

The Laplace Transform (Sect. 4.1).

- ▶ The definition of the Laplace Transform.
- ▶ Review: Improper integrals.
- ▶ Examples of Laplace Transforms.
- ▶ A table of Laplace Transforms.
- ▶ Properties of the Laplace Transform.
- ▶ Laplace Transform and differential equations.

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The definition of the Laplace Transform.

Definition

The function $F : D_F \rightarrow \mathbb{R}$ is the *Laplace transform* of a function $f : [0, \infty) \rightarrow \mathbb{R}$ iff for all $s \in D_F$ holds,

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt,$$

where $D_F \subset \mathbb{R}$ is the set where the integral converges.

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- ▶ This notation $\mathcal{L}[]$ emphasizes that the Laplace transform defines a map from a set of functions into a set of functions.
- ▶ Functions are denoted as $t \mapsto f(t)$.
- ▶ The Laplace transform is also a function: $f \mapsto \mathcal{L}[f]$.

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Recall: Improper integral are defined as a limit.

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Therefore $\mathcal{L}[1] = \frac{1}{s}$, for $s > 0$, and $\mathcal{L}[1]$ does not exist for $s \leq 0$.

In other words, $F(s) = \mathcal{L}[1]$ is the function $F : D_F \rightarrow \mathbb{R}$ given by

$$f(t) = 1, \quad F(s) = \frac{1}{s}, \quad D_F = (0, \infty). \quad \triangleleft$$

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$$\int_0^{\infty} e^{-(s-a)t} dt = \frac{1}{(s-a)} \quad \text{for } (s-a) > 0.$$

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We conclude that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s > a$. In other words,

$$f(t) = e^{at}, \quad F(s) = \frac{1}{(s-a)}, \quad s > a. \quad \triangleleft$$

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Integrating by parts twice it is not difficult to obtain:

$$\int_0^N e^{-st} \sin(at) dt =$$
$$-\frac{1}{s} [e^{-st} \sin(at)] \Big|_0^N - \frac{a}{s^2} [e^{-st} \cos(at)] \Big|_0^N - \frac{a^2}{s^2} \int_0^N e^{-st} \sin(at) dt.$$

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This identity implies

$$\left(1 + \frac{a^2}{s^2}\right) \int_0^N e^{-st} \sin(at) dt = -\frac{1}{s} [e^{-st} \sin(at)] \Big|_0^N - \frac{a}{s^2} [e^{-st} \cos(at)] \Big|_0^N.$$

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Hence, it is not difficult to see that

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which is equivalent to

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}, \quad s > 0. \quad \triangleleft$$

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A table of Laplace Transforms.

$$f(t) = 1 \qquad F(s) = \frac{1}{s} \qquad s > 0,$$

$$f(t) = e^{at} \qquad F(s) = \frac{1}{s - a} \qquad s > \max\{a, 0\},$$

$$f(t) = t^n \qquad F(s) = \frac{n!}{s^{(n+1)}} \qquad s > 0,$$

$$f(t) = \sin(at) \qquad F(s) = \frac{a}{s^2 + a^2} \qquad s > 0,$$

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$$f(t) = t^n e^{at} \qquad F(s) = \frac{n!}{(s - a)^{(n+1)}} \qquad s > \max\{a, 0\},$$

$$f(t) = e^{at} \sin(bt) \qquad F(s) = \frac{b}{(s - a)^2 + b^2} \qquad s > \max\{a, 0\}.$$

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Properties of the Laplace Transform.

Theorem (Sufficient conditions)

If the function $f : [0, \infty) \rightarrow \mathbb{R}$ is piecewise continuous and there exist positive constants k and a such that

$$|f(t)| \leq k e^{at},$$

then the Laplace Transform of f exists for all $s > a$.

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Theorem (Linear combination)

If the $\mathcal{L}[f]$ and $\mathcal{L}[g]$ are well-defined and a, b are constants, then

$$\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g].$$

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If the $\mathcal{L}[f]$ and $\mathcal{L}[f']$ are well-defined, then holds,

$$\mathcal{L}[f'] = s \mathcal{L}[f] - f(0). \quad (1)$$

Furthermore, if $\mathcal{L}[f'']$ is well-defined, then it also holds

$$\mathcal{L}[f''] = s^2 \mathcal{L}[f] - s f(0) - f'(0). \quad (2)$$

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Integrating by parts,

$$\lim_{n \rightarrow \infty} \int_0^n e^{-st} f'(t) dt = \lim_{n \rightarrow \infty} \left[\left(e^{-st} f(t) \right) \Big|_0^n - \int_0^n (-s) e^{-st} f(t) dt \right]$$

