\[
(f \ast g)(t) = \int_0^t f(\tau) g(t - \tau) \, d\tau.
\]

Change the integration variable: \( \hat{\tau} = t - \tau \), hence \( d\hat{\tau} = -d\tau \),

\[
(f \ast g)(t) = \int_t^0 f(t - \hat{\tau}) g(\hat{\tau})(-1) \, d\hat{\tau}
\]
Properties of convolutions.

Proof:

(1): Commutativity: \( f \ast g = g \ast f \).

The definition of convolution is,

\[
(f \ast g)(t) = \int_0^t f(\tau) g(t - \tau) \, d\tau.
\]

Change the integration variable: \( \hat{\tau} = t - \tau \), hence \( d\hat{\tau} = -d\tau \),

\[
(f \ast g)(t) = \int_t^0 f(t - \hat{\tau}) g(\hat{\tau})(-1) \, d\hat{\tau}
\]

\[
(f \ast g)(t) = \int_0^t g(\hat{\tau}) f(t - \hat{\tau}) \, d\hat{\tau}
\]
Properties of convolutions.

Proof:

(1): Commutativity: \( f \ast g = g \ast f \).

The definition of convolution is,

\[
(f \ast g)(t) = \int_0^t f(\tau) g(t - \tau) \, d\tau.
\]

Change the integration variable: \( \hat{\tau} = t - \tau \), hence \( d\hat{\tau} = -d\tau \),

\[
(f \ast g)(t) = \int_t^0 f(t - \hat{\tau}) g(\hat{\tau})(-1) \, d\hat{\tau}
\]

\[
(f \ast g)(t) = \int_0^t g(\hat{\tau}) f(t - \hat{\tau}) \, d\hat{\tau}
\]

We conclude: \( (f \ast g)(t) = (g \ast f)(t) \).
Convolution solutions (Sect. 4.5).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.
Laplace Transform of a convolution.

Theorem (Laplace Transform)

If \( f, g \) have well-defined Laplace Transforms \( \mathcal{L}[f], \mathcal{L}[g] \), then

\[
\mathcal{L}[f \ast g] = \mathcal{L}[f] \mathcal{L}[g].
\]
Laplace Transform of a convolution.

Theorem (Laplace Transform)

If \( f, g \) have well-defined Laplace Transforms \( \mathcal{L}[f], \mathcal{L}[g] \), then

\[
\mathcal{L}[f \ast g] = \mathcal{L}[f] \mathcal{L}[g].
\]

Proof: The key step is to interchange two integrals.
Laplace Transform of a convolution.

Theorem (Laplace Transform)

If \( f, g \) have well-defined Laplace Transforms \( \mathcal{L}[f], \mathcal{L}[g] \), then

\[
\mathcal{L}[f \ast g] = \mathcal{L}[f] \mathcal{L}[g].
\]

Proof: The key step is to interchange two integrals. We start with the product of the Laplace transforms,

\[
\mathcal{L}[f] \mathcal{L}[g] = \left[ \int_0^\infty e^{-st} f(t) \, dt \right] \left[ \int_0^\infty e^{-s\tilde{t}} g(\tilde{t}) \, d\tilde{t} \right],
\]
Laplace Transform of a convolution.

**Theorem (Laplace Transform)**

*If* $f$, $g$ *have well-defined Laplace Transforms* $\mathcal{L}[f]$, $\mathcal{L}[g]$, *then*

$$\mathcal{L}[f \ast g] = \mathcal{L}[f] \mathcal{L}[g].$$

**Proof:** The key step is to interchange two integrals. We start with the product of the Laplace transforms,

$$\mathcal{L}[f] \mathcal{L}[g] = \left[ \int_0^\infty e^{-st} f(t) \, dt \right] \left[ \int_0^\infty e^{-s\tilde{t}} g(\tilde{t}) \, d\tilde{t} \right],$$

$$\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-s\tilde{t}} g(\tilde{t}) \left( \int_0^\infty e^{-st} f(t) \, dt \right) d\tilde{t},$$
Laplace Transform of a convolution.

Theorem (Laplace Transform)

If $f$, $g$ have well-defined Laplace Transforms $L[f]$, $L[g]$, then

$$L[f * g] = L[f] L[g].$$

Proof: The key step is to interchange two integrals. We start with the product of the Laplace transforms,

$$L[f] L[g] = \left[ \int_0^\infty e^{-st} f(t) \, dt \right] \left[ \int_0^\infty e^{-s\tilde{t}} g(\tilde{t}) \, d\tilde{t} \right],$$

$$L[f] L[g] = \int_0^\infty e^{-s\tilde{t}} g(\tilde{t}) \left( \int_0^\infty e^{-st} f(t) \, dt \right) \, d\tilde{t},$$

$$L[f] L[g] = \int_0^\infty g(\tilde{t}) \left( \int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt \right) \, d\tilde{t}.$$
Laplace Transform of a convolution.

Proof: Recall: \[ \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt \right) d\tilde{t}. \]
Laplace Transform of a convolution.

Proof: Recall: \[ \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt \right) d\tilde{t}. \]

Change variables: \[ \tau = t + \tilde{t}, \]
Laplace Transform of a convolution.

Proof: Recall: \( \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t})(\int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt) \, d\tilde{t}. \)

Change variables: \( \tau = t + \tilde{t}, \) hence \( d\tau = dt; \)
Laplace Transform of a convolution.

Proof: Recall: \( \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt \right) d\tilde{t} \).

Change variables: \( \tau = t + \tilde{t} \), hence \( d\tau = dt \);

\[ \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_\tilde{t}^\infty e^{-s\tau} f(\tau - \tilde{t}) \, d\tau \right) d\tilde{t}. \]
Laplace Transform of a convolution.

Proof: Recall: \( \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt \right) d\tilde{t} \).

Change variables: \( \tau = t + \tilde{t} \), hence \( d\tau = dt \);

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_{\tilde{t}}^\infty e^{-s\tau} f(\tau - \tilde{t}) \, d\tau \right) d\tilde{t}.
\]

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty \int_{\tilde{t}}^\infty e^{-s\tau} g(\tilde{t}) f(\tau - \tilde{t}) \, d\tau \, d\tilde{t}.
\]
Laplace Transform of a convolution.

Proof: Recall: \( \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt \right) \, d\tilde{t} \).

Change variables: \( \tau = t + \tilde{t} \), hence \( d\tau = dt \);

\[ \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_\tilde{t}^\infty e^{-s\tau} f(\tau - \tilde{t}) \, d\tau \right) \, d\tilde{t}. \]

The key step: Switch the order of integration.
Laplace Transform of a convolution.

Proof: Recall: \( \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt \right) d\tilde{t}. \)

Change variables: \( \tau = t + \tilde{t} \), hence \( d\tau = dt; \)

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_{\tilde{t}}^\infty e^{-s\tau} f(\tau - \tilde{t}) \, d\tau \right) d\tilde{t}.
\]

The key step: Switch the order of integration.

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty \int_{\tilde{t}}^\infty e^{-s\tau} g(\tilde{t}) f(\tau - \tilde{t}) \, d\tau \, d\tilde{t}.
\]
Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt \right) \, d\tilde{t}$.

Change variables: $\tau = t + \tilde{t}$, hence $d\tau = dt$;

$$\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_{\tilde{t}}^\infty e^{-s\tau} f(\tau - \tilde{t}) \, d\tau \right) \, d\tilde{t}.$$ 

The key step: Switch the order of integration.

$$\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau} g(\tilde{t}) f(\tau - \tilde{t}) \, d\tau \, d\tilde{t}.$$
Laplace Transform of a convolution.

Proof: Recall:  \( \mathcal{L}[f] \mathcal{L}[g] = \int_{0}^{\infty} \int_{0}^{\tau} e^{-st} g(\tilde{t}) f(\tau - \tilde{t}) d\tilde{t} d\tau. \)
Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau} g(\tilde{t}) f(\tau - \tilde{t}) \, d\tilde{t} \, d\tau$.

Then, it is straightforward to check that

$$\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-s\tau} \left( \int_0^\tau g(\tilde{t}) f(\tau - \tilde{t}) \, d\tilde{t} \right) \, d\tau,$$

We conclude:

$$\mathcal{L}[f \ast g] = \mathcal{L}[f] \mathcal{L}[g].$$
Laplace Transform of a convolution.

Proof: Recall: \( \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau} g(\tilde{t}) f(\tau - \tilde{t}) d\tilde{t} d\tau. \)

Then, it is straightforward to check that

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-s\tau} \left( \int_0^\tau g(\tilde{t}) f(\tau - \tilde{t}) d\tilde{t} \right) d\tau,
\]

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-s\tau} (g * f)(\tau) d\tau
\]
Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau} g(\tilde{t}) f(\tau - \tilde{t}) \, d\tilde{t} \, d\tau$. 

Then, is straightforward to check that

$$\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-s\tau} \left( \int_0^\tau g(\tilde{t}) f(\tau - \tilde{t}) \, d\tilde{t} \right) \, d\tau,$$

$$\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-s\tau} (g \ast f)(\tau) \, d\tau$$

$$\mathcal{L}[f] \mathcal{L}[g] = \mathcal{L}[g \ast f]$$
Proof: Recall: \( \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau} g(\tilde{t}) f(\tau - \tilde{t}) \, d\tilde{t} \, d\tau. \)

Then, is straightforward to check that

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-s\tau} \left( \int_0^\tau g(\tilde{t}) f(\tau - \tilde{t}) \, d\tilde{t} \right) \, d\tau,
\]

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-s\tau} (g * f)(\tau) \, d\tau
\]

\[
\mathcal{L}[f] \mathcal{L}[g] = \mathcal{L}[g * f]
\]

We conclude: \( \mathcal{L}[f * g] = \mathcal{L}[f] \mathcal{L}[g] \).
Example

Use convolutions to find the inverse Laplace Transform of

$$F(s) = \frac{3}{s^3(s^2 - 3)}.$$
Example
Use convolutions to find the inverse Laplace Transform of

\[ F(s) = \frac{3}{s^3(s^2 - 3)}. \]

Solution: We express \( F \) as a product of two Laplace Transforms,
Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

\[ F(s) = \frac{3}{s^3(s^2 - 3)}. \]

Solution: We express \( F \) as a product of two Laplace Transforms,

\[ F(s) = 3 \frac{1}{s^3} \frac{1}{(s^2 - 3)}. \]
Laplace Transform of a convolution.

