Convolution solutions (Sect. 4.5).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.

Convolution solutions (Sect. 4.5).

- **Convolution of two functions.**
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.

Convolution of two functions.

Definition

The *convolution* of piecewise continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ is the function $f * g : \mathbb{R} \to \mathbb{R}$ given by

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

Convolution of two functions.

Definition

The *convolution* of piecewise continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ is the function $f * g : \mathbb{R} \to \mathbb{R}$ given by

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

Remarks:

- $f * g$ is also called the generalized product of f and g.

Convolution of two functions.

Definition

The *convolution* of piecewise continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ is the function $f * g : \mathbb{R} \to \mathbb{R}$ given by

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

Remarks:

- $f * g$ is also called the generalized product of f and g.
- The definition of convolution of two functions also holds in the case that one of the functions is a generalized function, like Dirac's delta.

Convolution of two functions.

Example

Find the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Convolution of two functions.

Example

Find the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Solution: By definition: $(f * g)(t) = \int_0^t e^{-\tau} \sin(t - \tau)\, d\tau.$

Convolution of two functions.

Example

Find the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Solution: By definition: $(f * g)(t) = \int_0^t e^{-\tau} \sin(t - \tau)\, d\tau.$

Integrate by parts twice:

Convolution of two functions.

Example

Find the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Solution: By definition: $(f * g)(t) = \displaystyle\int_0^t e^{-\tau} \sin(t - \tau)\, d\tau$.

Integrate by parts twice: $\displaystyle\int_0^t e^{-\tau}\, \sin(t - \tau)\, d\tau =$

$$\Big[e^{-\tau} \cos(t - \tau)\Big]\Big|_0^t - \Big[e^{-\tau} \sin(t - \tau)\Big]\Big|_0^t - \int_0^t e^{-\tau}\, \sin(t - \tau)\, d\tau,$$

Convolution of two functions.

Example

Find the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Solution: By definition: $(f * g)(t) = \int_0^t e^{-\tau} \sin(t - \tau)\, d\tau$.

Integrate by parts twice: $\int_0^t e^{-\tau} \sin(t - \tau)\, d\tau =$

$$\Big[e^{-\tau} \cos(t - \tau)\Big]\Big|_0^t - \Big[e^{-\tau} \sin(t - \tau)\Big]\Big|_0^t - \int_0^t e^{-\tau} \sin(t - \tau)\, d\tau,$$

$$2 \int_0^t e^{-\tau} \sin(t - \tau)\, d\tau = \Big[e^{-\tau} \cos(t - \tau)\Big]\Big|_0^t - \Big[e^{-\tau} \sin(t - \tau)\Big]\Big|_0^t,$$

Convolution of two functions.

Example

Find the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Solution: By definition: $(f * g)(t) = \int_0^t e^{-\tau} \sin(t - \tau)\, d\tau$.

Integrate by parts twice: $\displaystyle\int_0^t e^{-\tau} \sin(t-\tau)\, d\tau =$

$$\Big[e^{-\tau}\cos(t-\tau)\Big]\Big|_0^t - \Big[e^{-\tau}\sin(t-\tau)\Big]\Big|_0^t - \int_0^t e^{-\tau} \sin(t-\tau)\, d\tau,$$

$$2\int_0^t e^{-\tau} \sin(t-\tau)\, d\tau = \Big[e^{-\tau}\cos(t-\tau)\Big]\Big|_0^t - \Big[e^{-\tau}\sin(t-\tau)\Big]\Big|_0^t,$$

$$2(f * g)(t) = e^{-t} - \cos(t) - 0 + \sin(t).$$

Convolution of two functions.

Example

Find the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Solution: By definition: $(f * g)(t) = \int_0^t e^{-\tau} \sin(t - \tau)\, d\tau$.

Integrate by parts twice: $\displaystyle\int_0^t e^{-\tau} \sin(t-\tau)\, d\tau =$

$$\Big[e^{-\tau}\cos(t-\tau)\Big]\Big|_0^t - \Big[e^{-\tau}\sin(t-\tau)\Big]\Big|_0^t - \int_0^t e^{-\tau} \sin(t-\tau)\, d\tau,$$

$$2\int_0^t e^{-\tau} \sin(t-\tau)\, d\tau = \Big[e^{-\tau}\cos(t-\tau)\Big]\Big|_0^t - \Big[e^{-\tau}\sin(t-\tau)\Big]\Big|_0^t,$$

$$2(f * g)(t) = e^{-t} - \cos(t) - 0 + \sin(t).$$

We conclude: $(f * g)(t) = \dfrac{1}{2}\big[e^{-t} + \sin(t) - \cos(t)\big]$. $\triangleleft$

Convolution solutions (Sect. 4.5).

- Convolution of two functions.
- **Properties of convolutions.**
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.

Properties of convolutions.

Theorem (Properties)

For every piecewise continuous functions f, g, and h, hold:

(i) *Commutativity:* $f * g = g * f$;

(ii) *Associativity:* $f * (g * h) = (f * g) * h$;

(iii) *Distributivity:* $f * (g + h) = f * g + f * h$;

(iv) *Neutral element:* $f * 0 = 0$;

(v) *Identity element:* $f * \delta = f$.

Properties of convolutions.

Theorem (Properties)

For every piecewise continuous functions f, g, and h, hold:

(i) *Commutativity:* $f * g = g * f$;

(ii) *Associativity:* $f * (g * h) = (f * g) * h$;

(iii) *Distributivity:* $f * (g + h) = f * g + f * h$;

(iv) *Neutral element:* $f * 0 = 0$;

(v) *Identity element:* $f * \delta = f$.

Proof:

(v):

$$(f * \delta)(t) = \int_0^t f(\tau)\, \delta(t - \tau)\, d\tau$$

Properties of convolutions.

Theorem (Properties)

For every piecewise continuous functions f, g, and h, hold:

(i) *Commutativity:* $f * g = g * f$;

(ii) *Associativity:* $f * (g * h) = (f * g) * h$;

(iii) *Distributivity:* $f * (g + h) = f * g + f * h$;

(iv) *Neutral element:* $f * 0 = 0$;

(v) *Identity element:* $f * \delta = f$.

Proof:

(v):

$$(f * \delta)(t) = \int_0^t f(\tau)\,\delta(t - \tau)\,d\tau = \int_0^t f(\tau)\,\delta(\tau - t)\,d\tau$$

Properties of convolutions.

Theorem (Properties)

For every piecewise continuous functions f, g, and h, hold:

(i) *Commutativity:* $f * g = g * f$;

(ii) *Associativity:* $f * (g * h) = (f * g) * h$;

(iii) *Distributivity:* $f * (g + h) = f * g + f * h$;

(iv) *Neutral element:* $f * 0 = 0$;

(v) *Identity element:* $f * \delta = f$.

Proof:

(v):

$$(f * \delta)(t) = \int_0^t f(\tau)\,\delta(t - \tau)\,d\tau = \int_0^t f(\tau)\,\delta(\tau - t)\,d\tau = f(t).$$

Properties of convolutions.

Proof:

(1): Commutativity: $f * g = g * f$.

Properties of convolutions.

Proof:

(1): Commutativity: $f * g = g * f$.

The definition of convolution is,

$$(f * g)(t) = \int_0^t f(\tau)\, g(t - \tau)\, d\tau.$$

Properties of convolutions.

Proof:

(1): Commutativity: $f * g = g * f$.

The definition of convolution is,

$$(f * g)(t) = \int_0^t f(\tau)\, g(t - \tau)\, d\tau.$$

Change the integration variable: $\hat{\tau} = t - \tau$,

Properties of convolutions.

Proof:

(1): Commutativity: $f * g = g * f$.

The definition of convolution is,

$$(f * g)(t) = \int_0^t f(\tau)\, g(t - \tau)\, d\tau.$$

Change the integration variable: $\hat{\tau} = t - \tau$, hence $d\hat{\tau} = -d\tau$,

Properties of convolutions.

Proof:

(1): Commutativity: $f * g = g * f$.

The definition of convolution is,

$$(f * g)(t) = \int_0^t f(\tau)\, g(t - \tau)\, d\tau.$$

Change the integration variable: $\hat{\tau} = t - \tau$, hence $d\hat{\tau} = -d\tau$,

$$(f * g)(t) = \int_t^0 f(t - \hat{\tau})\, g(\hat{\tau})(-1)\, d\hat{\tau}$$

Properties of convolutions.

Proof:

(1): Commutativity: $f * g = g * f$.

The definition of convolution is,

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

Change the integration variable: $\hat{\tau} = t - \tau$, hence $d\hat{\tau} = -d\tau$,

$$(f * g)(t) = \int_t^0 f(t-\hat{\tau})\, g(\hat{\tau})(-1)\, d\hat{\tau}$$

$$(f * g)(t) = \int_0^t g(\hat{\tau})\, f(t-\hat{\tau})\, d\hat{\tau}$$

Properties of convolutions.

Proof:

(1): Commutativity: $f * g = g * f$.

The definition of convolution is,

$$(f * g)(t) = \int_0^t f(\tau)\, g(t - \tau)\, d\tau.$$

Change the integration variable: $\hat{\tau} = t - \tau$, hence $d\hat{\tau} = -d\tau$,

$$(f * g)(t) = \int_t^0 f(t - \hat{\tau})\, g(\hat{\tau})(-1)\, d\hat{\tau}$$

$$(f * g)(t) = \int_0^t g(\hat{\tau})\, f(t - \hat{\tau})\, d\hat{\tau}$$

We conclude: $(f * g)(t) = (g * f)(t)$. □

Convolution solutions (Sect. 4.5).