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$$\mathcal{L}[f'] = \int_0^{\infty} e^{-st} f'(t) dt = \lim_{n \rightarrow \infty} \int_0^n e^{-st} f'(t) dt$$

Integrating by parts,

$$\lim_{n \rightarrow \infty} \int_0^n e^{-st} f'(t) dt = \lim_{n \rightarrow \infty} \left[\left(e^{-st} f(t) \right) \Big|_0^n - \int_0^n (-s) e^{-st} f(t) dt \right]$$

$$\mathcal{L}[f'] = \lim_{n \rightarrow \infty} \left[e^{-sn} f(n) - f(0) \right] + s \int_0^{\infty} e^{-st} f(t) dt$$

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where we used that $\lim_{n \rightarrow \infty} e^{-sn} f(n) = 0$ for s big enough, and we also used that $\mathcal{L}[f]$ is well-defined.

We then conclude that $\mathcal{L}[f'] = s \mathcal{L}[f] - f(0)$.

The Laplace Transform (Sect. 4.1).

- ▶ The definition of the Laplace Transform.
- ▶ Review: Improper integrals.
- ▶ Examples of Laplace Transforms.
- ▶ A table of Laplace Transforms.
- ▶ Properties of the Laplace Transform.
- ▶ **Laplace Transform and differential equations.**

Laplace Transform and differential equations.

Remark: Laplace Transforms can be used to find solutions to differential equations with **constant coefficients**.

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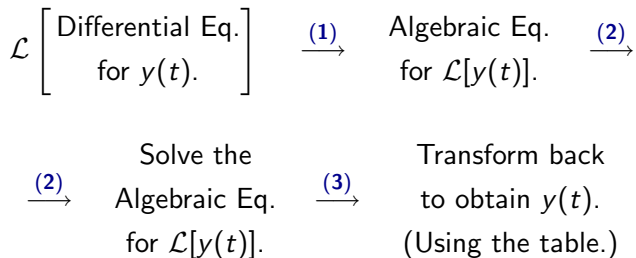
$$\mathcal{L} \left[\begin{array}{l} \text{Differential Eq.} \\ \text{for } y(t). \end{array} \right] \xrightarrow{(1)} \text{Algebraic Eq.} \xrightarrow{(2)} \text{for } \mathcal{L}[y(t)].$$

$$\xrightarrow{(2)} \begin{array}{l} \text{Solve the} \\ \text{Algebraic Eq.} \\ \text{for } \mathcal{L}[y(t)]. \end{array}$$

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Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \quad y(0) = 3.$$

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Solution: We know the solution: $y(t) = 3e^{-2t}$.

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Find an algebraic equation for $\mathcal{L}[y]$.

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$$\left[s\mathcal{L}[y] - y(0) \right] + 2\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s + 2)\mathcal{L}[y] = y(0).$$

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Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}]$

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Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}] \Rightarrow y(t) = 3e^{-2t}$.



The Laplace Transform and the IVP (Sect. 4.2).

- ▶ Solving differential equations using $\mathcal{L}[\]$.
 - ▶ Homogeneous IVP.
 - ▶ First, second, higher order equations.
 - ▶ Non-homogeneous IVP.

Solving differential equations using $\mathcal{L}[\]$.

Remark: The method works with:

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- ▶ Constant coefficient equations.

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Idea of the method:

Solving differential equations using $\mathcal{L}[\]$.

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$$\mathcal{L} \left[\begin{array}{l} \text{Differential Eq.} \\ \text{for } y(t). \end{array} \right] \xrightarrow{(1)} \begin{array}{l} \text{Algebraic Eq.} \\ \text{for } \mathcal{L}[y(t)]. \end{array}$$

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Solve the

$$\xrightarrow{(2)} \text{Algebraic Eq.}$$

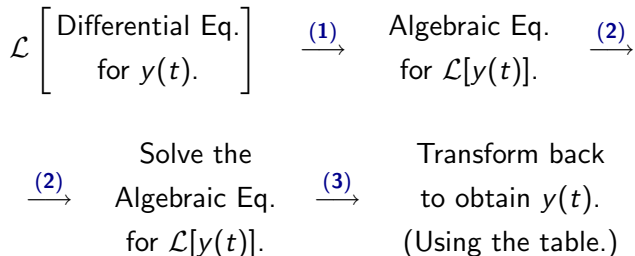
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Solving differential equations using $\mathcal{L}[\]$.