Example
Use convolutions to find the inverse Laplace Transform of

\[ F(s) = \frac{3}{s^3(s^2 - 3)}. \]

Solution: We express \( F \) as a product of two Laplace Transforms,

\[ F(s) = 3 \frac{1}{s^3} \frac{1}{(s^2 - 3)} = \frac{3}{2} \frac{1}{\sqrt{3}} \left( \frac{2}{s^3} \right) \left( \frac{\sqrt{3}}{s^2 - 3} \right) \]
Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

\[ F(s) = \frac{3}{s^3(s^2 - 3)}. \]

Solution: We express \( F \) as a product of two Laplace Transforms,

\[ F(s) = 3 \frac{1}{s^3} \frac{1}{(s^2 - 3)} = \frac{3}{2} \frac{1}{\sqrt{3}} \left( \frac{2}{s^3} \right) \left( \frac{\sqrt{3}}{s^2 - 3} \right). \]

Recalling that \( \mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \)
Example

Use convolutions to find the inverse Laplace Transform of

\[ F(s) = \frac{3}{s^3(s^2 - 3)}. \]

Solution: We express \( F \) as a product of two Laplace Transforms,

\[ F(s) = 3 \frac{1}{s^3} \frac{1}{(s^2 - 3)} = \frac{3}{2} \frac{1}{\sqrt{3}} \left( \frac{2}{s^3} \right) \left( \frac{\sqrt{3}}{s^2 - 3} \right). \]

Recalling that \( \mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \) and \( \mathcal{L}[\sinh(at)] = \frac{a}{s^2 - a^2}, \)
Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

\[ F(s) = \frac{3}{s^3(s^2 - 3)}. \]

Solution: We express \( F \) as a product of two Laplace Transforms,

\[ F(s) = 3 \frac{1}{s^3} \frac{1}{(s^2 - 3)} = 3 \frac{1}{2} \sqrt{3} \left( \frac{2}{s^3} \right) \left( \frac{\sqrt{3}}{s^2 - 3} \right) \]

Recalling that \( \mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \) and \( \mathcal{L}[\sinh(at)] = \frac{a}{s^2 - a^2} \),

\[ F(s) = \frac{\sqrt{3}}{2} \mathcal{L}[t^2] \mathcal{L}[\sinh(\sqrt{3} t)] \]
Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

\[ F(s) = \frac{3}{s^3(s^2 - 3)}. \]

Solution: We express \( F \) as a product of two Laplace Transforms,

\[
F(s) = 3 \frac{1}{s^3} \frac{1}{(s^2 - 3)} = \frac{3}{2} \frac{1}{\sqrt{3}} \left( \frac{2}{s^3} \right) \left( \frac{\sqrt{3}}{s^2 - 3} \right)
\]

Recalling that \( L[t^n] = \frac{n!}{s^{n+1}} \) and \( L[\sinh(at)] = \frac{a}{s^2 - a^2} \),

\[
F(s) = \frac{\sqrt{3}}{2} L[t^2] L[\sinh(\sqrt{3} t)] = \frac{\sqrt{3}}{2} L[t^2 * \sin(\sqrt{3} t)].
\]
Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

\[ F(s) = \frac{3}{s^3(s^2 - 3)}. \]

Solution: We express \( F \) as a product of two Laplace Transforms,

\[ F(s) = 3 \frac{1}{s^3} \frac{1}{(s^2 - 3)} = \frac{3}{2} \frac{1}{\sqrt{3}} \left( \frac{2}{s^3} \right) \left( \frac{\sqrt{3}}{s^2 - 3} \right). \]

Recalling that \( \mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \) and \( \mathcal{L}[\sinh(at)] = \frac{a}{s^2 - a^2}, \)

\[ F(s) = \frac{\sqrt{3}}{2} \mathcal{L}[t^2] \mathcal{L}[\sinh(\sqrt{3} t)] = \frac{\sqrt{3}}{2} \mathcal{L}[t^2 \ast \sin(\sqrt{3} t)]. \]

We conclude that \( f(t) = \frac{\sqrt{3}}{2} \int_0^t \tau^2 \sinh[\sqrt{3}(t - \tau))] \, d\tau. \) \( \diamondsuit \)
Laplace Transform of a convolution.

Example

Compute $L[f(t)]$ where $f(t) = \int_{0}^{t} e^{-3(t-\tau)} \cos(2\tau) \, d\tau$.
Laplace Transform of a convolution.

Example
Compute $L[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau) \, d\tau$.

Solution: The function $f$ is the convolution of two functions,
Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau) \, d\tau$.

Solution: The function $f$ is the convolution of two functions,

$$f(t) = (g \ast h)(t),$$
Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where

$$f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau) \, d\tau.$$  

Solution: The function $f$ is the convolution of two functions,

$$f(t) = (g \ast h)(t), \quad g(t) = \cos(2t),$$
Example

Compute $\mathcal{L}[f(t)]$ where

$$f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau) \, d\tau.$$  

Solution: The function $f$ is the convolution of two functions,

$$f(t) = (g \ast h)(t), \quad g(t) = \cos(2t), \quad h(t) = e^{-3t}.$$
Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau) \, d\tau$.

Solution: The function $f$ is the convolution of two functions,

$$f(t) = (g * h)(t), \quad g(t) = \cos(2t), \quad h(t) = e^{-3t}.$$

Since $\mathcal{L}[(g * h)(t)] = \mathcal{L}[g(t)] \mathcal{L}[h(t)]$, we conclude that

$$F(s) = s \left( s + 3 \right) \left( s^2 + 4 \right).$$
Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau) \, d\tau$.

Solution: The function $f$ is the convolution of two functions,

$$f(t) = (g \ast h)(t), \quad g(t) = \cos(2t), \quad h(t) = e^{-3t}.$$ 

Since $\mathcal{L}[(g \ast h)(t)] = \mathcal{L}[g(t)] \mathcal{L}[h(t)]$, then,

$$F(s) = \mathcal{L}\left[ \int_0^t e^{-3(t-\tau)} \cos(2\tau) \, d\tau \right]$$
Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau) d\tau$.

Solution: The function $f$ is the convolution of two functions,

$$f(t) = (g \ast h)(t), \quad g(t) = \cos(2t), \quad h(t) = e^{-3t}.$$

Since $\mathcal{L}[(g \ast h)(t)] = \mathcal{L}[g(t)] \mathcal{L}[h(t)]$, then,

$$F(s) = \mathcal{L} \left[ \int_0^t e^{-3(t-\tau)} \cos(2\tau) d\tau \right] = \mathcal{L} [e^{-3t}] \mathcal{L} [\cos(2t)].$$
Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau) \, d\tau$.

Solution: The function $f$ is the convolution of two functions,

$$f(t) = (g \ast h)(t), \quad g(t) = \cos(2t), \quad h(t) = e^{-3t}.$$ 

Since $\mathcal{L}[(g \ast h)(t)] = \mathcal{L}[g(t)] \mathcal{L}[h(t)]$, then,

$$F(s) = \mathcal{L} \left[ \int_0^t e^{-3(t-\tau)} \cos(2\tau) \, d\tau \right] = \mathcal{L} \left[ e^{-3t} \right] \mathcal{L} \left[ \cos(2t) \right].$$

We conclude that $F(s) = \frac{s}{(s + 3)(s^2 + 4)}$. \(\triangleq\)
Example
Solve the IVP
\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]
Example
Solve the IVP
\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Denote \( G(s) = \mathcal{L}[g(t)] \)
Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0.$$ 

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,
Laplace Transform of a convolution.

Example
Solve the IVP
\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Denote \( G(s) = \mathcal{L}[g(t)] \) and compute LT of the equation,
\[ (s^2 - 5s + 6) \mathcal{L}[y(t)] = \mathcal{L}[g(t)] \]
Example
Solve the IVP

\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Denote \( G(s) = \mathcal{L}[g(t)] \) and compute LT of the equation,

\[ (s^2 - 5s + 6) \mathcal{L}[y(t)] = \mathcal{L}[g(t)] \Rightarrow \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)} G(s). \]
Laplace Transform of a convolution.

Example
Solve the IVP

\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Denote \( G(s) = \mathcal{L}[g(t)] \) and compute LT of the equation,

\[ (s^2 - 5s + 6) \mathcal{L}[y(t)] = \mathcal{L}[g(t)] \quad \Rightarrow \quad \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)} G(s). \]

Denoting \( H(s) = \frac{1}{s^2 - 5s + 6} \),
Laplace Transform of a convolution.

Example
Solve the IVP
\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Denote \( G(s) = \mathcal{L}[g(t)] \) and compute LT of the equation,

\[(s^2 - 5s + 6) \mathcal{L}[y(t)] = \mathcal{L}[g(t)] \Rightarrow \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)} G(s).\]

Denoting \( H(s) = \frac{1}{s^2 - 5s + 6} \), and \( h(t) = \mathcal{L}^{-1}[H(s)] \),
Laplace Transform of a convolution.

Example
Solve the IVP

\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Denote \( G(s) = \mathcal{L}[g(t)] \) and compute LT of the equation,

\[ (s^2 - 5s + 6) \mathcal{L}[y(t)] = \mathcal{L}[g(t)] \Rightarrow \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)} \ G(s). \]

Denoting \( H(s) = \frac{1}{s^2 - 5s + 6} \), and \( h(t) = \mathcal{L}^{-1}[H(s)] \), then

\[ \mathcal{L}[y(t)] = H(s) \ G(s) \]
Example
Solve the IVP
\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Denote \( G(s) = \mathcal{L}[g(t)] \) and compute LT of the equation,

\[ (s^2 - 5s + 6) \mathcal{L}[y(t)] = \mathcal{L}[g(t)] \Rightarrow \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)} G(s). \]

Denoting \( H(s) = \frac{1}{s^2 - 5s + 6} \), and \( h(t) = \mathcal{L}^{-1}[H(s)] \), then

\[ \mathcal{L}[y(t)] = H(s) G(s) \Rightarrow y(t) = (h \ast g)(t). \]
Laplace Transform of a convolution.

Example
Solve the IVP
\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Denote \( G(s) = \mathcal{L}[g(t)] \) and compute LT of the equation,
\[ (s^2 - 5s + 6) \mathcal{L}[y(t)] = \mathcal{L}[g(t)] \Rightarrow \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)} G(s). \]

Denoting \( H(s) = \frac{1}{s^2 - 5s + 6} \), and \( h(t) = \mathcal{L}^{-1}[H(s)] \), then
\[ \mathcal{L}[y(t)] = H(s) G(s) \Rightarrow y(t) = (h \ast g)(t). \]

Function \( h \) is simple to compute:
Laplace Transform of a convolution.

Example
Solve the IVP
\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Denote \( G(s) = \mathcal{L}[g(t)] \) and compute LT of the equation,
\[ (s^2 - 5s + 6) \mathcal{L}[y(t)] = \mathcal{L}[g(t)] \implies \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)} G(s). \]
Denoting \( H(s) = \frac{1}{s^2 - 5s + 6} \), and \( h(t) = \mathcal{L}^{-1}[H(s)] \), then
\[ \mathcal{L}[y(t)] = H(s) G(s) \implies y(t) = (h \ast g)(t). \]

Function \( h \) is simple to compute:
\[ H(s) = \frac{1}{(s - 2)(s - 3)} \]
Laplace Transform of a convolution.

Example
Solve the IVP
\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Denote \( G(s) = \mathcal{L}[g(t)] \) and compute LT of the equation,

\[(s^2 - 5s + 6) \mathcal{L}[y(t)] = \mathcal{L}[g(t)] \Rightarrow \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)} \cdot G(s).\]

Denoting \( H(s) = \frac{1}{s^2 - 5s + 6} \), and \( h(t) = \mathcal{L}^{-1}[H(s)] \), then

\[ \mathcal{L}[y(t)] = H(s) \cdot G(s) \Rightarrow y(t) = (h \ast g)(t). \]

Function \( h \) is simple to compute:

\[ H(s) = \frac{1}{(s - 2)(s - 3)} = \frac{a}{s - 2} + \frac{b}{s - 3} \]
Laplace Transform of a convolution.

Example
Solve the IVP

\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Denote \( G(s) = \mathcal{L}[g(t)] \) and compute LT of the equation,

\[
(s^2 - 5s + 6) \mathcal{L}[y(t)] = \mathcal{L}[g(t)] \Rightarrow \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)} G(s).
\]

Denoting \( H(s) = \frac{1}{s^2 - 5s + 6} \), and \( h(t) = \mathcal{L}^{-1}[H(s)] \), then

\[
\mathcal{L}[y(t)] = H(s) G(s) \Rightarrow y(t) = (h \ast g)(t).
\]

Function \( h \) is simple to compute:

\[
H(s) = \frac{1}{(s-2)(s-3)} = \frac{a}{s-2} + \frac{b}{s-3} = \frac{a(s-3) + b(s-2)}{(s-2)(s-3)}
\]
Laplace Transform of a convolution.

Example
Solve the IVP

\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Then: 1 = a(s - 3) + b(s - 2).
Laplace Transform of a convolution.

Example

Solve the IVP

\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Then: \( 1 = a(s - 3) + b(s - 2) \). Evaluate at \( s = 2, 3 \).

\[ s = 2 \]
Laplace Transform of a convolution.