- Convolution of two functions.
- Properties of convolutions.
- **Laplace Transform of a convolution.**
- Impulse response solution.
- Solution decomposition theorem.

Laplace Transform of a convolution.

Theorem (Laplace Transform)

If f, g *have well-defined Laplace Transforms* $\mathcal{L}[f]$, $\mathcal{L}[g]$, *then*

$$\mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g].$$

Laplace Transform of a convolution.

Theorem (Laplace Transform)

If f, g *have well-defined Laplace Transforms* $\mathcal{L}[f]$, $\mathcal{L}[g]$, *then*

$$\mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g].$$

Proof: The key step is to interchange two integrals.

Laplace Transform of a convolution.

Theorem (Laplace Transform)

If f, g have well-defined Laplace Transforms $\mathcal{L}[f]$, $\mathcal{L}[g]$, then

$$\mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g].$$

Proof: The key step is to interchange two integrals. We start we the product of the Laplace transforms,

$$\mathcal{L}[f]\,\mathcal{L}[g] = \Big[\int_0^\infty e^{-st} f(t)\,dt\Big]\Big[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\,d\tilde{t}\Big],$$

Laplace Transform of a convolution.

Theorem (Laplace Transform)

If f, g have well-defined Laplace Transforms $\mathcal{L}[f]$, $\mathcal{L}[g]$, then

$$\mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g].$$

Proof: The key step is to interchange two integrals. We start we the product of the Laplace transforms,

$$\mathcal{L}[f]\,\mathcal{L}[g] = \Big[\int_0^\infty e^{-st} f(t)\,dt\Big]\,\Big[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\,d\tilde{t}\Big],$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\Big(\int_0^\infty e^{-st} f(t)\,dt\Big)\,d\tilde{t},$$

Laplace Transform of a convolution.

Theorem (Laplace Transform)

If f, g have well-defined Laplace Transforms $\mathcal{L}[f]$, $\mathcal{L}[g]$, then

$$\mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g].$$

Proof: The key step is to interchange two integrals. We start we the product of the Laplace transforms,

$$\mathcal{L}[f]\,\mathcal{L}[g] = \Big[\int_0^\infty e^{-st} f(t)\,dt\Big]\Big[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\,d\tilde{t}\Big],$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty e^{-s\tilde{t}} g(\tilde{t}) \Big(\int_0^\infty e^{-st} f(t)\,dt\Big)\,d\tilde{t},$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \Big(\int_0^\infty e^{-s(t+\tilde{t})} f(t)\,dt\Big)\,d\tilde{t}.$$

Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_0^\infty e^{-s(t+\tilde{t})} f(t)\,dt\Big)\,d\tilde{t}.$

Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_0^\infty e^{-s(t+\tilde{t})} f(t)\,dt\Big)\,d\tilde{t}.$

Change variables: $\tau = t + \tilde{t}$,

Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_0^\infty e^{-s(t+\tilde{t})} f(t)\,dt\Big)\,d\tilde{t}.$

Change variables: $\tau = t + \tilde{t}$, hence $d\tau = dt$;

Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_0^\infty e^{-s(t+\tilde{t})} f(t)\,dt\Big)\,d\tilde{t}.$

Change variables: $\tau = t + \tilde{t}$, hence $d\tau = dt$;

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_{\tilde{t}}^\infty e^{-s\tau} f(\tau - \tilde{t})\,d\tau\Big)\,d\tilde{t}.$$

Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_0^\infty e^{-s(t+\tilde{t})} f(t)\,dt\Big)\,d\tilde{t}.$

Change variables: $\tau = t + \tilde{t}$, hence $d\tau = dt$;

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_{\tilde{t}}^\infty e^{-s\tau} f(\tau - \tilde{t})\,d\tau\Big)\,d\tilde{t}.$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_{\tilde{t}}^\infty e^{-s\tau}\,g(\tilde{t})\,f(\tau - \tilde{t})\,d\tau\,d\tilde{t}.$$

Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_0^\infty e^{-s(t+\tilde{t})} f(t)\,dt\Big)\,d\tilde{t}.$

Change variables: $\tau = t + \tilde{t}$, hence $d\tau = dt$;

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_{\tilde{t}}^\infty e^{-s\tau} f(\tau - \tilde{t})\,d\tau\Big)\,d\tilde{t}.$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_{\tilde{t}}^\infty e^{-s\tau}\, g(\tilde{t})\, f(\tau - \tilde{t})\,d\tau\,d\tilde{t}.$$

The key step: Switch the order of integration.

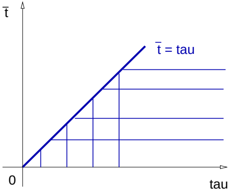
Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_0^\infty e^{-s(t+\tilde{t})} f(t)\,dt\Big)\,d\tilde{t}.$

Change variables: $\tau = t + \tilde{t}$, hence $d\tau = dt$;

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_{\tilde{t}}^\infty e^{-s\tau} f(\tau - \tilde{t})\,d\tau\Big)\,d\tilde{t}.$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_{\tilde{t}}^\infty e^{-s\tau}\,g(\tilde{t})\,f(\tau - \tilde{t})\,d\tau\,d\tilde{t}.$$



The key step: Switch the order of integration.

Laplace Transform of a convolution.

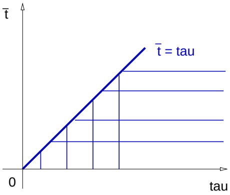
Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_0^\infty e^{-s(t+\tilde{t})} f(t)\,dt\Big)\,d\tilde{t}.$

Change variables: $\tau = t + \tilde{t}$, hence $d\tau = dt$;

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_{\tilde{t}}^\infty e^{-s\tau} f(\tau - \tilde{t})\,d\tau\Big)\,d\tilde{t}.$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_{\tilde{t}}^\infty e^{-s\tau}\,g(\tilde{t})\,f(\tau - \tilde{t})\,d\tau\,d\tilde{t}.$$

The key step: Switch the order of integration.



$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau}\,g(\tilde{t})\,f(\tau - \tilde{t})\,d\tilde{t}\,d\tau.$$

Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau}\, g(\tilde{t})\, f(\tau - \tilde{t})\, d\tilde{t}\, d\tau.$

Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau}\, g(\tilde{t})\, f(\tau - \tilde{t})\, d\tilde{t}\, d\tau.$

Then, is straightforward to check that

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty e^{-s\tau} \left(\int_0^\tau g(\tilde{t})\, f(\tau - \tilde{t})\, d\tilde{t} \right) d\tau,$$

Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^{\infty} \int_0^{\tau} e^{-s\tau}\, g(\tilde{t})\, f(\tau - \tilde{t})\, d\tilde{t}\, d\tau.$

Then, is straightforward to check that

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^{\infty} e^{-s\tau} \left(\int_0^{\tau} g(\tilde{t})\, f(\tau - \tilde{t})\, d\tilde{t} \right) d\tau,$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^{\infty} e^{-s\tau} (g * f)(\tau)\, d\tau$$

Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau}\, g(\tilde{t})\, f(\tau - \tilde{t})\, d\tilde{t}\, d\tau.$

Then, is straightforward to check that

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty e^{-s\tau} \left(\int_0^\tau g(\tilde{t})\, f(\tau - \tilde{t})\, d\tilde{t} \right) d\tau,$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty e^{-s\tau} (g * f)(\tau)\, d\tau$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \mathcal{L}[g * f]$$

Laplace Transform of a convolution.

Proof: Recall: $\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau}\, g(\tilde{t})\, f(\tau - \tilde{t})\, d\tilde{t}\, d\tau.$

Then, is straightforward to check that

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty e^{-s\tau} \left(\int_0^\tau g(\tilde{t})\, f(\tau - \tilde{t})\, d\tilde{t} \right) d\tau,$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty e^{-s\tau} (g * f)(\tau)\, d\tau$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \mathcal{L}[g * f]$$

We conclude: $\mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g].$ $\square$

Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

$$F(s) = \frac{3}{s^3(s^2 - 3)}.$$

Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

$$F(s) = \frac{3}{s^3(s^2 - 3)}.$$

Solution: We express F as a product of two Laplace Transforms,

Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

$$F(s) = \frac{3}{s^3(s^2 - 3)}.$$

Solution: We express F as a product of two Laplace Transforms,

$$F(s) = 3\,\frac{1}{s^3}\,\frac{1}{(s^2 - 3)}$$

Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

$$F(s) = \frac{3}{s^3(s^2 - 3)}.$$

Solution: We express F as a product of two Laplace Transforms,

$$F(s) = 3\,\frac{1}{s^3}\,\frac{1}{(s^2 - 3)} = \frac{3}{2}\,\frac{1}{\sqrt{3}}\left(\frac{2}{s^3}\right)\left(\frac{\sqrt{3}}{s^2 - 3}\right)$$

Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

$$F(s) = \frac{3}{s^3(s^2-3)}.$$

Solution: We express F as a product of two Laplace Transforms,

$$F(s) = 3\,\frac{1}{s^3}\,\frac{1}{(s^2-3)} = \frac{3}{2}\,\frac{1}{\sqrt{3}}\left(\frac{2}{s^3}\right)\left(\frac{\sqrt{3}}{s^2-3}\right)$$

Recalling that $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$

Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

$$F(s) = \frac{3}{s^3(s^2-3)}.$$

Solution: We express F as a product of two Laplace Transforms,

$$F(s) = 3\,\frac{1}{s^3}\,\frac{1}{(s^2-3)} = \frac{3}{2}\,\frac{1}{\sqrt{3}}\left(\frac{2}{s^3}\right)\left(\frac{\sqrt{3}}{s^2-3}\right)$$

Recalling that $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$ and $\mathcal{L}[\sinh(at)] = \dfrac{a}{s^2-a^2}$,

Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

$$F(s) = \frac{3}{s^3(s^2-3)}.$$

Solution: We express F as a product of two Laplace Transforms,

$$F(s) = 3\,\frac{1}{s^3}\,\frac{1}{(s^2-3)} = \frac{3}{2}\,\frac{1}{\sqrt{3}}\left(\frac{2}{s^3}\right)\left(\frac{\sqrt{3}}{s^2-3}\right)$$

Recalling that $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$ and $\mathcal{L}[\sinh(at)] = \dfrac{a}{s^2-a^2}$,

$$F(s) = \frac{\sqrt{3}}{2}\,\mathcal{L}[t^2]\,\mathcal{L}\big[\sinh(\sqrt{3}\,t)\big]$$

Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

$$F(s) = \frac{3}{s^3(s^2-3)}.$$

Solution: We express F as a product of two Laplace Transforms,

$$F(s) = 3\,\frac{1}{s^3}\,\frac{1}{(s^2-3)} = \frac{3}{2}\,\frac{1}{\sqrt{3}}\left(\frac{2}{s^3}\right)\left(\frac{\sqrt{3}}{s^2-3}\right)$$

Recalling that $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$ and $\mathcal{L}[\sinh(at)] = \dfrac{a}{s^2-a^2}$,

$$F(s) = \frac{\sqrt{3}}{2}\,\mathcal{L}[t^2]\,\mathcal{L}\big[\sinh(\sqrt{3}\,t)\big] = \frac{\sqrt{3}}{2}\,\mathcal{L}\big[t^2 * \sin(\sqrt{3}\,t)\big].$$

Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of

$$F(s) = \frac{3}{s^3(s^2-3)}.$$

Solution: We express F as a product of two Laplace Transforms,

$$F(s) = 3\,\frac{1}{s^3}\,\frac{1}{(s^2-3)} = \frac{3}{2}\,\frac{1}{\sqrt{3}}\left(\frac{2}{s^3}\right)\left(\frac{\sqrt{3}}{s^2-3}\right)$$

Recalling that $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$ and $\mathcal{L}[\sinh(at)] = \dfrac{a}{s^2-a^2}$,

$$F(s) = \frac{\sqrt{3}}{2}\,\mathcal{L}[t^2]\,\mathcal{L}\big[\sinh(\sqrt{3}\,t)\big] = \frac{\sqrt{3}}{2}\,\mathcal{L}\big[t^2 * \sin(\sqrt{3}\,t)\big].$$

We conclude that $f(t) = \dfrac{\sqrt{3}}{2}\displaystyle\int_0^t \tau^2 \sinh\big[\sqrt{3}(t-\tau)\big]\,d\tau.$ ◁

Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau$.

Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau$.

Solution: The function f is the convolution of two functions,

Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau$.

Solution: The function f is the convolution of two functions,

$$f(t) = (g * h)(t),$$

Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau$.

Solution: The function f is the convolution of two functions,

$$f(t) = (g * h)(t), \qquad g(t) = \cos(2t),$$

Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau$.

Solution: The function f is the convolution of two functions,

$$f(t) = (g * h)(t), \qquad g(t) = \cos(2t), \quad h(t) = e^{-3t}.$$

Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau$.

Solution: The function f is the convolution of two functions,

$$f(t) = (g * h)(t), \qquad g(t) = \cos(2t), \quad h(t) = e^{-3t}.$$

Since $\mathcal{L}[(g * h)(t)] = \mathcal{L}[g(t)]\, \mathcal{L}[h(t)]$,

Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau$.

Solution: The function f is the convolution of two functions,

$$f(t) = (g * h)(t), \qquad g(t) = \cos(2t), \quad h(t) = e^{-3t}.$$

Since $\mathcal{L}[(g * h)(t)] = \mathcal{L}[g(t)]\, \mathcal{L}[h(t)]$, then,

$$F(s) = \mathcal{L}\Big[\int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau\Big]$$

Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau$.

Solution: The function f is the convolution of two functions,

$$f(t) = (g * h)(t), \qquad g(t) = \cos(2t), \quad h(t) = e^{-3t}.$$

Since $\mathcal{L}[(g * h)(t)] = \mathcal{L}[g(t)]\, \mathcal{L}[h(t)]$, then,

$$F(s) = \mathcal{L}\Big[\int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau\Big] = \mathcal{L}\big[e^{-3t}\big]\, \mathcal{L}\big[\cos(2t)\big].$$

Laplace Transform of a convolution.

Example

Compute $\mathcal{L}[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau$.

Solution: The function f is the convolution of two functions,

$$f(t) = (g * h)(t), \qquad g(t) = \cos(2t), \quad h(t) = e^{-3t}.$$

Since $\mathcal{L}[(g * h)(t)] = \mathcal{L}[g(t)]\, \mathcal{L}[h(t)]$, then,

$$F(s) = \mathcal{L}\Big[\int_0^t e^{-3(t-\tau)} \cos(2\tau)\, d\tau\Big] = \mathcal{L}\big[e^{-3t}\big]\, \mathcal{L}\big[\cos(2t)\big].$$

We conclude that $F(s) = \dfrac{s}{(s+3)(s^2+4)}$. ◁

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,

$(s^2 - 5s + 6)\,\mathcal{L}[y(t)] = \mathcal{L}[g(t)]$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,

$$(s^2 - 5s + 6)\,\mathcal{L}[y(t)] = \mathcal{L}[g(t)] \;\Rightarrow\; \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)}\,G(s).$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,

$$(s^2 - 5s + 6)\,\mathcal{L}[y(t)] = \mathcal{L}[g(t)] \;\Rightarrow\; \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)}\,G(s).$$

Denoting $H(s) = \dfrac{1}{s^2 - 5s + 6}$,

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,

$$(s^2 - 5s + 6)\,\mathcal{L}[y(t)] = \mathcal{L}[g(t)] \;\Rightarrow\; \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)}\,G(s).$$

Denoting $H(s) = \dfrac{1}{s^2 - 5s + 6}$, and $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$,

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,

$$(s^2 - 5s + 6)\,\mathcal{L}[y(t)] = \mathcal{L}[g(t)] \;\Rightarrow\; \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)}\,G(s).$$

Denoting $H(s) = \dfrac{1}{s^2 - 5s + 6}$, and $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, then

$$\mathcal{L}[y(t)] = H(s)\,G(s)$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,

$$(s^2 - 5s + 6)\,\mathcal{L}[y(t)] = \mathcal{L}[g(t)] \;\Rightarrow\; \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)}\,G(s).$$

Denoting $H(s) = \dfrac{1}{s^2 - 5s + 6}$, and $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, then

$$\mathcal{L}[y(t)] = H(s)\,G(s) \quad \Rightarrow \quad y(t) = (h * g)(t).$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,

$$(s^2 - 5s + 6)\,\mathcal{L}[y(t)] = \mathcal{L}[g(t)] \;\Rightarrow\; \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)}\,G(s).$$

Denoting $H(s) = \dfrac{1}{s^2 - 5s + 6}$, and $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, then

$$\mathcal{L}[y(t)] = H(s)\,G(s) \quad \Rightarrow \quad y(t) = (h * g)(t).$$

Function h is simple to compute:

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,

$$(s^2 - 5s + 6)\,\mathcal{L}[y(t)] = \mathcal{L}[g(t)] \;\Rightarrow\; \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)}\,G(s).$$

Denoting $H(s) = \dfrac{1}{s^2 - 5s + 6}$, and $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, then

$$\mathcal{L}[y(t)] = H(s)\,G(s) \quad \Rightarrow \quad y(t) = (h * g)(t).$$

Function h is simple to compute:

$$H(s) = \frac{1}{(s-2)(s-3)}$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,

$$(s^2 - 5s + 6)\,\mathcal{L}[y(t)] = \mathcal{L}[g(t)] \;\Rightarrow\; \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)}\,G(s).$$

Denoting $H(s) = \dfrac{1}{s^2 - 5s + 6}$, and $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, then

$$\mathcal{L}[y(t)] = H(s)\,G(s) \quad \Rightarrow \quad y(t) = (h * g)(t).$$

Function h is simple to compute:

$$H(s) = \frac{1}{(s-2)(s-3)} = \frac{a}{(s-2)} + \frac{b}{(s-3)}$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Denote $G(s) = \mathcal{L}[g(t)]$ and compute LT of the equation,

$$(s^2 - 5s + 6)\,\mathcal{L}[y(t)] = \mathcal{L}[g(t)] \;\Rightarrow\; \mathcal{L}[y(t)] = \frac{1}{(s^2 - 5s + 6)}\,G(s).$$

Denoting $H(s) = \dfrac{1}{s^2 - 5s + 6}$, and $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, then

$$\mathcal{L}[y(t)] = H(s)\,G(s) \quad \Rightarrow \quad y(t) = (h * g)(t).$$

Function h is simple to compute:

$$H(s) = \frac{1}{(s-2)(s-3)} = \frac{a}{(s-2)} + \frac{b}{(s-3)} = \frac{a(s-3) + b(s-2)}{(s-2)(s-3)}$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Then: $1 = a(s-3) + b(s-2)$.