Example
Solve the IVP

\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Then: \( 1 = a(s - 3) + b(s - 2) \). Evaluate at \( s = 2, \ 3 \).

\[ s = 2 \quad \Rightarrow \quad a = -1. \]
Laplace Transform of a convolution.

Example

Solve the IVP

\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Then: \[ 1 = a(s - 3) + b(s - 2). \] Evaluate at \( s = 2, \ 3. \)

\[ s = 2 \quad \Rightarrow \quad a = -1. \quad s = 3 \]
Laplace Transform of a convolution.

Example

Solve the IVP

\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Then: \( 1 = a(s - 3) + b(s - 2) \). Evaluate at \( s = 2, \ 3 \).

\[ s = 2 \quad \Rightarrow \quad a = -1. \quad s = 3 \quad \Rightarrow \quad b = 1. \]
Laplace Transform of a convolution.

Example
Solve the IVP
\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Then: \( 1 = a(s - 3) + b(s - 2) \). Evaluate at \( s = 2, 3 \).

\[ s = 2 \implies a = -1. \quad s = 3 \implies b = 1. \]

Therefore \( H(s) = -\frac{1}{(s - 2)} + \frac{1}{(s - 3)}. \)
Laplace Transform of a convolution.

Example
Solve the IVP

\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Then: \(1 = a(s - 3) + b(s - 2)\). Evaluate at \(s = 2, 3\).

\[ s = 2 \implies a = -1. \quad s = 3 \implies b = 1. \]

Therefore \(H(s) = -\frac{1}{(s - 2)} + \frac{1}{(s - 3)}\). Then

\[ h(t) = -e^{2t} + e^{3t}. \]
Example

Solve the IVP

\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Then: \(1 = a(s - 3) + b(s - 2)\). Evaluate at \(s = 2, 3\).

\[ s = 2 \quad \Rightarrow \quad a = -1. \quad s = 3 \quad \Rightarrow \quad b = 1. \]

Therefore \(H(s) = -\frac{1}{(s - 2)} + \frac{1}{(s - 3)}\). Then

\[ h(t) = -e^{2t} + e^{3t}. \]

Recalling the formula \(y(t) = (h * g)(t)\),
Laplace Transform of a convolution.

Example
Solve the IVP
\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Then: \( 1 = a(s - 3) + b(s - 2). \) Evaluate at \( s = 2, \ 3. \)

\[ s = 2 \quad \Rightarrow \quad a = -1. \quad s = 3 \quad \Rightarrow \quad b = 1. \]

Therefore \( H(s) = -\frac{1}{(s - 2)} + \frac{1}{(s - 3)}. \) Then

\[ h(t) = -e^{2t} + e^{3t}. \]

Recalling the formula \( y(t) = (h \ast g)(t), \) we get

\[ y(t) = \int_{0}^{t} (-e^{2\tau} + e^{3\tau}) g(t - \tau) \, d\tau. \]
Convolution solutions (Sect. 4.5).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.
Impulse response solution.

Definition

The *impulse response solution* is the solution $y_\delta$ to the IVP

\[ y''_\delta + a_1 y'_\delta + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0. \]
Impulse response solution.

Definition
The *impulse response solution* is the solution $y_\delta$ to the IVP

$$y_\delta'' + a_1 y_\delta' + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$ 

Computing Laplace Transforms,

$$(s^2 + a_1 s + a_0) \mathcal{L}[y_\delta] = 1$$
Impulse response solution.

Definition
The *impulse response solution* is the solution $y_\delta$ to the IVP

$$y_\delta'' + a_1 y_\delta' + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$ 

Computing Laplace Transforms,

$$(s^2 + a_1 s + a_0) \mathcal{L}[y_\delta] = 1 \quad \Rightarrow \quad y_\delta(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + a_1 s + a_0}\right].$$
Impulse response solution.

Definition

The *impulse response solution* is the solution $y_\delta$ to the IVP

$$y''_\delta + a_1 y'_\delta + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0.$$  

Computing Laplace Transforms,

$$(s^2 + a_1 s + a_0) \mathcal{L}[y_\delta] = 1 \quad \Rightarrow \quad y_\delta(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + a_1 s + a_0}\right].$$

Denoting the characteristic polynomial by $p(s) = s^2 + a_1 s + a_0$,  

$$y_\delta(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right].$$
Impulse response solution.

Definition
The *impulse response solution* is the solution \( y_\delta \) to the IVP

\[
y''_\delta + a_1 y'_\delta + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0.
\]

Computing Laplace Transforms,

\[
(s^2 + a_1 s + a_0) \mathcal{L}[y_\delta] = 1 \quad \Rightarrow \quad y_\delta(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + a_1 s + a_0}\right].
\]

Denoting the characteristic polynomial by \( p(s) = s^2 + a_1 s + a_0 \),

\[
y_\delta = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right].
\]
Impulse response solution.

Definition
The *impulse response solution* is the solution $y_\delta$ to the IVP

$$y''_\delta + a_1 y'_\delta + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0.$$ 

Computing Laplace Transforms,

$$(s^2 + a_1 s + a_0) \mathcal{L}[y_\delta] = 1 \quad \Rightarrow \quad y_\delta(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + a_1 s + a_0}\right].$$

Denoting the characteristic polynomial by $p(s) = s^2 + a_1 s + a_0$,

$$y_\delta = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right].$$

**Summary:** The impulse response solution is the inverse Laplace Transform of the reciprocal of the equation characteristic polynomial.
Impulse response solution.

Recall: The impulse response solution is $y_\delta$ solution of the IVP

$$y_\delta'' + a_1 y_\delta' + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$
Impulse response solution.

Recall: The impulse response solution is \( y_\delta \) solution of the IVP

\[
y''_\delta + a_1 y'_\delta + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0.
\]

Example
Find the solution (impulse response at \( t = c \)) of the IVP

\[
y''_c + 2 y'_c + 2 y_c = \delta(t - c), \quad y_c(0) = 0, \quad y'_c(0) = 0, \quad c \in \mathbb{R}.
\]
Impulse response solution.

Recall: The impulse response solution is $y_\delta$ solution of the IVP

\[ y''_\delta + a_1 y'_\delta + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0. \]

Example

Find the solution (impulse response at $t = c$) of the IVP

\[ y''_{\delta_c} + 2 y'_{\delta_c} + 2 y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}. \]

Solution: $\mathcal{L}[y''_{\delta_c}] + 2 \mathcal{L}[y'_{\delta_c}] + 2 \mathcal{L}[y_{\delta_c}] = \mathcal{L}[\delta(t - c)]$. 
Impulse response solution.

Recall: The impulse response solution is \( y_\delta \) solution of the IVP

\[
y''_\delta + a_1 y'_\delta + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0.
\]

Example
Find the solution (impulse response at \( t = c \)) of the IVP

\[
y''_{\delta c} + 2 y'_{\delta c} + 2 y_{\delta c} = \delta(t - c), \quad y_{\delta c}(0) = 0, \quad y'_{\delta c}(0) = 0, \quad c \in \mathbb{R}.
\]

Solution: \( \mathcal{L}[y''_{\delta c}] + 2 \mathcal{L}[y'_{\delta c}] + 2 \mathcal{L}[y_{\delta c}] = \mathcal{L}[\delta(t - c)]. \)

\[
(s^2 + 2s + 2) \mathcal{L}[y_{\delta c}] = e^{-cs}
\]
Impulse response solution.

Recall: The impulse response solution is \( y_\delta \) solution of the IVP

\[
y''_\delta + a_1 y'_\delta + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0.
\]

Example
Find the solution (impulse response at \( t = c \)) of the IVP

\[
y''_{\delta_c} + 2 y'_{\delta_c} + 2 y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.
\]

Solution: \( \mathcal{L}[y''_{\delta_c}] + 2 \mathcal{L}[y'_{\delta_c}] + 2 \mathcal{L}[y_{\delta_c}] = \mathcal{L}[\delta(t - c)]. \)

\[
(s^2 + 2s + 2) \mathcal{L}[y_{\delta_c}] = e^{-cs} \quad \Rightarrow \quad \mathcal{L}[y_{\delta_c}] = \frac{e^{-cs}}{(s^2 + 2s + 2)}.
\]
Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2y'_{\delta_c} + 2y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recall:  

$$\mathcal{L}[y_{\delta_c}] = \frac{e^{-cs}}{(s^2 + 2s + 2)}.$$
Impulse response solution.

Example
Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2y'_{\delta_c} + 2y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$  

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \frac{e^{-cs}}{(s^2 + 2s + 2)}$.

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0$$
Impulse response solution.

Example
Find the solution (impulse response at \( t = c \)) of the IVP

\[
y''_{\delta c} + 2 y'_{\delta c} + 2 y_{\delta c} = \delta(t - c), \quad y_{\delta c}(0) = 0, \quad y'_{\delta c}(0) = 0, \quad c \in \mathbb{R}.
\]

Solution: Recall: \( \mathcal{L}[y_{\delta c}] = \frac{e^{-cs}}{(s^2 + 2s + 2)}. \)

Find the roots of the denominator,

\[
s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2} [-2 \pm \sqrt{4 - 8}]
\]
Impulse response solution.

Example
Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta c} + 2y'_{\delta c} + 2y_{\delta c} = \delta(t - c), \quad y_{\delta c}(0) = 0, \quad y'_{\delta c}(0) = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recall: $\mathcal{L}[y_{\delta c}] = \frac{e^{-cs}}{(s^2 + 2s + 2)}$. 

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2} \left[ -2 \pm \sqrt{4 - 8} \right]$$

Complex roots.
Impulse response solution.

Example
Find the solution (impulse response at \( t = c \)) of the IVP

\[
y''_{\delta c} + 2 y'_{\delta c} + 2 y_{\delta c} = \delta(t - c), \quad y_{\delta c}(0) = 0, \quad y'_{\delta c}(0) = 0, \quad c \in \mathbb{R}.
\]

Solution: Recall: \( \mathcal{L}[y_{\delta c}] = \frac{e^{-cs}}{(s^2 + 2s + 2)} \).

Find the roots of the denominator,

\[
s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2} \left[ -2 \pm \sqrt{4 - 8} \right]
\]

Complex roots. We complete the square:
Impulse response solution.

Example
Find the solution (impulse response at \( t = c \)) of the IVP

\[
y''_{\delta c} + 2y'_{\delta c} + 2y_{\delta c} = \delta(t - c), \quad y_{\delta c}(0) = 0, \quad y'_{\delta c}(0) = 0, \quad c \in \mathbb{R}.
\]

Solution: Recall: \( \mathcal{L}[y_{\delta c}] = \frac{e^{-cs}}{(s^2 + 2s + 2)}. \)

Find the roots of the denominator,

\[
s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2} \left[-2 \pm \sqrt{4 - 8}\right]
\]

Complex roots. We complete the square:

\[
s^2 + 2s + 2 = \left[s^2 + 2\left(\frac{2}{2}\right)s + 1\right] - 1 + 2
\]
Impulse response solution.

Example
Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta c} + 2y'_{\delta c} + 2y_{\delta c} = \delta(t - c), \quad y_{\delta c}(0) = 0, \quad y'_{\delta c}(0) = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recall: $\mathcal{L}[y_{\delta c}] = \frac{e^{-cs}}{(s^2 + 2s + 2)}$.

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \implies s_{\pm} = \frac{1}{2} \left[-2 \pm \sqrt{4 - 8}\right]$$

Complex roots. We complete the square:

$$s^2 + 2s + 2 = \left[s^2 + 2\left(\frac{2}{2}\right)s + 1\right] - 1 + 2 = (s + 1)^2 + 1.$$
Impulse response solution.