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Then: $1 = a(s-3) + b(s-2)$. Evaluate at $s = 2,\ 3$.

$$s = 2$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Then: $1 = a(s-3) + b(s-2)$. Evaluate at $s = 2,\ 3$.

$$s = 2 \quad \Rightarrow \quad a = -1.$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Then: $1 = a(s-3) + b(s-2)$. Evaluate at $s = 2,\ 3$.

$$s = 2 \quad \Rightarrow \quad a = -1. \qquad s = 3$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Then: $1 = a(s-3) + b(s-2)$. Evaluate at $s = 2,\ 3$.

$$s = 2 \quad \Rightarrow \quad a = -1. \qquad s = 3 \quad \Rightarrow \quad b = 1.$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Then: $1 = a(s-3) + b(s-2)$. Evaluate at $s = 2,\ 3$.

$$s = 2 \quad \Rightarrow \quad a = -1. \qquad s = 3 \quad \Rightarrow \quad b = 1.$$

Therefore $H(s) = -\dfrac{1}{(s-2)} + \dfrac{1}{(s-3)}$.

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Then: $1 = a(s-3) + b(s-2)$. Evaluate at $s = 2,\ 3$.

$$s = 2 \quad \Rightarrow \quad a = -1. \qquad s = 3 \quad \Rightarrow \quad b = 1.$$

Therefore $H(s) = -\dfrac{1}{(s-2)} + \dfrac{1}{(s-3)}$. Then

$$h(t) = -e^{2t} + e^{3t}.$$

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Then: $1 = a(s-3) + b(s-2)$. Evaluate at $s = 2,\ 3$.

$$s = 2 \quad \Rightarrow \quad a = -1. \qquad s = 3 \quad \Rightarrow \quad b = 1.$$

Therefore $H(s) = -\dfrac{1}{(s-2)} + \dfrac{1}{(s-3)}$. Then

$$h(t) = -e^{2t} + e^{3t}.$$

Recalling the formula $y(t) = (h * g)(t)$,

Laplace Transform of a convolution.

Example

Solve the IVP

$$y'' - 5y' + 6y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Then: $1 = a(s-3) + b(s-2)$. Evaluate at $s = 2,\ 3$.

$$s = 2 \quad \Rightarrow \quad a = -1. \qquad s = 3 \quad \Rightarrow \quad b = 1.$$

Therefore $H(s) = -\dfrac{1}{(s-2)} + \dfrac{1}{(s-3)}$. Then

$$h(t) = -e^{2t} + e^{3t}.$$

Recalling the formula $y(t) = (h * g)(t)$, we get

$$y(t) = \int_0^t \left(-e^{2\tau} + e^{3\tau}\right) g(t-\tau)\, d\tau. \qquad \triangleleft$$

Convolution solutions (Sect. 4.5).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- **Impulse response solution.**
- Solution decomposition theorem.

Impulse response solution.

Definition

The *impulse response solution* is the solution y_δ to the IVP

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Impulse response solution.

Definition

The *impulse response solution* is the solution y_δ to the IVP

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Computing Laplace Transforms,

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y_\delta] = 1$$

Impulse response solution.

Definition

The *impulse response solution* is the solution y_δ to the IVP

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Computing Laplace Transforms,

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y_\delta] = 1 \quad \Rightarrow \quad y_\delta(t) = \mathcal{L}^{-1}\Big[\frac{1}{s^2 + a_1 s + a_0}\Big].$$

Impulse response solution.

Definition

The *impulse response solution* is the solution y_δ to the IVP

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Computing Laplace Transforms,

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y_\delta] = 1 \quad \Rightarrow \quad y_\delta(t) = \mathcal{L}^{-1}\Big[\frac{1}{s^2 + a_1 s + a_0}\Big].$$

Denoting the characteristic polynomial by $p(s) = s^2 + a_1 s + a_0$,

Impulse response solution.

Definition

The *impulse response solution* is the solution y_δ to the IVP

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Computing Laplace Transforms,

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y_\delta] = 1 \quad \Rightarrow \quad y_\delta(t) = \mathcal{L}^{-1}\Big[\frac{1}{s^2 + a_1 s + a_0}\Big].$$

Denoting the characteristic polynomial by $p(s) = s^2 + a_1 s + a_0$,

$$y_\delta = \mathcal{L}^{-1}\Big[\frac{1}{p(s)}\Big].$$

Impulse response solution.

Definition

The *impulse response solution* is the solution y_δ to the IVP

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Computing Laplace Transforms,

$$(s^2 + a_1 s + a_0)\,\mathcal{L}[y_\delta] = 1 \quad \Rightarrow \quad y_\delta(t) = \mathcal{L}^{-1}\Big[\frac{1}{s^2 + a_1 s + a_0}\Big].$$

Denoting the characteristic polynomial by $p(s) = s^2 + a_1 s + a_0$,

$$y_\delta = \mathcal{L}^{-1}\Big[\frac{1}{p(s)}\Big].$$

Summary: The impulse reponse solution is the inverse Laplace Transform of the reciprocal of the equation characteristic polynomial.

Impulse response solution.

Recall: The impulse response solution is y_δ solution of the IVP

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Impulse response solution.

Recall: The impulse response solution is y_δ solution of the IVP

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y_{\delta_c}'' + 2\, y_{\delta_c}' + 2\, y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y_{\delta_c}'(0) = 0, \quad c \in \mathbb{R}.$$

Impulse response solution.

Recall: The impulse response solution is y_δ solution of the IVP

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y_{\delta_c}'' + 2\, y_{\delta_c}' + 2\, y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y_{\delta_c}'(0) = 0, \quad c \in \mathbb{R}.$$

Solution: $\mathcal{L}[y_{\delta_c}''] + 2\,\mathcal{L}[y_{\delta_c}'] + 2\,\mathcal{L}[y_{\delta_c}] = \mathcal{L}[\delta(t-c)]$.

Impulse response solution.

Recall: The impulse response solution is y_δ solution of the IVP

$$y''_\delta + a_1\, y'_\delta + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0.$$

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\, y'_{\delta_c} + 2\, y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: $\mathcal{L}[y''_{\delta_c}] + 2\,\mathcal{L}[y'_{\delta_c}] + 2\,\mathcal{L}[y_{\delta_c}] = \mathcal{L}[\delta(t - c)].$

$$(s^2 + 2s + 2)\,\mathcal{L}[y_{\delta_c}] = e^{-cs}$$

Impulse response solution.

Recall: The impulse response solution is y_δ solution of the IVP

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y_{\delta_c}'' + 2\, y_{\delta_c}' + 2\, y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y_{\delta_c}'(0) = 0, \quad c \in \mathbb{R}.$$

Solution: $\mathcal{L}[y_{\delta_c}''] + 2\,\mathcal{L}[y_{\delta_c}'] + 2\,\mathcal{L}[y_{\delta_c}] = \mathcal{L}[\delta(t - c)].$

$$(s^2 + 2s + 2)\,\mathcal{L}[y_{\delta_c}] = e^{-cs} \quad \Rightarrow \quad \mathcal{L}[y_{\delta_c}] = \frac{e^{-cs}}{(s^2 + 2s + 2)}.$$

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s^2+2s+2)}$.

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s^2 + 2s + 2)}$.

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0$$

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s^2 + 2s + 2)}$.

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4 - 8}\right]$$

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s^2+2s+2)}$.

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\left[-2 \pm \sqrt{4-8}\right]$$

Complex roots.

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s^2 + 2s + 2)}$.

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\left[-2 \pm \sqrt{4-8}\right]$$

Complex roots. We complete the square:

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s^2 + 2s + 2)}$.

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4-8}\right]$$

Complex roots. We complete the square:

$$s^2 + 2s + 2 = \left[s^2 + 2\left(\frac{2}{2}\right)s + 1\right] - 1 + 2$$

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s^2+2s+2)}$.

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4-8}\right]$$

Complex roots. We complete the square:

$$s^2 + 2s + 2 = \left[s^2 + 2\left(\frac{2}{2}\right)s + 1\right] - 1 + 2 = (s+1)^2 + 1.$$

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s^2 + 2s + 2)}$.

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\left[-2 \pm \sqrt{4-8}\right]$$

Complex roots. We complete the square:

$$s^2 + 2s + 2 = \left[s^2 + 2\left(\frac{2}{2}\right) s + 1\right] - 1 + 2 = (s+1)^2 + 1.$$

Therefore, $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s+1)^2 + 1}$.

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s+1)^2 + 1}$.

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s+1)^2+1}$.

Recall: $\mathcal{L}[\sin(t)] = \dfrac{1}{s^2+1}$,

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\, y'_{\delta_c} + 2\, y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s+1)^2 + 1}$.

Recall: $\mathcal{L}[\sin(t)] = \dfrac{1}{s^2 + 1}$, and $\mathcal{L}[f](s - c) = \mathcal{L}[e^{ct}\, f(t)]$.

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s+1)^2+1}$.

Recall: $\mathcal{L}[\sin(t)] = \dfrac{1}{s^2+1}$, and $\mathcal{L}[f](s-c) = \mathcal{L}[e^{ct}\,f(t)]$.

$$\frac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\sin(t)]$$

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s+1)^2+1}.$

Recall: $\mathcal{L}[\sin(t)] = \dfrac{1}{s^2+1}$, and $\mathcal{L}[f](s-c) = \mathcal{L}[e^{ct}\,f(t)]$.

$$\frac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_{\delta_c}] = e^{-cs}\,\mathcal{L}[e^{-t}\,\sin(t)].$$

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\,y'_{\delta_c} + 2\,y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s+1)^2+1}$.

Recall: $\mathcal{L}[\sin(t)] = \dfrac{1}{s^2+1}$, and $\mathcal{L}[f](s-c) = \mathcal{L}[e^{ct}\,f(t)]$.

$$\frac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_{\delta_c}] = e^{-cs}\,\mathcal{L}[e^{-t}\,\sin(t)].$$

Since $e^{-cs}\,\mathcal{L}[f](s) = \mathcal{L}[u(t-c)\,f(t-c)]$,

Impulse response solution.

Example

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2\, y'_{\delta_c} + 2\, y_{\delta_c} = \delta(t-c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$

Solution: Recall: $\mathcal{L}[y_{\delta_c}] = \dfrac{e^{-cs}}{(s+1)^2 + 1}$.

Recall: $\mathcal{L}[\sin(t)] = \dfrac{1}{s^2+1}$, and $\mathcal{L}[f](s-c) = \mathcal{L}[e^{ct} f(t)]$.

$$\frac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t} \sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_{\delta_c}] = e^{-cs}\, \mathcal{L}[e^{-t} \sin(t)].$$

Since $e^{-cs}\, \mathcal{L}[f](s) = \mathcal{L}[u(t-c)\, f(t-c)]$,

we conclude $y_{\delta_c}(t) = u(t-c)\, e^{-(t-c)} \sin(t-c)$. ◁

Convolution solutions (Sect. 4.5).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- **Solution decomposition theorem.**

Solution decomposition theorem.

Theorem (Solution decomposition)

The solution y to the IVP

$$y'' + a_1\, y' + a_0\, y = g(t), \quad y(0) = y_0, \quad y'(0) = y_1,$$

can be decomposed as

$$y(t) = y_h(t) + (y_\delta * g)(t),$$

where y_h is the solution of the homogeneous IVP

$$y_h'' + a_1\, y_h' + a_0\, y_h = 0, \quad y_h(0) = y_0, \quad y_h'(0) = y_1,$$

and y_δ is the impulse response solution, that is,

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: $\mathcal{L}[y''] + 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[\sin(at)]$,

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: $\mathcal{L}[y''] + 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[\sin(at)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,(1) - (-1),$$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: $\mathcal{L}[y''] + 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[\sin(at)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,(1) - (-1), \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - 1.$$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: $\mathcal{L}[y''] + 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[\sin(at)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,(1) - (-1), \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - 1.$$

$$(s^2 + 2s + 2)\,\mathcal{L}[y] - s + 1 - 2 = \mathcal{L}[\sin(at)].$$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: $\mathcal{L}[y''] + 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[\sin(at)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,(1) - (-1), \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - 1.$$

$$(s^2 + 2s + 2)\,\mathcal{L}[y] - s + 1 - 2 = \mathcal{L}[\sin(at)].$$

$$\mathcal{L}[y] = \frac{(s+1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)}\,\mathcal{L}[\sin(at)].$$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2 + 2s + 2)} + \dfrac{1}{(s^2 + 2s + 2)}\,\mathcal{L}[\sin(at)].$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2+2s+2)} + \dfrac{1}{(s^2+2s+2)}\,\mathcal{L}[\sin(at)].$

But: $\mathcal{L}[y_h] = \dfrac{(s+1)}{(s^2+2s+2)}$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2+2s+2)} + \dfrac{1}{(s^2+2s+2)}\,\mathcal{L}[\sin(at)]$.

But: $\mathcal{L}[y_h] = \dfrac{(s+1)}{(s^2+2s+2)} = \dfrac{(s+1)}{(s+1)^2+1}$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2+2s+2)} + \dfrac{1}{(s^2+2s+2)}\,\mathcal{L}[\sin(at)]$.

But: $\mathcal{L}[y_h] = \dfrac{(s+1)}{(s^2+2s+2)} = \dfrac{(s+1)}{(s+1)^2+1} = \mathcal{L}[e^{-t}\cos(t)]$,

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2+2s+2)} + \dfrac{1}{(s^2+2s+2)}\,\mathcal{L}[\sin(at)]$.

But: $\mathcal{L}[y_h] = \dfrac{(s+1)}{(s^2+2s+2)} = \dfrac{(s+1)}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\cos(t)]$,

and: $\mathcal{L}[y_\delta] = \dfrac{1}{(s^2+2s+2)}$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2+2s+2)} + \dfrac{1}{(s^2+2s+2)}\,\mathcal{L}[\sin(at)].$

But: $\mathcal{L}[y_h] = \dfrac{(s+1)}{(s^2+2s+2)} = \dfrac{(s+1)}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\cos(t)],$

and: $\mathcal{L}[y_\delta] = \dfrac{1}{(s^2+2s+2)} = \dfrac{1}{(s+1)^2+1}$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2+2s+2)} + \dfrac{1}{(s^2+2s+2)}\,\mathcal{L}[\sin(at)]$.

But: $\mathcal{L}[y_h] = \dfrac{(s+1)}{(s^2+2s+2)} = \dfrac{(s+1)}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\cos(t)]$,

and: $\mathcal{L}[y_\delta] = \dfrac{1}{(s^2+2s+2)} = \dfrac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\sin(t)]$.

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2+2s+2)} + \dfrac{1}{(s^2+2s+2)}\,\mathcal{L}[\sin(at)]$.

But: $\mathcal{L}[y_h] = \dfrac{(s+1)}{(s^2+2s+2)} = \dfrac{(s+1)}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\cos(t)]$,

and: $\mathcal{L}[y_\delta] = \dfrac{1}{(s^2+2s+2)} = \dfrac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\sin(t)]$. So,

$$\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\,\mathcal{L}[g(t)]$$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2+2s+2)} + \dfrac{1}{(s^2+2s+2)}\,\mathcal{L}[\sin(at)]$.

But: $\mathcal{L}[y_h] = \dfrac{(s+1)}{(s^2+2s+2)} = \dfrac{(s+1)}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\cos(t)]$,

and: $\mathcal{L}[y_\delta] = \dfrac{1}{(s^2+2s+2)} = \dfrac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\sin(t)]$. So,

$$\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\,\mathcal{L}[g(t)] \quad \Rightarrow \quad y(t) = y_h(t) + (y_\delta * g)(t),$$

Solution decomposition theorem.

Example

Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2+2s+2)} + \dfrac{1}{(s^2+2s+2)}\,\mathcal{L}[\sin(at)].$

But: $\mathcal{L}[y_h] = \dfrac{(s+1)}{(s^2+2s+2)} = \dfrac{(s+1)}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\cos(t)],$

and: $\mathcal{L}[y_\delta] = \dfrac{1}{(s^2+2s+2)} = \dfrac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\sin(t)].$ So,

$$\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\,\mathcal{L}[g(t)] \quad \Rightarrow \quad y(t) = y_h(t) + (y_\delta * g)(t),$$

So: $y(t) = e^{-t}\,\cos(t) + \displaystyle\int_0^t e^{-\tau}\sin(\tau)\,\sin[a(t-\tau)]\,d\tau.$ $\triangleleft$

Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1\,\mathcal{L}[y'] + a_0\,\mathcal{L}[y] = \mathcal{L}[g(t)]$,

Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1\,\mathcal{L}[y'] + a_0\,\mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - sy_0 - y_1,$$

Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1\,\mathcal{L}[y'] + a_0\,\mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y_0.$$

Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1\,\mathcal{L}[y'] + a_0\,\mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y_0.$$

$$(s^2 + a_1 s + a_0)\,\mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1\,\mathcal{L}[y'] + a_0\,\mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y_0.$$

$$(s^2 + a_1 s + a_0)\,\mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)}\,\mathcal{L}[g(t)].$$

Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1\,\mathcal{L}[y'] + a_0\,\mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y_0.$$

$$(s^2 + a_1 s + a_0)\,\mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)}\,\mathcal{L}[g(t)].$$

Recall: $\mathcal{L}[y_h] = \dfrac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)}$,

Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1\,\mathcal{L}[y'] + a_0\,\mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y_0.$$

$$(s^2 + a_1 s + a_0)\,\mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)}\,\mathcal{L}[g(t)].$$

Recall: $\mathcal{L}[y_h] = \dfrac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)}$, and $\mathcal{L}[y_\delta] = \dfrac{1}{(s^2 + a_1 s + a_0)}$.

Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1\,\mathcal{L}[y'] + a_0\,\mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y_0.$$

$$(s^2 + a_1 s + a_0)\,\mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)}\,\mathcal{L}[g(t)].$$

Recall: $\mathcal{L}[y_h] = \dfrac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)}$, and $\mathcal{L}[y_\delta] = \dfrac{1}{(s^2 + a_1 s + a_0)}$.

Since, $\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\,\mathcal{L}[g(t)]$,

Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1\,\mathcal{L}[y'] + a_0\,\mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y_0.$$

$$(s^2 + a_1 s + a_0)\,\mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)}\,\mathcal{L}[g(t)].$$

Recall: $\mathcal{L}[y_h] = \dfrac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)}$, and $\mathcal{L}[y_\delta] = \dfrac{1}{(s^2 + a_1 s + a_0)}$.

Since, $\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\,\mathcal{L}[g(t)]$, so $y(t) = y_h(t) + (y_\delta * g)(t)$.

Solution decomposition theorem.

Proof: Compute: $\mathcal{L}[y''] + a_1\,\mathcal{L}[y'] + a_0\,\mathcal{L}[y] = \mathcal{L}[g(t)]$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y_0.$$

$$(s^2 + a_1 s + a_0)\,\mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

$$\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)}\,\mathcal{L}[g(t)].$$

Recall: $\mathcal{L}[y_h] = \dfrac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)}$, and $\mathcal{L}[y_\delta] = \dfrac{1}{(s^2 + a_1 s + a_0)}$.

Since, $\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\,\mathcal{L}[g(t)]$, so $y(t) = y_h(t) + (y_\delta * g)(t)$.

Equivalently: $y(t) = y_h(t) + \displaystyle\int_0^t y_\delta(\tau)g(t - \tau)\,d\tau$. $\square$

Systems of linear differential equations (Sect. 5.1).

- $n \times n$ systems of linear differential equations.
- Second order equations and first order systems.
- Main concepts from Linear Algebra.