Example

Find the solution (impulse response at \( t = c \)) of the IVP

\[
y''_\delta c + 2 y'_\delta c + 2 y_\delta c = \delta(t - c), \quad y_\delta c(0) = 0, \quad y'_\delta c(0) = 0, \quad c \in \mathbb{R}.
\]

Solution: Recall: \( \mathcal{L}[y_\delta c] = \frac{e^{-cs}}{(s^2 + 2s + 2)}. \)

Find the roots of the denominator,

\[
s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2} \left[ -2 \pm \sqrt{4 - 8} \right]
\]

Complex roots. We complete the square:

\[
s^2 + 2s + 2 = \left[ s^2 + 2\left(\frac{2}{2}\right)s + 1 \right] - 1 + 2 = (s + 1)^2 + 1.
\]

Therefore, \( \mathcal{L}[y_\delta c] = \frac{e^{-cs}}{(s + 1)^2 + 1}. \)
Impulse response solution.

Example
Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta c} + 2y'_{\delta c} + 2y_{\delta c} = \delta(t - c), \quad y_{\delta c}(0) = 0, \quad y'_{\delta c}(0) = 0, \quad c \in \mathbb{R}.$$  

Solution: Recall: $\mathcal{L}[y_{\delta c}] = \frac{e^{-cs}}{(s + 1)^2 + 1}$. 
Impulse response solution.

Example
Find the solution (impulse response at \( t = c \)) of the IVP

\[
 y''_\delta_c + 2 y'_\delta_c + 2 y_\delta_c = \delta(t - c), \quad y_\delta_c(0) = 0, \quad y'_\delta_c(0) = 0, \quad c \in \mathbb{R}.
\]

Solution: Recall: \( \mathcal{L}[y_\delta_c] = \frac{e^{-cs}}{(s + 1)^2 + 1} \).

Recall: \( \mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1} \).
Impulse response solution.

**Example**

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta c} + 2y'_{\delta c} + 2y_{\delta c} = \delta(t - c), \quad y_{\delta c}(0) = 0, \quad y'_{\delta c}(0) = 0, \quad c \in \mathbb{R}.$$  

**Solution:** Recall: $\mathcal{L}[y_{\delta c}] = \frac{e^{-cs}}{(s + 1)^2 + 1}$.

Recall: $\mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1}$, and $\mathcal{L}[f](s - c) = \mathcal{L}[e^{ct} f(t)]$. 

Impulse response solution.

Example
Find the solution (impulse response at $t = c$) of the IVP

$$y''_δ + 2y'_δ + 2y_δ = \delta(t - c), \quad y_δ(0) = 0, \quad y'_δ(0) = 0, \quad c \in \mathbb{R}.$$ 

Solution: Recall: $\mathcal{L}[y_δ] = \frac{e^{-cs}}{(s + 1)^2 + 1}$. 

Recall: $\mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1}$, and $\mathcal{L}[f](s - c) = \mathcal{L}[e^{ct} f(t)]$.

$$\frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)]$$
Impulse response solution.

Example

Find the solution (impulse response at \( t = c \)) of the IVP

\[
y''_\delta c + 2y'_\delta c + 2y_\delta c = \delta(t - c), \quad y_\delta c(0) = 0, \quad y'_\delta c(0) = 0, \quad c \in \mathbb{R}.
\]

Solution: Recall: \( \mathcal{L}[y_\delta c] = \frac{e^{-cs}}{(s + 1)^2 + 1} \).

Recall: \( \mathcal{L}[^\sin(t)] = \frac{1}{s^2 + 1} \), and \( \mathcal{L}[f](s - c) = \mathcal{L}[e^{ct} f(t)] \).

\[
\frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_\delta c] = e^{-cs} \mathcal{L}[e^{-t} \sin(t)].
\]
Impulse response solution.

Example

Find the solution (impulse response at \( t = c \)) of the IVP

\[ y''_{\delta_c} + 2y'_{\delta_c} + 2y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}. \]

Solution: Recall: \( \mathcal{L}[y_{\delta_c}] = \frac{e^{-cs}}{(s+1)^2 + 1}. \)

Recall: \( \mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1}, \) and \( \mathcal{L}[f](s-c) = \mathcal{L}[e^{ct}f(t)]. \)

\[ \frac{1}{(s+1)^2 + 1} = \mathcal{L}[e^{-t}\sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_{\delta_c}] = e^{-cs}\mathcal{L}[e^{-t}\sin(t)]. \]

Since \( e^{-cs}\mathcal{L}[f](s) = \mathcal{L}[u(t-c)f(t-c)], \)
Impulse response solution.

Example
Find the solution (impulse response at $t = c$) of the IVP

\[ y''_\delta + 2y'_\delta + 2y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0, \quad c \in \mathbb{R}. \]

Solution: Recall: \( \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s + 1)^2 + 1}. \)

Recall: \( \mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1}, \) and \( \mathcal{L}[f](s - c) = \mathcal{L}[e^{ct} f(t)]. \)

\[
\frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_\delta] = e^{-cs} \mathcal{L}[e^{-t} \sin(t)].
\]

Since \( e^{-cs} \mathcal{L}[f](s) = \mathcal{L}[u(t - c) f(t - c)], \)

we conclude \( y_\delta(t) = u(t - c) e^{-(t-c)} \sin(t - c). \)
Convolution solutions (Sect. 4.5).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.
Solution decomposition theorem.

Theorem (Solution decomposition)

The solution $y$ to the IVP

$$y'' + a_1 y' + a_0 y = g(t), \quad y(0) = y_0, \quad y'(0) = y_1,$$

can be decomposed as

$$y(t) = y_h(t) + (y_\delta * g)(t),$$

where $y_h$ is the solution of the homogeneous IVP

$$y_h'' + a_1 y_h' + a_0 y_h = 0, \quad y_h(0) = y_0, \quad y_h'(0) = y_1,$$

and $y_\delta$ is the impulse response solution, that is,

$$y''_\delta + a_1 y_\delta' + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$
Solution decomposition theorem.

Example
Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$
Solution decomposition theorem.

Example
Use the Solution Decomposition Theorem to express the solution of
\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: \[ \mathcal{L}[y''] + 2 \mathcal{L}[y'] + 2 \mathcal{L}[y] = \mathcal{L}[\sin(at)], \]
Solution decomposition theorem.

Example
Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: \( \mathcal{L}[y''] + 2\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[\sin(at)], \) and recall,

\[ \mathcal{L}[y''] = s^2 \mathcal{L}[y] - s(1) - (-1), \]
Example

Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: \( L[y''] + 2L[y'] + 2L[y] = L[\sin(at)] \), and recall,

\[ L[y''] = s^2L[y] - s(1) - (-1), \quad L[y'] = sL[y] - 1. \]
Solution decomposition theorem.

Example
Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: \( \mathcal{L}[y''] + 2\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[\sin(at)], \) and recall,

\[ \mathcal{L}[y''] = s^2 \mathcal{L}[y] - s(1) - (-1), \quad \mathcal{L}[y'] = s \mathcal{L}[y] - 1. \]

\[ (s^2 + 2s + 2) \mathcal{L}[y] - s + 1 - 2 = \mathcal{L}[\sin(at)]. \]
Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: \( \mathcal{L}[y''] + 2\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[\sin(at)], \) and recall,

\[
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - s(1) - (-1), \quad \mathcal{L}[y'] = s \mathcal{L}[y] - 1.
\]

\[
(s^2 + 2s + 2) \mathcal{L}[y] - s + 1 - 2 = \mathcal{L}[\sin(at)].
\]

\[
\mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)].
\]
Solution decomposition theorem.

Example
Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: Recall: \( \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \)
Example

Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: Recall: \( \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \)

But: \( \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} \)
Solution decomposition theorem.

Example
Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution:
Recall: \[ \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \]
But: \[ \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1}. \]
Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: Recall: \[ \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \]

But: \[ \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \cos(t)], \]
Example
Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$  

Solution: Recall: \( \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \)

But: \( \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \cos(t)], \)

and: \( \mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} \)
**Example**

Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: Recall: \( \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \)

But: \( \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \cos(t)], \)

and: \( \mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s + 1)^2 + 1} \)
Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: Recall: \( \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \)

But: \( \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \cos(t)], \)

and: \( \mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)]. \)
Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: Recall: \[ \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \]

But: \[ \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \cos(t)], \]

and: \[ \mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)]. \] So,

\[ \mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)] \]
Example
Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: Recall: \( \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \)

But: \( \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \cos(t)], \)

and: \( \mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)]. \) So,

\[ \mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)] \quad \Rightarrow \quad y(t) = y_h(t) + (y_\delta \ast g)(t), \]
Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: Recall: \( \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \)

But: \( \mathcal{L}[y_h] = \frac{(s + 1)}{(s^2 + 2s + 2)} = \frac{(s + 1)}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \cos(t)], \)

and: \( \mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)]. \) So,

\[ \mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)] \quad \Rightarrow \quad y(t) = y_h(t) + (y_\delta \ast g)(t), \]

So: \( y(t) = e^{-t} \cos(t) + \int_0^t e^{-\tau} \sin(\tau) \sin[a(t - \tau)] d\tau. \) \( \square \)
Solution decomposition theorem.

Proof: Compute: \( \mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)], \)
Solution decomposition theorem.

**Proof:** Compute: \( \mathcal{L}[y'''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)] \), and recall,

\[
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1,
\]
Solution decomposition theorem.

Proof: Compute: \( \mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)] \), and recall,

\[
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y_0.
\]
Solution decomposition theorem.

Proof: Compute: \( \mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)] \), and recall,

\[
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1,
\mathcal{L}[y'] = s \mathcal{L}[y] - y_0.
\]

\[
(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1y_0 = \mathcal{L}[g(t)].
\]
Solution decomposition theorem.

Proof: Compute: \( \mathcal{L}[y'''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)] \), and recall,

\[
\mathcal{L}[y'''] = s^2 \mathcal{L}[y] - sy_0 - y_1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y_0.
\]

\[
(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].
\]

\[
\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} \mathcal{L}[g(t)].
\]
Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y_0.$$

$$(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} \mathcal{L}[g(t)].$$

Recall: $\mathcal{L}[y_h] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)}$, 

$$y(t) = y_h(t) + \int_0^t y_\delta(\tau) g(t - \tau) d\tau.$$
Solution decomposition theorem.

Proof: Compute: \( \mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)] \), and recall,

\[
\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y_0.
\]

\[
(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].
\]

\[
\mathcal{L}[y] = \frac{(s + a_1) y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} \mathcal{L}[g(t)].
\]

Recall: \( \mathcal{L}[y_h] = \frac{(s + a_1) y_0 + y_1}{(s^2 + a_1 s + a_0)} \), and \( \mathcal{L}[y_\delta] = \frac{1}{(s^2 + a_1 s + a_0)} \).
Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y'''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y'''] = s^2 \mathcal{L}[y] - sy_0 - y_1,$$
$$\mathcal{L}[y'] = s \mathcal{L}[y] - y_0.$$

$$(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$\mathcal{L}[y] = \frac{(s + a_1) y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} \mathcal{L}[g(t)].$$

Recall: $\mathcal{L}[y_h] = \frac{(s + a_1) y_0 + y_1}{(s^2 + a_1 s + a_0)}$, and $\mathcal{L}[y_\delta] = \frac{1}{(s^2 + a_1 s + a_0)}$.

Since, $\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)]$, 

$$y(t) = y_h(t) + \int_0^t y_\delta(\tau) g(t - \tau) d\tau.$$
Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y_0.$$ 

$$(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} \mathcal{L}[g(t)].$$

Recall: $\mathcal{L}[y_h] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)}$, and $\mathcal{L}[y_\delta] = \frac{1}{(s^2 + a_1 s + a_0)}$.

Since, $\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)]$, so $y(t) = y_h(t) + (y_\delta * g)(t)$. 
Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy_0 - y_1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - y_0.$$ 

$$(s^2 + a_1 s + a_0) \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$ 

$$\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} \mathcal{L}[g(t)].$$ 

Recall: $\mathcal{L}[y_h] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)}$, and $\mathcal{L}[y_\delta] = \frac{1}{(s^2 + a_1 s + a_0)}$.

Since, $\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta] \mathcal{L}[g(t)]$, so $y(t) = y_h(t) + (y_\delta \ast g)(t)$.

Equivalently: $y(t) = y_h(t) + \int_0^t y_\delta(\tau)g(t - \tau) \, d\tau$.  \hfill \Box
Systems of linear differential equations (Sect. 5.1).

- $n \times n$ systems of linear differential equations.
- Second order equations and first order systems.
- Main concepts from Linear Algebra.
$n \times n$ systems of linear differential equations.

**Remark:** Many physical systems must be described with more than one differential equation.
$n \times n$ systems of linear differential equations.

**Remark:** Many physical systems must be described with more than one differential equation.

**Example**

Newton’s law of motion for a particle of mass $m$ moving in space.
$n \times n$ systems of linear differential equations.

Remark: Many physical systems must be described with more than one differential equation.

Example

Newton’s law of motion for a particle of mass $m$ moving in space. The unknown and the force are vector-valued functions,
\( n \times n \) systems of linear differential equations.

**Remark:** Many physical systems must be described with more than one differential equation.

**Example**

Newton’s law of motion for a particle of mass \( m \) moving in space. The unknown and the force are vector-valued functions,

\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},
\]

\[
F(t) = \begin{bmatrix} F_1(t, x(t)) \\ F_2(t, x(t)) \\ F_3(t, x(t)) \end{bmatrix}.
\]

The equation of motion are:

\[
m \frac{d^2 x}{dt^2} = F(t, x(t)).
\]
Remark: Many physical systems must be described with more than one differential equation.

Example
Newton’s law of motion for a particle of mass $m$ moving in space. The unknown and the force are vector-valued functions,

\[
\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \mathbf{F}(t) = \begin{bmatrix} F_1(t, \mathbf{x}) \\ F_2(t, \mathbf{x}) \\ F_3(t, \mathbf{x}) \end{bmatrix}.
\]
Remark: Many physical systems must be described with more than one differential equation.

Example

Newton’s law of motion for a particle of mass $m$ moving in space. The unknown and the force are vector-valued functions,

$$
\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \mathbf{F}(t) = \begin{bmatrix} F_1(t, \mathbf{x}) \\ F_2(t, \mathbf{x}) \\ F_3(t, \mathbf{x}) \end{bmatrix}.
$$

The equation of motion are:

$$
m \frac{d^2 \mathbf{x}}{dt^2} = \mathbf{F}(t, \mathbf{x}(t)).$$
$n \times n$ systems of linear differential equations.

**Remark:** Many physical systems must be described with more than one differential equation.

**Example**

Newton’s law of motion for a particle of mass $m$ moving in space. The unknown and the force are vector-valued functions,

$$
\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad 
\mathbf{F}(t) = \begin{bmatrix} F_1(t, \mathbf{x}) \\ F_2(t, \mathbf{x}) \\ F_3(t, \mathbf{x}) \end{bmatrix}.
$$

The equation of motion are:

$$m \frac{d^2 \mathbf{x}}{dt^2} = \mathbf{F}(t, \mathbf{x}(t)).$$

These are three differential equations,

$$m \frac{d^2 x_1}{dt^2} = F_1(t, \mathbf{x}(t)), \quad m \frac{d^2 x_2}{dt^2} = F_2(t, \mathbf{x}(t)), \quad m \frac{d^2 x_3}{dt^2} = F_3(t, \mathbf{x}(t)).$$
$n \times n$ systems of linear differential equations.

**Definition**

An $n \times n$ system of linear first order differential equations is the following: Given the functions $a_{ij}, g_i : [a, b] \to \mathbb{R}$, where $i, j = 1, \cdots, n$, find $n$ functions $x_j : [a, b] \to \mathbb{R}$ solutions of the $n$ linear differential equations

$$x_1' = a_{11}(t) x_1 + \cdots + a_{1n}(t) x_n + g_1(t)$$

$$\vdots$$

$$x_n' = a_{n1}(t) x_1 + \cdots + a_{nn}(t) x_n + g_n(t).$$

The system is called *homogeneous* iff the source functions satisfy that $g_1 = \cdots = g_n = 0$. 
$n \times n$ systems of linear differential equations.

Example

$n = 1$: Single differential equation: Find $x_1(t)$ solution of

$$x_1' = a_{11}(t)x_1 + g_1(t).$$
$n \times n$ systems of linear differential equations.

Example

$n = 1$: Single differential equation: Find $x_1(t)$ solution of

$$x_1' = a_{11}(t)x_1 + g_1(t).$$

Example

$n = 2$: $2 \times 2$ linear system: Find $x_1(t)$ and $x_2(t)$ solutions of

$$x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + g_1(t),$$
$$x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + g_2(t).$$
$n \times n$ systems of linear differential equations.

Example

$n = 1$: Single differential equation: Find $x_1(t)$ solution of

$$x_1' = a_{11}(t) x_1 + g_1(t).$$

Example

$n = 2$: $2 \times 2$ linear system: Find $x_1(t)$ and $x_2(t)$ solutions of

$$x_1' = a_{11}(t) x_1 + a_{12}(t) x_2 + g_1(t),$$
$$x_2' = a_{21}(t) x_1 + a_{22}(t) x_2 + g_2(t).$$

Example

$n = 2$: $2 \times 2$ homogeneous linear system: Find $x_1(t)$ and $x_2(t)$,

$$x_1' = a_{11}(t) x_1 + a_{12}(t) x_2$$
$$x_2' = a_{21}(t) x_1 + a_{22}(t) x_2.$$
\[ n \times n \text{ systems of linear differential equations.} \]

**Example**

Find \( x_1(t), x_2(t) \) solutions of the \( 2 \times 2 \), constant coefficients, homogeneous system

\[
\begin{align*}
  x_1' &= x_1 - x_2, \\
  x_2' &= -x_1 + x_2.
\end{align*}
\]
$n \times n$ systems of linear differential equations.

Example
Find $x_1(t)$, $x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

\[ x'_1 = x_1 - x_2, \]
\[ x'_2 = -x_1 + x_2. \]

Solution: Add up the equations, and subtract the equations,
$n \times n$ systems of linear differential equations.

Example

Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

\[ x_1' = x_1 - x_2, \quad x_2' = -x_1 + x_2. \]

Solution: Add up the equations, and subtract the equations,

\[
(x_1 + x_2)' = 0,
\]

\[
(x_1 - x_2)' = 2(x_1 - x_2).
\]
$n \times n$ systems of linear differential equations.

Example
Find $x_{1}(t)$, $x_{2}(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

\[
\begin{align*}
x_{1}' &= x_{1} - x_{2}, \\
x_{2}' &= -x_{1} + x_{2}.
\end{align*}
\]

Solution: Add up the equations, and subtract the equations,

\[
\begin{align*}
(x_{1} + x_{2})' &= 0, \\
(x_{1} - x_{2})' &= 2(x_{1} - x_{2}).
\end{align*}
\]
$n \times n$ systems of linear differential equations.

Example
Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

$$x'_1 = x_1 - x_2,$$
$$x'_2 = -x_1 + x_2.$$  

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0,$$
$$(x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$,  

$$x_1(t) = \frac{1}{2}(c_1 + c_2 e^{2t}),$$
$$x_2(t) = \frac{1}{2}(c_1 - c_2 e^{2t}).$$
Example

Find \( x_1(t), x_2(t) \) solutions of the 2 \( \times \) 2, constant coefficients, homogeneous system

\[
\begin{align*}
x'_1 &= x_1 - x_2, \\
x'_2 &= -x_1 + x_2.
\end{align*}
\]

Solution: Add up the equations, and subtract the equations,

\[
\begin{align*}
(x_1 + x_2)' &= 0, \\
(x_1 - x_2)' &= 2(x_1 - x_2).
\end{align*}
\]

Introduce the unknowns \( v = x_1 + x_2, \ w = x_1 - x_2, \)
$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

\[ \begin{align*}
    x_1' &= x_1 - x_2, \\
    x_2' &= -x_1 + x_2.
\end{align*} \]

Solution: Add up the equations, and subtract the equations,

\[ (x_1 + x_2)' = 0, \quad (x_1 - x_2)' = 2(x_1 - x_2). \]

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

\[ v' = 0 \]

\[ w' = 2w, \]

We conclude:

\[ x_1(t) = \frac{1}{2} (c_1 + c_2 e^{2t}), \quad x_2(t) = \frac{1}{2} (c_1 - c_2 e^{2t}). \]
$n \times n$ systems of linear differential equations.

**Example**

Find $x_1(t)$, $x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

\[
x_1' = x_1 - x_2,
\]

\[
x_2' = -x_1 + x_2.
\]

**Solution:** Add up the equations, and subtract the equations,

\[
(x_1 + x_2)' = 0,
\]

\[
(x_1 - x_2)' = 2(x_1 - x_2).
\]

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

\[
v' = 0 \quad \Rightarrow \quad v = c_1,
\]
$n \times n$ systems of linear differential equations.

**Example**

Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

\[
\begin{align*}
    x'_1 &= x_1 - x_2, \\
    x'_2 &= -x_1 + x_2.
\end{align*}
\]

**Solution:** Add up the equations, and subtract the equations,

\[
\begin{align*}
    (x_1 + x_2)' &= 0, \\
    (x_1 - x_2)' &= 2(x_1 - x_2).
\end{align*}
\]

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

\[
\begin{align*}
    v' &= 0 \quad \Rightarrow \quad v = c_1, \\
    w' &= 2w
\end{align*}
\]
$n \times n$ systems of linear differential equations.

**Example**

Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

\[ x'_1 = x_1 - x_2, \quad x'_2 = -x_1 + x_2. \]

**Solution:** Add up the equations, and subtract the equations,

\[ (x_1 + x_2)' = 0, \quad (x_1 - x_2)' = 2(x_1 - x_2). \]

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

\[ v' = 0 \quad \Rightarrow \quad v = c_1, \]

\[ w' = 2w \quad \Rightarrow \quad w = c_2 e^{2t}. \]
$n \times n$ systems of linear differential equations.

Example
Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

\[
x_1' = x_1 - x_2, \quad x_2' = -x_1 + x_2.
\]

Solution: Add up the equations, and subtract the equations,

\[
(x_1 + x_2)' = 0, \quad (x_1 - x_2)' = 2(x_1 - x_2).
\]

Introduce the unknowns $v = x_1 + x_2, \ w = x_1 - x_2$, then

\[
v' = 0 \quad \Rightarrow \quad v = c_1,
\]

\[
w' = 2w \quad \Rightarrow \quad w = c_2 e^{2t}.
\]

Back to $x_1$ and $x_2$: 
\( n \times n \) systems of linear differential equations.

Example

Find \( x_1(t) \), \( x_2(t) \) solutions of the \( 2 \times 2 \), constant coefficients, homogeneous system

\[
\begin{align*}
  x_1' &= x_1 - x_2, \\
  x_2' &= -x_1 + x_2.
\end{align*}
\]

Solution: Add up the equations, and subtract the equations,

\[
\begin{align*}
  (x_1 + x_2)' &= 0, \\
  (x_1 - x_2)' &= 2(x_1 - x_2).
\end{align*}
\]

Introduce the unknowns \( v = x_1 + x_2 \), \( w = x_1 - x_2 \), then

\[
\begin{align*}
  v' &= 0 \quad \Rightarrow \quad v = c_1, \\
  w' &= 2w \quad \Rightarrow \quad w = c_2 e^{2t}.
\end{align*}
\]

Back to \( x_1 \) and \( x_2 \):

\[
x_1 = \frac{1}{2} (v + w),
\]
$n \times n$ systems of linear differential equations.