$n \times n$ systems of linear differential equations.

Remark: Many physical systems must be described with more than one differential equation.

$n \times n$ systems of linear differential equations.

Remark: Many physical systems must be described with more than one differential equation.

Example

Newton's law of motion for a particle of mass m moving in space.

$n \times n$ systems of linear differential equations.

Remark: Many physical systems must be described with more than one differential equation.

Example

Newton's law of motion for a particle of mass m moving in space. The unknown and the force are vector-valued functions,

$n \times n$ systems of linear differential equations.

Remark: Many physical systems must be described with more than one differential equation.

Example

Newton's law of motion for a particle of mass m moving in space. The unknown and the force are vector-valued functions,

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},$$

$n \times n$ systems of linear differential equations.

Remark: Many physical systems must be described with more than one differential equation.

Example

Newton's law of motion for a particle of mass m moving in space. The unknown and the force are vector-valued functions,

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \qquad \mathbf{F}(t) = \begin{bmatrix} F_1(t, \mathbf{x}) \\ F_2(t, \mathbf{x}) \\ F_3(t, \mathbf{x}) \end{bmatrix}.$$

$n \times n$ systems of linear differential equations.

Remark: Many physical systems must be described with more than one differential equation.

Example

Newton's law of motion for a particle of mass m moving in space. The unknown and the force are vector-valued functions,

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \qquad \mathbf{F}(t) = \begin{bmatrix} F_1(t, \mathbf{x}) \\ F_2(t, \mathbf{x}) \\ F_3(t, \mathbf{x}) \end{bmatrix}.$$

The equation of motion are: $m \dfrac{d^2\mathbf{x}}{dt^2} = \mathbf{F}\big(t, \mathbf{x}(t)\big).$

$n \times n$ systems of linear differential equations.

Remark: Many physical systems must be described with more than one differential equation.

Example

Newton's law of motion for a particle of mass m moving in space. The unknown and the force are vector-valued functions,

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \qquad \mathbf{F}(t) = \begin{bmatrix} F_1(t, \mathbf{x}) \\ F_2(t, \mathbf{x}) \\ F_3(t, \mathbf{x}) \end{bmatrix}.$$

The equation of motion are: $m \dfrac{d^2\mathbf{x}}{dt^2} = \mathbf{F}\big(t, \mathbf{x}(t)\big).$

These are three differential equations,

$$m \frac{d^2 x_1}{dt^2} = F_1\big(t, \mathbf{x}(t)\big), \quad m \frac{d^2 x_2}{dt^2} = F_2\big(t, \mathbf{x}(t)\big), \quad m \frac{d^2 x_3}{dt^2} = F_3\big(t, \mathbf{x}(t)\big).$$

◁

$n \times n$ systems of linear differential equations.

Definition

An *$n \times n$ system of linear first order differential equations* is the following: Given the functions a_{ij}, $g_i : [a, b] \to \mathbb{R}$, where $i, j = 1, \cdots, n$, find n functions $x_j : [a, b] \to \mathbb{R}$ solutions of the n linear differential equations

$$\begin{aligned} x_1' &= a_{11}(t)\, x_1 + \cdots + a_{1n}(t)\, x_n + g_1(t) \\ &\vdots \\ x_n' &= a_{n1}(t)\, x_1 + \cdots + a_{nn}(t)\, x_n + g_n(t). \end{aligned}$$

The system is called *homogeneous* iff the source functions satisfy that $g_1 = \cdots = g_n = 0$.

$n \times n$ systems of linear differential equations.

Example

$n = 1$: Single differential equation: Find $x_1(t)$ solution of

$$x_1' = a_{11}(t)\, x_1 + g_1(t).$$

$n \times n$ systems of linear differential equations.

Example

$n = 1$: Single differential equation: Find $x_1(t)$ solution of

$$x_1' = a_{11}(t)\, x_1 + g_1(t).$$

Example

$n = 2$: 2×2 linear system: Find $x_1(t)$ and $x_2(t)$ solutions of

$$x_1' = a_{11}(t)\, x_1 + a_{12}(t)\, x_2 + g_1(t),$$
$$x_2' = a_{21}(t)\, x_1 + a_{22}(t)\, x_2 + g_2(t).$$

$n \times n$ systems of linear differential equations.

Example

$n = 1$: Single differential equation: Find $x_1(t)$ solution of

$$x_1' = a_{11}(t)\, x_1 + g_1(t).$$

Example

$n = 2$: 2×2 linear system: Find $x_1(t)$ and $x_2(t)$ solutions of

$$\begin{aligned} x_1' &= a_{11}(t)\, x_1 + a_{12}(t)\, x_2 + g_1(t), \\ x_2' &= a_{21}(t)\, x_1 + a_{22}(t)\, x_2 + g_2(t). \end{aligned}$$

Example

$n = 2$: 2×2 homogeneous linear system: Find $x_1(t)$ and $x_2(t)$,

$$\begin{aligned} x_1' &= a_{11}(t)\, x_1 + a_{12}(t)\, x_2 \\ x_2' &= a_{21}(t)\, x_1 + a_{22}(t)\, x_2. \end{aligned}$$

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0,$$

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$,

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$,

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

$$v' = 0$$

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

$$v' = 0 \quad \Rightarrow \quad v = c_1,$$

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

$$v' = 0 \quad \Rightarrow \quad v = c_1,$$

$$w' = 2w$$

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

$$v' = 0 \quad \Rightarrow \quad v = c_1,$$

$$w' = 2w \quad \Rightarrow \quad w = c_2 e^{2t}.$$

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

$$v' = 0 \quad \Rightarrow \quad v = c_1,$$

$$w' = 2w \quad \Rightarrow \quad w = c_2 e^{2t}.$$

Back to x_1 and x_2:

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

$$v' = 0 \quad \Rightarrow \quad v = c_1,$$

$$w' = 2w \quad \Rightarrow \quad w = c_2 e^{2t}.$$

Back to x_1 and x_2: $\qquad x_1 = \frac{1}{2}(v + w),$

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

$$v' = 0 \quad \Rightarrow \quad v = c_1,$$

$$w' = 2w \quad \Rightarrow \quad w = c_2 e^{2t}.$$

Back to x_1 and x_2: $\qquad x_1 = \dfrac{1}{2}(v + w), \qquad x_2 = \dfrac{1}{2}(v - w).$

$n \times n$ systems of linear differential equations.

Example

Find $x_1(t)$, $x_2(t)$ solutions of the 2×2, constant coefficients, homogeneous system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: Add up the equations, and subtract the equations,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

Introduce the unknowns $v = x_1 + x_2$, $w = x_1 - x_2$, then

$$v' = 0 \quad \Rightarrow \quad v = c_1,$$

$$w' = 2w \quad \Rightarrow \quad w = c_2 e^{2t}.$$

Back to x_1 and x_2: $\quad x_1 = \dfrac{1}{2}(v + w), \qquad x_2 = \dfrac{1}{2}(v - w).$

We conclude: $\quad x_1(t) = \dfrac{1}{2}(c_1 + c_2 e^{2t}), \qquad x_2(t) = \dfrac{1}{2}(c_1 - c_2 e^{2t}).$ $\triangleleft$

Systems of linear differential equations (Sect. 5.1).

- $n \times n$ systems of linear differential equations.
- **Second order equations and first order systems.**
- Main concepts from Linear Algebra.

Second order equations and first order systems.

Theorem (Reduction to first order)

Every solution y *to the second order linear equation*

$$y'' + p(t)\,y' + q(t)\,y = g(t), \tag{1}$$

defines a solution $x_1 = y$ *and* $x_2 = y'$ *of the* 2×2 *first order linear differential system*

$$x_1' = x_2, \tag{2}$$

$$x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t). \tag{3}$$

Conversely, every solution x_1, x_2 *of the* 2×2 *first order linear system in Eqs. (2)-(3) defines a solution* $y = x_1$ *of the second order differential equation in (1).*

Second order equations and first order systems.

Proof:

($\Rightarrow$) Given y solution of $y'' + p(t)\,y' + q(t)\,y = g(t)$,

Second order equations and first order systems.

Proof:

$(\Rightarrow)$ Given y solution of $y'' + p(t)\,y' + q(t)\,y = g(t)$,

introduce $x_1 = y$ and $x_2 = y'$,

Second order equations and first order systems.

Proof:

($\Rightarrow$) Given y solution of $y'' + p(t)\,y' + q(t)\,y = g(t)$,

introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$,

Second order equations and first order systems.

Proof:

$(\Rightarrow)$ Given y solution of $y'' + p(t)\,y' + q(t)\,y = g(t)$,

introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$

Second order equations and first order systems.

Proof:

($\Rightarrow$) Given y solution of $y'' + p(t)\,y' + q(t)\,y = g(t)$,

introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$

Then, $x_2' = y''$

Second order equations and first order systems.

Proof:

($\Rightarrow$) Given y solution of $y'' + p(t)\,y' + q(t)\,y = g(t)$,

introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$

Then, $x_2' = y'' = -q(t)\,y - p(t)\,y' + g(t)$.

Second order equations and first order systems.

Proof:

($\Rightarrow$) Given y solution of $y'' + p(t)\,y' + q(t)\,y = g(t)$,

introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$

Then, $x_2' = y'' = -q(t)\,y - p(t)\,y' + g(t)$. That is,

$$x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t).$$

Second order equations and first order systems.

Proof:

($\Rightarrow$) Given y solution of $y'' + p(t)\,y' + q(t)\,y = g(t)$,

introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$

Then, $x_2' = y'' = -q(t)\,y - p(t)\,y' + g(t)$. That is,

$$x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t).$$

($\Leftarrow$) Introduce $x_2 = x_1'$ into $x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t)$.

Second order equations and first order systems.

Proof:

$(\Rightarrow)$ Given y solution of $y'' + p(t)\,y' + q(t)\,y = g(t)$,

introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$

Then, $x_2' = y'' = -q(t)\,y - p(t)\,y' + g(t)$. That is,

$$x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t).$$

$(\Leftarrow)$ Introduce $x_2 = x_1'$ into $x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t)$.

$$x_1'' = -q(t)\,x_1 - p(t)\,x_1' + g(t),$$

Second order equations and first order systems.

Proof:

($\Rightarrow$) Given y solution of $y'' + p(t)\,y' + q(t)\,y = g(t)$,

introduce $x_1 = y$ and $x_2 = y'$, hence $x_1' = y' = x_2$, that is,

$$x_1' = x_2.$$

Then, $x_2' = y'' = -q(t)\,y - p(t)\,y' + g(t)$. That is,

$$x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t).$$

($\Leftarrow$) Introduce $x_2 = x_1'$ into $x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t)$.

$$x_1'' = -q(t)\,x_1 - p(t)\,x_1' + g(t),$$

that is

$$x_1'' + p(t)\,x_1' + q(t)\,x_1 = g(t).$$

$\square$

Second order equations and first order systems.

Example

Express as a first order system the equation

$$y'' + 2y' + 2y = \sin(at).$$

Second order equations and first order systems.

Example

Express as a first order system the equation

$$y'' + 2y' + 2y = \sin(at).$$

Solution: Introduce the new unknowns

$$x_1 = y, \quad x_2 = y'$$

Second order equations and first order systems.

Example

Express as a first order system the equation

$$y'' + 2y' + 2y = \sin(at).$$

Solution: Introduce the new unknowns

$$x_1 = y, \quad x_2 = y' \quad \Rightarrow \quad x_1' = x_2.$$

Second order equations and first order systems.

Example

Express as a first order system the equation

$$y'' + 2y' + 2y = \sin(at).$$

Solution: Introduce the new unknowns

$$x_1 = y, \quad x_2 = y' \quad \Rightarrow \quad x_1' = x_2.$$

Then, the differential equation can be written as

$$x_2' + 2x_2 + 2x_1 = \sin(at).$$

Second order equations and first order systems.

Example

Express as a first order system the equation

$$y'' + 2y' + 2y = \sin(at).$$

Solution: Introduce the new unknowns

$$x_1 = y, \quad x_2 = y' \quad \Rightarrow \quad x_1' = x_2.$$

Then, the differential equation can be written as

$$x_2' + 2x_2 + 2x_1 = \sin(at).$$

We conclude that

$$x_1' = x_2.$$

$$x_2' = -2x_1 - 2x_2 + \sin(at). \qquad \triangleleft$$

Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Compute x_1 from the second equation:

Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Compute x_1 from the second equation: $x_1 = x_2' + x_2$.

Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Compute x_1 from the second equation: $x_1 = x_2' + x_2$. Introduce this expression into the first equation,

Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Compute x_1 from the second equation: $x_1 = x_2' + x_2$. Introduce this expression into the first equation,

$$(x_2' + x_2)' = -(x_2' + x_2) + 3x_2,$$

Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Compute x_1 from the second equation: $x_1 = x_2' + x_2$. Introduce this expression into the first equation,

$$(x_2' + x_2)' = -(x_2' + x_2) + 3x_2,$$

$$x_2'' + x_2' = -x_2' - x_2 + 3x_2,$$

Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Compute x_1 from the second equation: $x_1 = x_2' + x_2$. Introduce this expression into the first equation,

$$(x_2' + x_2)' = -(x_2' + x_2) + 3x_2,$$

$$x_2'' + x_2' = -x_2' - x_2 + 3x_2,$$

$$x_2'' + 2x_2' - 2x_2 = 0.$$

Second order equations and first order systems.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Recall: $x_2'' + 2x_2' - 2x_2 = 0.$

Second order equations and first order systems.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Recall: $x_2'' + 2x_2' - 2x_2 = 0$.

$r^2 + 2r - 2 = 0$

Second order equations and first order systems.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Recall: $x_2'' + 2x_2' - 2x_2 = 0$.

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4+8}\right]$$

Second order equations and first order systems.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Recall: $x_2'' + 2x_2' - 2x_2 = 0$.

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4+8}\right] \quad \Rightarrow \quad r_\pm = -1 \pm \sqrt{3}.$$

Second order equations and first order systems.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Recall: $x_2'' + 2x_2' - 2x_2 = 0$.

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4+8}\right] \quad \Rightarrow \quad r_\pm = -1 \pm \sqrt{3}.$$

Therefore, $x_2 = c_1\, e^{r_+ t} + c_2\, e^{r_- t}$.

Second order equations and first order systems.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Recall: $x_2'' + 2x_2' - 2x_2 = 0$.

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2}\left[-2 \pm \sqrt{4+8}\right] \quad \Rightarrow \quad r_{\pm} = -1 \pm \sqrt{3}.$$

Therefore, $x_2 = c_1\, e^{r_+ t} + c_2\, e^{r_- t}$. Since $x_1 = x_2' + x_2$,

Second order equations and first order systems.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Recall: $x_2'' + 2x_2' - 2x_2 = 0.$

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4+8}\right] \quad \Rightarrow \quad r_\pm = -1 \pm \sqrt{3}.$$

Therefore, $x_2 = c_1\, e^{r_+ t} + c_2\, e^{r_- t}$. Since $x_1 = x_2' + x_2$,

$$x_1 = \left(c_1 r_+\, e^{r_+ t} + c_2 r_-\, e^{r_- t}\right) + \left(c_1\, e^{r_+ t} + c_2\, e^{r_- t}\right),$$

Second order equations and first order systems.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Recall: $x_2'' + 2x_2' - 2x_2 = 0$.

$$r^2+2r-2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[-2\pm\sqrt{4+8}\right] \quad \Rightarrow \quad r_\pm = -1\pm\sqrt{3}.$$

Therefore, $x_2 = c_1\, e^{r_+ t} + c_2\, e^{r_- t}$. Since $x_1 = x_2' + x_2$,

$$x_1 = \left(c_1 r_+\, e^{r_+ t} + c_2 r_-\, e^{r_- t}\right) + \left(c_1\, e^{r_+ t} + c_2\, e^{r_- t}\right),$$

We conclude: $x_1 = c_1(1 + r_+)\, e^{r_+ t} + c_2(1 + r_-)\, e^{r_- t}$. $\triangleleft$

Systems of linear differential equations (Sect. 5.1).

- $n \times n$ systems of linear differential equations.
- Second order equations and first order systems.
- **Main concepts from Linear Algebra.**

Main concepts from Linear Algebra.

Remark: Ideas from Linear Algebra are useful to study systems of linear differential equations.

Main concepts from Linear Algebra.

Remark: Ideas from Linear Algebra are useful to study systems of linear differential equations.

We review:

- Matrices $m \times n$.
- Matrix operations.
- n-vectors, dot product.
- matrix-vector product.

Main concepts from Linear Algebra.

Remark: Ideas from Linear Algebra are useful to study systems of linear differential equations.

We review:

- Matrices $m \times n$.
- Matrix operations.
- n-vectors, dot product.
- matrix-vector product.

Definition

An $m \times n$ matrix, A, is an array of numbers

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \qquad \begin{array}{c} m \text{ rows}, \\ n \text{ columns}. \end{array}$$

where $a_{ij} \in \mathbb{C}$ and $i = 1, \cdots, m$, and $j = 1, \cdots, n$. An $n \times n$ matrix is called a square matrix.

Main concepts from Linear Algebra.

Example

(a) 2×2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Main concepts from Linear Algebra.

Example

(a) 2×2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(b) 2×3 matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Main concepts from Linear Algebra.

Example

(a) 2×2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(b) 2×3 matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

(c) 3×2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

Main concepts from Linear Algebra.

Example

(a) 2×2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(b) 2×3 matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

(c) 3×2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

(d) 2×2 complex-valued matrix: $A = \begin{bmatrix} 1+i & 2-i \\ 3 & 4i \end{bmatrix}$.

Main concepts from Linear Algebra.

Example

(a) 2×2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(b) 2×3 matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

(c) 3×2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

(d) 2×2 complex-valued matrix: $A = \begin{bmatrix} 1+i & 2-i \\ 3 & 4i \end{bmatrix}$.

(e) The coefficients of a linear system can be grouped in a matrix,

$$x_1' = -x_1 + 3x_2$$
$$x_2' = x_1 - x_2$$

Main concepts from Linear Algebra.

Example

(a) 2×2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(b) 2×3 matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

(c) 3×2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

(d) 2×2 complex-valued matrix: $A = \begin{bmatrix} 1+i & 2-i \\ 3 & 4i \end{bmatrix}$.

(e) The coefficients of a linear system can be grouped in a matrix,

$$\left.\begin{aligned} x_1' &= -x_1 + 3x_2 \\ x_2' &= x_1 - x_2 \end{aligned}\right\} \quad \Rightarrow \quad A = \begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Remark: An $m \times 1$ matrix is called an m-vector.

Main concepts from Linear Algebra.

Remark: An $m \times 1$ matrix is called an m-vector.

Definition

An m-vector, $\mathbf{v}$, is the array of numbers $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, where the vector components $v_i \in \mathbb{C}$, with $i = 1, \cdots, m$.

Main concepts from Linear Algebra.

Remark: An $m \times 1$ matrix is called an m-vector.