Example

Find $x_1(t), x_2(t)$ solutions of the $2 \times 2$, constant coefficients, homogeneous system

\[ x_1' = x_1 - x_2, \]
\[ x_2' = -x_1 + x_2. \]

Solution: Add up the equations, and subtract the equations,

\[ (x_1 + x_2)' = 0, \quad (x_1 - x_2)' = 2(x_1 - x_2). \]

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

\[ v' = 0 \quad \Rightarrow \quad v = c_1, \]
\[ w' = 2w \quad \Rightarrow \quad w = c_2 e^{2t}. \]

Back to $x_1$ and $x_2$:

\[ x_1 = \frac{1}{2} (v + w), \quad x_2 = \frac{1}{2} (v - w). \]
Example

Find \( x_1(t) \), \( x_2(t) \) solutions of the \( 2 \times 2 \), constant coefficients, homogeneous system

\[
\begin{align*}
x_1' &= x_1 - x_2, \\
x_2' &= -x_1 + x_2.
\end{align*}
\]

Solution: Add up the equations, and subtract the equations,

\[
(x_1 + x_2)' = 0, \\
(x_1 - x_2)' = 2(x_1 - x_2).
\]

Introduce the unknowns \( v = x_1 + x_2 \), \( w = x_1 - x_2 \), then

\[
\begin{align*}
v' &= 0 \quad \Rightarrow \quad v = c_1, \\
w' &= 2w \quad \Rightarrow \quad w = c_2 e^{2t}.
\end{align*}
\]

Back to \( x_1 \) and \( x_2 \):

\[
\begin{align*}
x_1 &= \frac{1}{2} (v + w), \\
x_2 &= \frac{1}{2} (v - w).
\end{align*}
\]

We conclude:

\[
\begin{align*}
x_1(t) &= \frac{1}{2} (c_1 + c_2 e^{2t}), \\
x_2(t) &= \frac{1}{2} (c_1 - c_2 e^{2t}).
\end{align*}
\]
Systems of linear differential equations (Sect. 5.1).

- $n \times n$ systems of linear differential equations.
- **Second order equations and first order systems.**
- Main concepts from Linear Algebra.
Second order equations and first order systems.

Theorem (Reduction to first order)

Every solution $y$ to the second order linear equation

$$y'' + p(t) y' + q(t) y = g(t), \quad (1)$$

defines a solution $x_1 = y$ and $x_2 = y'$ of the $2 \times 2$ first order linear differential system

$$x_1' = x_2, \quad \quad \quad (2)$$
$$x_2' = -q(t) x_1 - p(t) x_2 + g(t). \quad \quad \quad (3)$$

Conversely, every solution $x_1, x_2$ of the $2 \times 2$ first order linear system in Eqs. (2)-(3) defines a solution $y = x_1$ of the second order differential equation in (1).
Second order equations and first order systems.

Proof:

\( \Rightarrow \) Given \( y \) solution of \( y'' + p(t)y' + q(t)y = g(t) \),
Second order equations and first order systems.

Proof:

$(\Rightarrow)$ Given $y$ solution of $y'' + p(t) y' + q(t) y = g(t)$,
introduce $x_1 = y$ and $x_2 = y'$,
Second order equations and first order systems.

Proof:

$(\Rightarrow)$ Given $y$ solution of $y'' + p(t) y' + q(t) y = g(t)$,
introduce $x_1 = y$ and $x_2 = y'$, hence $x'_1 = y' = x_2$, 

$(\Leftarrow)$ Introduce $x_2 = x'_1$ into $x''_1 = -q(t) x_1 - p(t) x_2 + g(t)$. 

that is, $x''_1 + p(t) x'_1 + q(t) x_1 = g(t)$. 

Second order equations and first order systems.

Proof:

($\Rightarrow$) Given $y$ solution of $y'' + p(t)y' + q(t)y = g(t)$, introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$
Second order equations and first order systems.

Proof:

$(\Rightarrow)$ Given $y$ solution of $y'' + p(t) y' + q(t) y = g(t)$, introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$ 

Then, $x_2' = y''$
Second order equations and first order systems.

Proof:

(⇒) Given $y$ solution of \[ y'' + p(t) y' + q(t) y = g(t), \]
introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

\[ x_1' = x_2. \]

Then, $x_2' = y'' = -q(t) y - p(t) y' + g(t)$. 

Second order equations and first order systems.

Proof:

\((\Rightarrow)\) Given \(y\) solution of \(y'' + p(t)\,y' + q(t)\,y = g(t)\), introduce \(x_1 = y\) and \(x_2 = y'\), hence \(x_1' = y' = x_2\), that is,

\[ x_1' = x_2. \]

Then, \(x_2' = y'' = -q(t)\,y - p(t)\,y' + g(t)\). That is,

\[ x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t). \]
Second order equations and first order systems.

Proof:

$(\Rightarrow)$ Given $y$ solution of $y'' + p(t) y' + q(t) y = g(t)$, introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$

Then, $x_2' = y'' = -q(t) y - p(t) y' + g(t)$. That is,

$$x_2' = -q(t) x_1 - p(t) x_2 + g(t).$$

$(\Leftarrow)$ Introduce $x_2 = x_1'$ into $x_2' = -q(t) x_1 - p(t) x_2 + g(t)$. 

Second order equations and first order systems.

Proof:

(⇒) Given $y$ solution of $y'' + p(t) y' + q(t) y = g(t)$, introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$

Then, $x_2' = y'' = -q(t) y - p(t) y' + g(t)$. That is,

$$x_2' = -q(t) x_1 - p(t) x_2 + g(t).$$

(⇐) Introduce $x_2 = x_1'$ into $x_2' = -q(t) x_1 - p(t) x_2 + g(t)$.

$$x_1'' = -q(t) x_1 - p(t) x_1' + g(t),$$
Second order equations and first order systems.

Proof:
(⇒) Given \( y \) solution of \( y'' + p(t) y' + q(t) y = g(t) \), introduce \( x_1 = y \) and \( x_2 = y' \), hence \( x'_1 = y' = x_2 \), that is,
\[
x'_1 = x_2.
\]

Then, \( x'_2 = y''' = -q(t) y - p(t) y' + g(t) \). That is,
\[
x'_2 = -q(t) x_1 - p(t) x_2 + g(t).
\]

(⇐) Introduce \( x_2 = x'_1 \) into \( x'_2 = -q(t) x_1 - p(t) x_2 + g(t) \).

\[
x''_1 = -q(t) x_1 - p(t) x'_1 + g(t),
\]
that is
\[
x''_1 + p(t) x'_1 + q(t) x_1 = g(t).
\]
\[
\boxed{}
\]
Second order equations and first order systems.

Example
Express as a first order system the equation

\[ y'' + 2y' + 2y = \sin(at). \]
Second order equations and first order systems.

Example
Express as a first order system the equation

\[ y'' + 2y' + 2y = \sin(at). \]

Solution: Introduce the new unknowns

\[ x_1 = y, \quad x_2 = y' \]
Second order equations and first order systems.

**Example**
Express as a first order system the equation

\[ y'' + 2y' + 2y = \sin(\alpha t). \]

**Solution:** Introduce the new unknowns

\[ x_1 = y, \quad x_2 = y' \quad \Rightarrow \quad x_1' = x_2. \]
Example

Express as a first order system the equation

\[ y'' + 2y' + 2y = \sin(at) \].

Solution: Introduce the new unknowns

\[ x_1 = y, \quad x_2 = y' \quad \Rightarrow \quad x_1' = x_2. \]

Then, the differential equation can be written as

\[ x_2' + 2x_2 + 2x_1 = \sin(at). \]
Second order equations and first order systems.

Example
Express as a first order system the equation
\[ y'' + 2y' + 2y = \sin(at). \]

Solution: Introduce the new unknowns
\[ x_1 = y, \quad x_2 = y' \implies x_1' = x_2. \]

Then, the differential equation can be written as
\[ x_2' + 2x_2 + 2x_1 = \sin(at). \]

We conclude that
\[ x_1' = x_2. \]
\[ x_2' = -2x_1 - 2x_2 + \sin(at). \]
Second order equations and first order systems.

**Remark:** Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example: Express as a single second order equation the $2 \times 2$ system and solve it,

\[
x_1' = -x_1 + 3x_2,
\]
\[
x_2' = x_1 - x_2.
\]

Solution:

Compute $x_1$ from the second equation:

\[
x_1 = x_2' + x_2.
\]

Introduce this expression into the first equation,

\[
(x_2' + x_2)' = - (x_2' + x_2) + 3x_2,
\]

\[
x_2'' + 2x_2' - 2x_2 = 0.
\]
Second order equations and first order systems.

**Remark:** Systems of first order equations can, sometimes, be transformed into a second order single equation.

**Example**

Express as a single second order equation the $2 \times 2$ system and solve it,

\[
\begin{align*}
  x_1' &= -x_1 + 3x_2, \\
  x_2' &= x_1 - x_2.
\end{align*}
\]
Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example

Express as a single second order equation the $2 \times 2$ system and solve it,

$$x_1' = -x_1 + 3x_2, \quad x_2' = x_1 - x_2.$$

Solution: Compute $x_1$ from the second equation:
Second order equations and first order systems.

**Remark:** Systems of first order equations can, sometimes, be transformed into a second order single equation.

**Example**

Express as a single second order equation the $2 \times 2$ system and solve it,

\[ x_1' = -x_1 + 3x_2, \]
\[ x_2' = x_1 - x_2. \]

**Solution:** Compute $x_1$ from the second equation:

\[ x_1 = x_2' + x_2. \]
Second order equations and first order systems.

**Remark:** Systems of first order equations can, sometimes, be transformed into a second order single equation.

**Example**

Express as a single second order equation the $2 \times 2$ system and solve it,

\[
\begin{align*}
x_1' &= -x_1 + 3x_2, \\
x_2' &= x_1 - x_2.
\end{align*}
\]

**Solution:** Compute $x_1$ from the second equation:

\[
x_1 = x_2' + x_2.
\]

Introduce this expression into the first equation,
Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example

Express as a single second order equation the $2 \times 2$ system and solve it,

\[
\begin{align*}
    x_1' &= -x_1 + 3x_2, \\
    x_2' &= x_1 - x_2. 
\end{align*}
\]

Solution: Compute $x_1$ from the second equation: $x_1 = x_2' + x_2$. Introduce this expression into the first equation,

\[
(x_2' + x_2)' = -(x_2' + x_2) + 3x_2, 
\]
Second order equations and first order systems.

**Remark:** Systems of first order equations can, sometimes, be transformed into a second order single equation.

**Example**
Express as a single second order equation the $2 \times 2$ system and solve it,

$x_1' = -x_1 + 3x_2,$
$x_2' = x_1 - x_2.$

**Solution:** Compute $x_1$ from the second equation:

$x_1 = x_2' + x_2.$

Introduce this expression into the first equation,

$(x_2' + x_2)' = -(x_2' + x_2) + 3x_2,$

$x_2'' + x_2' = -x_2' - x_2 + 3x_2,$


Second order equations and first order systems.

**Remark:** Systems of first order equations can, sometimes, be transformed into a second order single equation.

**Example**

Express as a single second order equation the $2 \times 2$ system and solve it,

$x_1' = -x_1 + 3x_2,$

$x_2' = x_1 - x_2.$

**Solution:** Compute $x_1$ from the second equation:

$x_1 = x_2' + x_2.$

Introduce this expression into the first equation,

\[
(x_2' + x_2)' = -(x_2' + x_2) + 3x_2,
\]

\[
x_2'' + x_2' = -x_2' - x_2 + 3x_2,
\]

\[
x_2'' + 2x_2' - 2x_2 = 0.
\]
Second order equations and first order systems.