Definition

An m-vector, $\mathbf{v}$, is the array of numbers $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, where the vector components $v_i \in \mathbb{C}$, with $i = 1, \cdots, m$.

Example

The unknowns of a 2×2 linear system can be grouped in a 2-vector,

Main concepts from Linear Algebra.

Remark: An $m \times 1$ matrix is called an m-vector.

Definition

An m-vector, $\mathbf{v}$, is the array of numbers $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, where the vector components $v_i \in \mathbb{C}$, with $i = 1, \cdots, m$.

Example

The unknowns of a 2×2 linear system can be grouped in a 2-vector, for example,

$$x_1' = -x_1 + 3x_2$$
$$x_2' = x_1 - x_2$$

Main concepts from Linear Algebra.

Remark: An $m \times 1$ matrix is called an m-vector.

Definition

An m-vector, $\mathbf{v}$, is the array of numbers $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, where the vector components $v_i \in \mathbb{C}$, with $i = 1, \cdots, m$.

Example

The unknowns of a 2×2 linear system can be grouped in a 2-vector, for example,

$$\left.\begin{aligned} x_1' &= -x_1 + 3x_2 \\ x_2' &= x_1 - x_2 \end{aligned}\right\} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) A-transpose: Interchange rows with columns:

Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2+i & 2 \\ -1+2i & 1 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2+i & 2 \\ -1+2i & 1 \end{bmatrix}. \qquad \text{Notice that: } \left(A^T\right)^T = A.$$

Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2+i & 2 \\ -1+2i & 1 \end{bmatrix}. \qquad \text{Notice that: } (A^T)^T = A.$$

(b) *A*-conjugate: Conjugate every matrix coefficient:

Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) A-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2+i & 2 \\ -1+2i & 1 \end{bmatrix}. \qquad \text{Notice that: } (A^T)^T = A.$$

(b) A-conjugate: Conjugate every matrix coefficient:

$$\overline{A} = \begin{bmatrix} 1 & 2-i & -1-2i \\ -3i & 2 & 1 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2+i & 2 \\ -1+2i & 1 \end{bmatrix}. \qquad \text{Notice that: } (A^T)^T = A.$$

(b) *A*-conjugate: Conjugate every matrix coefficient:

$$\overline{A} = \begin{bmatrix} 1 & 2-i & -1-2i \\ -3i & 2 & 1 \end{bmatrix}. \qquad \text{Notice that: } \overline{(\overline{A})} = A.$$

Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2+i & 2 \\ -1+2i & 1 \end{bmatrix}. \qquad \text{Notice that: } (A^T)^T = A.$$

(b) *A*-conjugate: Conjugate every matrix coefficient:

$$\overline{A} = \begin{bmatrix} 1 & 2-i & -1-2i \\ -3i & 2 & 1 \end{bmatrix}. \qquad \text{Notice that: } \overline{(\overline{A})} = A.$$

Matrix A is real iff $\overline{A} = A$.

Main concepts from Linear Algebra.

Remark: We present only examples of *matrix operations*.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-transpose: Interchange rows with columns:

$$A^T = \begin{bmatrix} 1 & 3i \\ 2+i & 2 \\ -1+2i & 1 \end{bmatrix}. \qquad \text{Notice that: } (A^T)^T = A.$$

(b) *A*-conjugate: Conjugate every matrix coefficient:

$$\overline{A} = \begin{bmatrix} 1 & 2-i & -1-2i \\ -3i & 2 & 1 \end{bmatrix}. \qquad \text{Notice that: } \overline{(\overline{A})} = A.$$

Matrix A is real iff $\overline{A} = A$. Matrix A is imaginary iff $\overline{A} = -A$.

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) A-adjoint: Conjugate and transpose:

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2-i & 2 \\ -1-2i & 1 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2-i & 2 \\ -1-2i & 1 \end{bmatrix}. \qquad \text{Notice that: } (A^*)^* = A.$$

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2-i & 2 \\ -1-2i & 1 \end{bmatrix}. \qquad \text{Notice that: } (A^*)^* = A.$$

(b) Addition of two $m \times n$ matrices is performed component-wise:

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2-i & 2 \\ -1-2i & 1 \end{bmatrix}. \qquad \text{Notice that: } \left(A^*\right)^* = A.$$

(b) Addition of two $m \times n$ matrices is performed component-wise:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$$

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2-i & 2 \\ -1-2i & 1 \end{bmatrix}. \qquad \text{Notice that: } (A^*)^* = A.$$

(b) Addition of two $m \times n$ matrices is performed component-wise:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3+5) & (4+1) \end{bmatrix}$$

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2-i & 2 \\ -1-2i & 1 \end{bmatrix}. \qquad \text{Notice that: } \left(A^*\right)^* = A.$$

(b) Addition of two $m \times n$ matrices is performed component-wise:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3+5) & (4+1) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 8 & 5 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}$.

(a) *A*-adjoint: Conjugate and transpose:

$$A^* = \begin{bmatrix} 1 & -3i \\ 2-i & 2 \\ -1-2i & 1 \end{bmatrix}. \qquad \text{Notice that: } (A^*)^* = A.$$

(b) Addition of two $m \times n$ matrices is performed component-wise:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3+5) & (4+1) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 8 & 5 \end{bmatrix}.$$

The addition $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is not defined.

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix},$$

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}, \qquad \begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix}$$

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}, \quad \begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix} = 4\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}, \quad \begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix} = 4\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

Also:

$$\frac{A}{3}$$

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}, \quad \begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix} = 4 \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

Also:

$$\frac{A}{3} = \frac{1}{3} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Main concepts from Linear Algebra.

Example

Consider a 2×3 matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Multiplication of a matrix by a number is performed component-wise:

$$2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}, \quad \begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix} = 4 \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

Also:

$$\frac{A}{3} = \frac{1}{3} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{2}{3} & \frac{4}{3} & 2 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Example

(a) Matrix multiplication.

Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

$$\begin{array}{ccccc} A & \text{times} & B & \text{defines} & AB \\ m \times n & & n \times \ell & & m \times \ell \end{array}$$

Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

A	times	B	defines	AB
$m \times n$		$n \times \ell$		$m \times \ell$

Example: A is 2×2, B is 2×3, so AB is 2×3:

Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

$$\begin{array}{ccccc} A & \text{times} & B & \text{defines} & AB \\ m \times n & & n \times \ell & & m \times \ell \end{array}$$

Example: A is 2×2, B is 2×3, so AB is 2×3:

$$AB = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 16 & 23 & 30 \\ 6 & 9 & 12 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

$$\begin{array}{ccccc} A & \text{times} & B & \text{defines} & AB \\ m \times n & & n \times \ell & & m \times \ell \end{array}$$

Example: A is 2×2, B is 2×3, so AB is 2×3:

$$AB = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 16 & 23 & 30 \\ 6 & 9 & 12 \end{bmatrix}.$$

Notice B is 2×3, A is 2×2, so BA is not defined.

Main concepts from Linear Algebra.

Example

(a) Matrix multiplication. The matrix sizes is important:

$$\begin{array}{ccccc} A & \text{times} & B & \text{defines} & AB \\ m \times n & & n \times \ell & & m \times \ell \end{array}$$

Example: A is 2×2, B is 2×3, so AB is 2×3:

$$AB = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 16 & 23 & 30 \\ 6 & 9 & 12 \end{bmatrix}.$$

Notice B is 2×3, A is 2×2, so BA is not defined.

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \quad \text{not defined.}$$

Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example

Find AB and BA for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example

Find AB and BA for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$$

Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example

Find AB and BA for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6-2) & (0+1) \\ (-3+4) & (0-2) \end{bmatrix}$$

Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example

Find AB and BA for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6-2) & (0+1) \\ (-3+4) & (0-2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example

Find AB and BA for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6-2) & (0+1) \\ (-3+4) & (0-2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example

Find AB and BA for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6-2) & (0+1) \\ (-3+4) & (0-2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (6+0) & (-3+0) \\ (4+1) & (-2-2) \end{bmatrix}$$

Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example

Find AB and BA for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6-2) & (0+1) \\ (-3+4) & (0-2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (6+0) & (-3+0) \\ (4+1) & (-2-2) \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 5 & -4 \end{bmatrix}.$$

Main concepts from Linear Algebra.

Remark: The matrix product is not commutative, that is, in general holds $AB \neq BA$.

Example

Find AB and BA for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} (6-2) & (0+1) \\ (-3+4) & (0-2) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (6+0) & (-3+0) \\ (4+1) & (-2-2) \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 5 & -4 \end{bmatrix}.$$

So $AB \neq BA$. ◁

Main concepts from Linear Algebra.

Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Main concepts from Linear Algebra.

Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Example

Find AB for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Main concepts from Linear Algebra.

Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Example

Find AB for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Main concepts from Linear Algebra.

Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Example

Find AB for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (1-1) & (-1+1) \\ (-1+1) & (1-1) \end{bmatrix}$$

Main concepts from Linear Algebra.

Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Example

Find AB for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (1-1) & (-1+1) \\ (-1+1) & (1-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\lhd$

Main concepts from Linear Algebra.

Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Example

Find AB for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (1-1) & (-1+1) \\ (-1+1) & (1-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\lhd$

Recall: If $a, b \in \mathbb{R}$ and $ab = 0$, then either $a = 0$ or $b = 0$.

Main concepts from Linear Algebra.

Remark: There exist matrices $A \neq 0$ and $B \neq 0$ with $AB = 0$.

Example

Find AB for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (1-1) & (-1+1) \\ (-1+1) & (1-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\triangleleft$

Recall: If $a, b \in \mathbb{R}$ and $ab = 0$, then either $a = 0$ or $b = 0$.

We have just shown that this statement is not true for matrices.