Example

Express as a single second order equation the $2 \times 2$ system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Recall: $x_2'' + 2x_2' - 2x_2 = 0$. 

\[
\begin{align*}
    x_2'' & = -2x_2' + 2x_2 \\
    & = r_1x_1 + r_2x_2 \\
\end{align*}
\]

\[
\begin{align*}
    r_1 & = 1 + r_2 \\
    r_2 & = 1 - r_2 \\
\end{align*}
\]

\[
\begin{align*}
    x_1 & = c_1(r_1 + 1)e^{rt} + c_2(r_2 - 1)e^{-rt} \\
    x_2 & = c_1(r_1 + 1)e^{rt} + c_2(r_2 - 1)e^{-rt} \\
\end{align*}
\]

\[
\begin{align*}
    x_1 & = c_1(1 + r_2)e^{rt} + c_2(1 - r_2)e^{-rt} \\
    x_2 & = c_1(1 + r_2)e^{rt} + c_2(1 - r_2)e^{-rt} \\
\end{align*}
\]
Second order equations and first order systems.

Example
Express as a single second order equation the 2 × 2 system and solve it,

\[ x_1' = -x_1 + 3x_2, \]
\[ x_2' = x_1 - x_2. \]

Solution: Recall:
\[ x''_2 + 2x'_2 - 2x_2 = 0. \]
\[ r^2 + 2r - 2 = 0 \]
Example

Express as a single second order equation the $2 \times 2$ system and solve it,

\[ x_1' = -x_1 + 3x_2, \]
\[ x_2' = x_1 - x_2. \]

Solution: Recall: \[ x_2'' + 2x_2' - 2x_2 = 0. \]

\[ r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} \left[ -2 \pm \sqrt{4 + 8} \right] \]
Example

Express as a single second order equation the 2 × 2 system and solve it,

\[ x'_1 = -x_1 + 3x_2, \]
\[ x'_2 = x_1 - x_2. \]

Solution: Recall: \[ x''_2 + 2x'_2 - 2x_2 = 0. \]

\[ r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} \left[-2 \pm \sqrt{4 + 8}\right] \quad \Rightarrow \quad r_{\pm} = -1 \pm \sqrt{3}. \]
Example

Express as a single second order equation the $2 \times 2$ system and solve it,

$x'_1 = -x_1 + 3x_2,$
$x'_2 = x_1 - x_2.$

Solution: Recall: $x''_2 + 2x'_2 - 2x_2 = 0.$

$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} \left[ -2 \pm \sqrt{4 + 8} \right] \quad \Rightarrow \quad r_{\pm} = -1 \pm \sqrt{3}.$

Therefore, $x_2 = c_1 e^{r_+ t} + c_2 e^{r_- t}.$
Second order equations and first order systems.

Example
Express as a single second order equation the $2 \times 2$ system and solve it,

\[ x_1' = -x_1 + 3x_2, \]
\[ x_2' = x_1 - x_2. \]

Solution: Recall: \[ x_2'' + 2x_2' - 2x_2 = 0. \]

\[ r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r = \frac{1}{2} \left[ -2 \pm \sqrt{4 + 8} \right] \quad \Rightarrow \quad r = -1 \pm \sqrt{3}. \]

Therefore, \[ x_2 = c_1 e^{r_+ t} + c_2 e^{r_- t}. \] Since \[ x_1 = x_2' + x_2, \]
Second order equations and first order systems.

Example
Express as a single second order equation the 2 × 2 system and solve it,

\[
x_1' = -x_1 + 3x_2,
\]
\[
x_2' = x_1 - x_2.
\]

Solution: Recall: \(x_2'' + 2x_2' - 2x_2 = 0\).

\[
r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} \left[-2 \pm \sqrt{4 + 8}\right] \quad \Rightarrow \quad r_{\pm} = -1 \pm \sqrt{3}.
\]

Therefore, \(x_2 = c_1 e^{r_+ t} + c_2 e^{r_- t}\). Since \(x_1 = x_2' + x_2\),

\[
x_1 = (c_1 r_+ e^{r_+ t} + c_2 r_- e^{r_- t}) + (c_1 e^{r_+ t} + c_2 e^{r_- t}),
\]
Second order equations and first order systems.

Example

Express as a single second order equation the $2 \times 2$ system and solve it,

\[
x_1' = -x_1 + 3x_2, \\
x_2' = x_1 - x_2.
\]

Solution: Recall: \( x''_2 + 2x'_2 - 2x_2 = 0 \).

\[
r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2} \left[ -2 \pm \sqrt{4 + 8} \right] \quad \Rightarrow \quad r_{\pm} = -1 \pm \sqrt{3}.
\]

Therefore, \( x_2 = c_1 e^{r_+ t} + c_2 e^{r_- t} \). Since \( x_1 = x'_2 + x_2 \),

\[
x_1 = \left( c_1 r_+ e^{r_+ t} + c_2 r_- e^{r_- t} \right) + \left( c_1 e^{r_+ t} + c_2 e^{r_- t} \right),
\]

We conclude: \( x_1 = c_1 (1 + r_+) e^{r_+ t} + c_2 (1 + r_-) e^{r_- t} \). \( \blacksquare \)
Systems of linear differential equations (Sect. 5.1).

- $n \times n$ systems of linear differential equations.
- Second order equations and first order systems.
- Main concepts from Linear Algebra.
Main concepts from Linear Algebra.

Remark: Ideas from Linear Algebra are useful to study systems of linear differential equations.
Main concepts from Linear Algebra.

**Remark:** Ideas from Linear Algebra are useful to study systems of linear differential equations.

We review:

- Matrices $m \times n$.
- Matrix operations.
- $n$-vectors, dot product.
- Matrix-vector product.
Main concepts from Linear Algebra.

Remark: Ideas from Linear Algebra are useful to study systems of linear differential equations.

We review:
- Matrices $m \times n$.
- Matrix operations.
- $n$-vectors, dot product.
- Matrix-vector product.

Definition
An $m \times n$ matrix, $A$, is an array of numbers

$$A = \begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{bmatrix},$$

where $a_{ij} \in \mathbb{C}$ and $i = 1, \cdots, m$, and $j = 1, \cdots, n$. An $n \times n$ matrix is called a square matrix.
Main concepts from Linear Algebra.

Example

(a) $2 \times 2$ matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. 
Main concepts from Linear Algebra.

Example

(a) $2 \times 2$ matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(b) $2 \times 3$ matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.
Main concepts from Linear Algebra.

Example

(a) $2 \times 2$ matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(b) $2 \times 3$ matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

(c) $3 \times 2$ matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$. 
Main concepts from Linear Algebra.

Example

(a) $2 \times 2$ matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(b) $2 \times 3$ matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

(c) $3 \times 2$ matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

(d) $2 \times 2$ complex-valued matrix: $A = \begin{bmatrix} 1 + i & 2 - i \\ 3 & 4i \end{bmatrix}$.
Main concepts from Linear Algebra.

Example

(a) 2 × 2 matrix: \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \).

(b) 2 × 3 matrix: \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \).

(c) 3 × 2 matrix: \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \).

(d) 2 × 2 complex-valued matrix: \( A = \begin{bmatrix} 1 + i & 2 - i \\ 3 & 4i \end{bmatrix} \).

(e) The coefficients of a linear system can be grouped in a matrix,

\[
\begin{align*}
    x'_1 &= -x_1 + 3x_2 \\
    x'_2 &= x_1 - x_2
\end{align*}
\]
Main concepts from Linear Algebra.

Example

(a) 2 × 2 matrix: \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \).

(b) 2 × 3 matrix: \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \).

(c) 3 × 2 matrix: \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \).

(d) 2 × 2 complex-valued matrix: \( A = \begin{bmatrix} 1 + i & 2 - i \\ 3 & 4i \end{bmatrix} \).

(e) The coefficients of a linear system can be grouped in a matrix,

\[
\begin{align*}
\begin{cases}
 x'_1 &= -x_1 + 3x_2 \\
 x'_2 &= x_1 - x_2
\end{cases} 
\end{align*}
\]

\[ \Rightarrow \quad A = \begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}. \]
Main concepts from Linear Algebra.

Remark: An $m \times 1$ matrix is called an $m$-vector.
Main concepts from Linear Algebra.

Remark: An $m \times 1$ matrix is called an $m$-vector.

Definition

An $m$-vector, $\mathbf{v}$, is the array of numbers $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, where the vector components $v_i \in \mathbb{C}$, with $i = 1, \ldots, m$. 

Example: The unknowns of a $2 \times 2$ linear system can be grouped in a $2$-vector, for example,

$\begin{align*}
    x_1' &= -x_1 + 3x_2 \\
    x_2' &= x_1 - x_2
\end{align*}$

$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. 

Main concepts from Linear Algebra.

Remark: An $m \times 1$ matrix is called an $m$-vector.

Definition

An $m$-vector, $\mathbf{v}$, is the array of numbers $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, where the vector components $v_i \in \mathbb{C}$, with $i = 1, \ldots, m$.

Example

The unknowns of a $2 \times 2$ linear system can be grouped in a 2-vector,
Main concepts from Linear Algebra.

Remark: An $m \times 1$ matrix is called an $m$-vector.

Definition

An $m$-vector, $\mathbf{v}$, is the array of numbers $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, where the vector components $v_i \in \mathbb{C}$, with $i = 1, \ldots, m$.

Example

The unknowns of a $2 \times 2$ linear system can be grouped in a 2-vector, for example,

\[
\begin{align*}
x'_1 &= -x_1 + 3x_2 \\
x'_2 &= x_1 - x_2
\end{align*}
\]
Main concepts from Linear Algebra.

Remark: An $m \times 1$ matrix is called an $m$-vector.

Definition

An $m$-vector, $\mathbf{v}$, is the array of numbers $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, where the vector components $v_i \in \mathbb{C}$, with $i = 1, \ldots, m$.

Example

The unknowns of a $2 \times 2$ linear system can be grouped in a 2-vector, for example,

$$\begin{aligned}
    x'_1 &= -x_1 + 3x_2 \\
    x'_2 &= x_1 - x_2
\end{aligned} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$
Main concepts from Linear Algebra.

Remark: We present only examples of matrix operations.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 + 2i & -1 + 2i \\ 3i & 2 - 1i \end{bmatrix}$.

(a) $A^\text{transpose}$: Interchange rows with columns:

$A^T = \begin{bmatrix} 1 & 3i \\ 2 + i & -1 + 2i \end{bmatrix}$. Notice that: $(A^T)^T = A$.

(b) $A^\text{conjugate}$: Conjugate every matrix coefficient:

$A^* = \begin{bmatrix} 1 & 2 - i \\ -1 - 1i & -2 - 2i \\ -3i & 1 \end{bmatrix}$. Notice that: $(A^*)^* = A$.

Matrix $A$ is real iff $A = A^*$. Matrix $A$ is imaginary iff $A = -A^*$.
Main concepts from Linear Algebra.

Remark: We present only examples of matrix operations.

Example
Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-transpose: Interchange rows with columns: $A^T = \begin{bmatrix} 1 & 3i \\ 2 + i & 2 \\ -1 + 2i & 1 \end{bmatrix}$.

Notice that: $(A^T)^T = A$.

(b) $A$-conjugate: Conjugate every matrix coefficient: $A = \begin{bmatrix} 1 & 2 - i & -1 - 2i \\ 3i & 2 & 1 \end{bmatrix}$.

Notice that: $(A)^* = A$.

Matrix $A$ is real iff $A = A$.

Matrix $A$ is imaginary iff $A = -A$. 
Main concepts from Linear Algebra.

Remark: We present only examples of matrix operations.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2 + i & 2 \\ -1 + 2i & 1 \end{bmatrix}.$$
Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

**Example**

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) **$A$-transpose**: Interchange rows with columns:

$A^T = \begin{bmatrix} 1 & 3i \\ 2 + i & 2 \\ -1 + 2i & 1 \end{bmatrix}$.
Main concepts from Linear Algebra.

Remark: We present only examples of matrix operations.

Example
Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2 + i & 2 \\ -1 + 2i & 1 \end{bmatrix}.$$  Notice that: $(A^T)^T = A$. 

Matrix $A$ is real iff $A = -A$. Matrix $A$ is imaginary iff $A = A$. 

Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2 + i & 2 \\ -1 + 2i & 1 \end{bmatrix}. \text{ Notice that: } (A^T)^T = A.$$

(b) $A$-conjugate: Conjugate every matrix coefficient:
Main concepts from Linear Algebra.

Remark: We present only examples of matrix operations.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2+i & 2 \\ -1+2i & 1 \end{bmatrix}.$$ Notice that: $(A^T)^T = A$.

(b) $A$-conjugate: Conjugate every matrix coefficient:

$$\overline{A} = \begin{bmatrix} 1 & 2-i & -1-2i \\ -3i & 2 & 1 \end{bmatrix}.$$
Main concepts from Linear Algebra.

Remark: We present only examples of matrix operations.

Example
Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2 + i & 2 \\ -1 + 2i & 1 \end{bmatrix}.$$  Notice that: $(A^T)^T = A$.

(b) $A$-conjugate: Conjugate every matrix coefficient:

$$\overline{A} = \begin{bmatrix} 1 & 2 - i & -1 - 2i \\ -3i & 2 & 1 \end{bmatrix}.$$  Notice that: $(\overline{A}) = A$. 

Matrix $A$ is real iff $A = \overline{A}$.
Matrix $A$ is imaginary iff $A = -\overline{A}$. 
Main concepts from Linear Algebra.

Remark: We present only examples of matrix operations.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2 + i & 2 \\ -1 + 2i & 1 \end{bmatrix}.$$ Notice that: $(A^T)^T = A$.

(b) $A$-conjugate: Conjugate every matrix coefficient:

$$\bar{A} = \begin{bmatrix} 1 & 2 - i & -1 - 2i \\ -3i & 2 & 1 \end{bmatrix}.$$ Notice that: $(\bar{A}) = A$.

Matrix $A$ is real iff $\bar{A} = A$. 
Main concepts from Linear Algebra.

Remark: We present only examples of matrix operations.

Example
Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2 + i & 2 \\ -1 + 2i & 1 \end{bmatrix}.$$ Notice that: $(A^T)^T = A$.

(b) $A$-conjugate: Conjugate every matrix coefficient:

$$\overline{A} = \begin{bmatrix} 1 & 2 - i & -1 - 2i \\ -3i & 2 & 1 \end{bmatrix}.$$ Notice that: $\overline{(A)} = A$.

Matrix $A$ is real iff $\overline{A} = A$. Matrix $A$ is imaginary iff $\overline{A} = -A$. 
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-adjoint: Conjugate and transpose:
Main concepts from Linear Algebra.

Example
Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2 - i & 2 \\ -1 - 2i & 1 \end{bmatrix}.$$
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2 - i & 2 \\ -1 - 2i & 1 \end{bmatrix}.$$ Notice that: $(A^*)^* = A.$
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2 - i & 2 \\ -1 - 2i & 1 \end{bmatrix}.$$ Notice that: $(A^*)^* = A$.

(b) Addition of two $m \times n$ matrices is performed component-wise:
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2 - i & 2 \\ -1 - 2i & 1 \end{bmatrix}.$$ Notice that: $(A^*)^* = A$.

(b) Addition of two $m \times n$ matrices is performed component-wise:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$$
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2 - i & 2 \\ -1 - 2i & 1 \end{bmatrix}.$$ 

Notice that: $\left(A^*\right)^* = A$.

(b) Addition of two $m \times n$ matrices is performed component-wise:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3+5) & (4+1) \end{bmatrix}.$$
Main concepts from Linear Algebra.

Example
Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2 - i & 2 \\ -1 - 2i & 1 \end{bmatrix}.$$ Notice that: $(A^*)^* = A$.

(b) Addition of two $m \times n$ matrices is performed component-wise:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} (1 + 2) & (2 + 3) \\ (3 + 5) & (4 + 1) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 8 & 5 \end{bmatrix}.$$
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 2 + i & -1 + 2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) $A$-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2 - i & 2 \\ -1 - 2i & 1 \end{bmatrix}.$$ Notice that: $(A^*)^* = A$.

(b) Addition of two $m \times n$ matrices is performed component-wise:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} (1 + 2) & (2 + 3) \\ (3 + 5) & (4 + 1) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 8 & 5 \end{bmatrix}.$$

The addition $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is not defined.
Main concepts from Linear Algebra.

Example
Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:
Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix},$$
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}, \quad \begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix}$$
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

\[
2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}, \quad \begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix} = 4 \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.
\]
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}, \quad \begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix} = 4 \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$  

Also:

$$A \div 3$$
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix} , \quad \frac{8}{3} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = 4 \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} .$$

Also:

$$\frac{A}{3} = \frac{1}{3} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$
Main concepts from Linear Algebra.

Example

Consider a $2 \times 3$ matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}, \quad \begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix} = 4 \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

Also:

$$\frac{A}{3} = \frac{1}{3} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{2}{3} & \frac{4}{3} & 2 \end{bmatrix}.$$
Main concepts from Linear Algebra.

Example

(a) Matrix multiplication.
Main concepts from Linear Algebra.

Example

(a) **Matrix multiplication.** The matrix sizes is important:

\[
\begin{bmatrix}
4 & 3 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
= 
\begin{bmatrix}
16 & 23 & 30 \\
6 & 9 & 12
\end{bmatrix}
\]

Notice \(BA\) is not defined.
Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

\[
A \quad \text{times} \quad B \quad \text{defines} \quad AB \\
\begin{array}{ccc}
m \times n & n \times l & m \times l
\end{array}
\]

Example:

\[
A \quad \text{is} \quad 2 \times 2, \quad B \quad \text{is} \quad 2 \times 3, \quad \text{so} \quad AB \quad \text{is} \quad 2 \times 3:
\]

\[
\begin{bmatrix}
4 & 3 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
= 
\begin{bmatrix}
16 & 23 & 30 \\
6 & 9 & 12
\end{bmatrix}
\]

Notice \(B\) is \(2 \times 3\), \(A\) is \(2 \times 2\), so \(BA\) is not defined.

\[
BA = 
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\begin{bmatrix}
4 & 3 \\
2 & 1
\end{bmatrix}
\text{not defined}
\]
Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

\[
A \quad \text{times} \quad B \quad \text{defines} \quad AB \\
m \times n \quad n \times \ell \quad m \times \ell
\]

Example: \(A\) is 2 \(\times\) 2, \(B\) is 2 \(\times\) 3, so \(AB\) is 2 \(\times\) 3:
Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

\[
A \text{ times } B \text{ defines } AB \\
\begin{array}{c|c|c}
| & | & | \\
| m \times n | n \times \ell | m \times \ell \\
\end{array}
\]

Example: A is 2 \times 2, B is 2 \times 3, so AB is 2 \times 3:

\[
AB = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 16 & 23 & 30 \\ 6 & 9 & 12 \end{bmatrix}.
\]
Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

\[ A \times B \text{ defines } AB \]

\[ m \times n \quad n \times \ell \quad m \times \ell \]

Example: \(A\) is \(2 \times 2\), \(B\) is \(2 \times 3\), so \(AB\) is \(2 \times 3\):

\[
AB = \begin{bmatrix}
4 & 3 \\
2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{bmatrix}
= \begin{bmatrix}
16 & 23 & 30 \\
6 & 9 & 12 \\
\end{bmatrix}.
\]

Notice \(B\) is \(2 \times 3\), \(A\) is \(2 \times 2\), so \(BA\) is not defined.
Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

\[
A \times B \quad \text{defines} \quad AB
\]

\[
m \times n \quad n \times \ell \quad m \times \ell
\]

Example: $A$ is $2 \times 2$, $B$ is $2 \times 3$, so $AB$ is $2 \times 3$:

\[
AB = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 16 & 23 & 30 \\ 6 & 9 & 12 \end{bmatrix}.
\]

Notice $B$ is $2 \times 3$, $A$ is $2 \times 2$, so $BA$ is not defined.

\[
BA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \quad \text{not defined.}
\]
Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$. 

Example: Find $AB$ and $BA$ for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6 - 2) & (0 + 1) \\ (-3 + 4) & (0 - 2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$.

$BA = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (6 + 0) & (-3 + 0) \\ (4 + 1) & (-2 - 2) \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 5 & -4 \end{bmatrix}$.

So $AB \neq BA$. \boxed{\quad}$
Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example

Find $AB$ and $BA$ for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.
Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example
Find $AB$ and $BA$ for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6 - 2) & (0 + 1) \\ (-3 + 4) & (0 - 2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}.$$
Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example

Find $AB$ and $BA$ for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6 - 2) & (0 + 1) \\ (-3 + 4) & (0 - 2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$.
Main concepts from Linear Algebra.

**Remark:** The matrix product is not commutative, that is, in general holds $AB \neq BA$.

**Example**

Find $AB$ and $BA$ for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

**Solution:**

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6 - 2) & (0 + 1) \\ (-3 + 4) & (0 - 2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}.$$
Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example

Find $AB$ and $BA$ for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6 - 2) & (0 + 1) \\ (-3 + 4) & (0 - 2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$.

$BA = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$
Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds \( AB \neq BA \).

Example

Find \( AB \) and \( BA \) for \( A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \).

Solution:

\[
\begin{align*}
AB &= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6 - 2) & (0 + 1) \\ (-3 + 4) & (0 - 2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}.
\end{align*}
\]

\[
\begin{align*}
BA &= \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (6 + 0) & (-3 + 0) \\ (4 + 1) & (-2 - 2) \end{bmatrix}.
\end{align*}
\]
Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds \( AB \neq BA \).

Example

Find \( AB \) and \( BA \) for \( A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \).

Solution:

\[
AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6 - 2) & (0 + 1) \\ (-3 + 4) & (0 - 2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}.
\]

\[
BA = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (6 + 0) & (-3 + 0) \\ (4 + 1) & (-2 - 2) \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 5 & -4 \end{bmatrix}.
\]

So \( AB \neq BA \).
Main concepts from Linear Algebra.

**Remark:** The matrix product is not commutative, that is, in general holds $AB \neq BA$.

**Example**

Find $AB$ and $BA$ for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

**Solution:**

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6-2) & (0+1) \\ (-3+4) & (0-2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}.$$  

$$BA = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (6+0) & (-3+0) \\ (4+1) & (-2-2) \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 5 & -4 \end{bmatrix}.$$  

So $AB \neq BA$.  

\[ \triangleleft \]
Main concepts from Linear Algebra.

Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$. 

Example: Find $AB$ for

$$A = \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}$$

and

$$B = \begin{bmatrix}
1 & -1 \\
1 & -1
\end{bmatrix}.$$ 

Solution:

$$AB = \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
1 & -1 \\
1 & -1
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}. \quad \Box$$

Recall: If $a, b \in \mathbb{R}$ and $ab = 0$, then either $a = 0$ or $b = 0$. 

We have just shown that this statement is not true for matrices.
Main concepts from Linear Algebra.

Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Example
Find $AB$ for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

\[
AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]
Main concepts from Linear Algebra.

Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Example

Find $AB$ for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$
Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Example

Find $AB$ for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (1-1) & (-1+1) \\ (-1+1) & (1-1) \end{bmatrix}$$
Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Example

Find $AB$ for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (1-1) & (-1+1) \\ (-1+1) & (1-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$
Main concepts from Linear Algebra.

Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Example

Find $AB$ for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (1-1) & (-1+1) \\ (-1+1) & (1-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$  

Recall: If $a, b \in \mathbb{R}$ and $ab = 0$, then either $a = 0$ or $b = 0$.  

Main concepts from Linear Algebra.

**Remark:** There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

**Example**

Find $AB$ for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

**Solution:**

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (1 - 1) & (-1 + 1) \\ (-1 + 1) & (1 - 1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$ 

**Recall:** If $a, b \in \mathbb{R}$ and $ab = 0$, then either $a = 0$ or $b = 0$.

We have just shown that this statement is not true for matrices.