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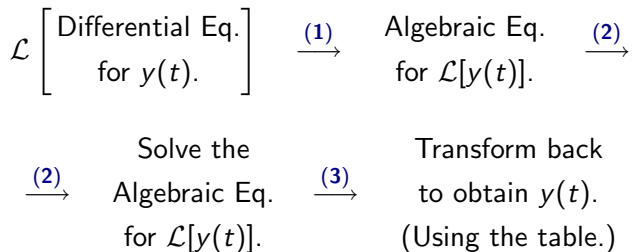
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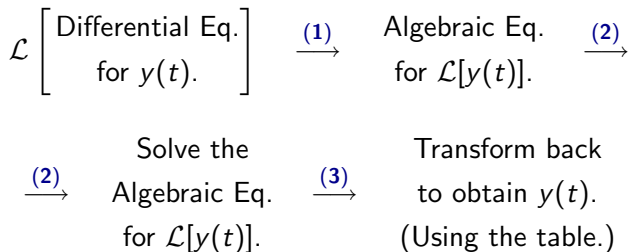
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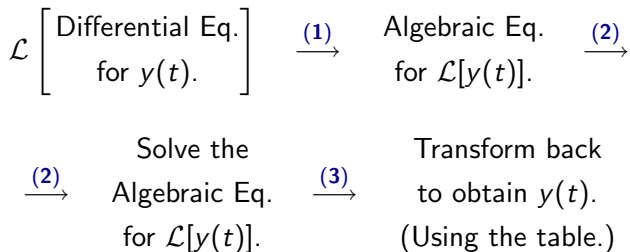


Recall:

$$(a) \quad \mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)];$$

Solving differential equations using $\mathcal{L}[\]$.

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Recall:

$$(a) \quad \mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)];$$

$$(b) \quad \mathcal{L}[y^{(n)}] = s^n \mathcal{L}[y] - s^{(n-1)} y(0) - s^{(n-2)} y'(0) - \dots - y^{(n-1)}(0).$$

The Laplace Transform and the IVP (Sect. 4.2).

- ▶ Solving differential equations using $\mathcal{L}[\]$.
 - ▶ **Homogeneous IVP.**
 - ▶ First, second, higher order equations.
 - ▶ Non-homogeneous IVP.

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

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Solution: Compute the $\mathcal{L}[\]$ of the differential equation,

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The $\mathcal{L}[\]$ is a linear function,

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The $\mathcal{L}[\]$ is a linear function, so

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] = 0.$$

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$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] = 0.$$

Derivatives are transformed into power functions,

$$\left[s^2 \mathcal{L}[y] - s y(0) - y'(0) \right] - \left[s \mathcal{L}[y] - y(0) \right] - 2 \mathcal{L}[y] = 0,$$

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Derivatives are transformed into power functions,

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We then obtain $(s^2 - s - 2) \mathcal{L}[y] = (s - 1) y(0) + y'(0)$.

Homogeneous IVP.

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$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

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Differential equation for $y \xrightarrow{\mathcal{L}[\]}$ Algebraic equation for $\mathcal{L}[y]$.

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Introduce the initial condition,

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Introduce the initial condition,

$$(s^2 - s - 2) \mathcal{L}[y] = (s - 1).$$

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$$(s^2 - s - 2) \mathcal{L}[y] = (s - 1).$$

We can solve for the unknown $\mathcal{L}[y]$ as follows,

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

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The partial fraction method: Find the zeros of the denominator,

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Therefore, we rewrite: $\mathcal{L}[y] = \frac{(s-1)}{(s-2)(s+1)}$.

Find constants a and b such that

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}.$$

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$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1)}$$

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Hence, $a = \frac{1}{3}$ and $b = \frac{2}{3}$.

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$$(s-1) = s(a+b) + (a-2b) \Rightarrow \begin{cases} a+b=1, \\ a-2b=-1 \end{cases}$$

Hence, $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Then, $\mathcal{L}[y] = \frac{1}{3} \frac{1}{(s-2)} + \frac{2}{3} \frac{1}{(s+1)}.$

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Solution: Recall: $\mathcal{L}[y] = \frac{1}{3} \frac{1}{(s-2)} + \frac{2}{3} \frac{1}{(s+1)}$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

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So we arrive at the equation

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$$\mathcal{L}[y] = \frac{1}{3} \mathcal{L}[e^{2t}] + \frac{2}{3} \mathcal{L}[e^{-t}] = \mathcal{L}\left[\frac{1}{3}(e^{2t} + 2e^{-t})\right]$$

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So we arrive at the equation

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We conclude that: $y(t) = \frac{1}{3}(e^{2t} + 2e^{-t})$.



Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Compute the $\mathcal{L}[\]$ of the differential equation,

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Solution: Compute the $\mathcal{L}[\]$ of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[0] = 0.$$

The $\mathcal{L}[\]$ is a linear function,

$$\mathcal{L}[y''] - 4\mathcal{L}[y'] + 4\mathcal{L}[y] = 0.$$

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Derivatives are transformed into power functions,

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$$\left[s^2 \mathcal{L}[y] - s y(0) - y'(0) \right] - 4 \left[s \mathcal{L}[y] - y(0) \right] + 4 \mathcal{L}[y] = 0,$$

Therefore, $(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4) y(0) + y'(0)$.

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Introduce the initial conditions,

Homogeneous IVP.

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Solution: Recall: $(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4)y(0) + y'(0)$.

Introduce the initial conditions, $(s^2 - 4s + 4) \mathcal{L}[y] = s - 3$.

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Solve for $\mathcal{L}[y]$ as follows:

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Solve for $\mathcal{L}[y]$ as follows: $\mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)}$.

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The partial fraction method: Find the roots of the denominator,

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$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 16}]$$

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$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2} [4 \pm \sqrt{16 - 16}] \quad \Rightarrow \quad s_+ = s_- = 2.$$

We obtain: $\mathcal{L}[y] = \frac{(s - 3)}{(s - 2)^2}$.

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If $s = 2$, then $b = -1$.

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If $s = 2$, then $b = -1$. If $s = 3$, then $a = 1$. Hence

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

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If $s = 2$, then $b = -1$. If $s = 3$, then $a = 1$. Hence

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

From the Laplace transforms table:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \frac{(s-3)}{(s-2)^2}$. We find the partial fraction,

$$\frac{(s-3)}{(s-2)^2} = \frac{a}{(s-2)} + \frac{b}{(s-2)^2} \Rightarrow s-3 = a(s-2) + b$$

If $s = 2$, then $b = -1$. If $s = 3$, then $a = 1$. Hence

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

From the Laplace transforms table:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \Rightarrow \frac{1}{s-2} = \mathcal{L}[e^{2t}],$$

Homogeneous IVP.

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Solution: Recall: $\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2}$ and

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So we arrive at the equation

$$\mathcal{L}[y] = \mathcal{L}[e^{2t}] - \mathcal{L}[te^{2t}] = \mathcal{L}[e^{2t} - te^{2t}].$$

We conclude that $y(t) = e^{2t} - te^{2t}$.



The Laplace Transform and the IVP (Sect. 4.2).

- ▶ Solving differential equations using $\mathcal{L}[]$.
 - ▶ Homogeneous IVP.
 - ▶ **First, second, higher order equations.**
 - ▶ Non-homogeneous IVP.

First, second, higher order equations.

Example

Use the Laplace Transform to find the solution of $y^{(4)} - 4y = 0$,

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$$

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$$\mathcal{L}[y^{(4)}] - 4\mathcal{L}[y] = 0.$$

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$$[s^4 \mathcal{L}[y] - s^3 + 2s] - 4\mathcal{L}[y] = 0$$

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$$[s^4 \mathcal{L}[y] - s^3 + 2s] - 4\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s^4 - 4)\mathcal{L}[y] = s^3 - 2s,$$

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$$[s^4 \mathcal{L}[y] - s^3 + 2s] - 4\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s^4 - 4)\mathcal{L}[y] = s^3 - 2s,$$

We obtain,
$$\mathcal{L}[y] = \frac{s^3 - 2s}{(s^4 - 4)}.$$

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Solution: Recall: $\mathcal{L}[y] = \frac{s^3 - 2s}{(s^4 - 4)}$.

$$\mathcal{L}[y] = \frac{s(s^2 - 2)}{(s^2 - 2)(s^2 + 2)}$$

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The last expression is in the table of Laplace Transforms,

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The last expression is in the table of Laplace Transforms,

$$\mathcal{L}[y] = \frac{s}{(s^2 + [\sqrt{2}]^2)}$$

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$$\mathcal{L}[y] = \frac{s}{(s^2 + [\sqrt{2}]^2)} = \mathcal{L}[\cos(\sqrt{2} t)].$$

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We conclude that $y(t) = \cos(\sqrt{2} t)$.



The Laplace Transform and the IVP (Sect. 4.2).

- ▶ Solving differential equations using $\mathcal{L}[]$.
 - ▶ Homogeneous IVP.
 - ▶ First, second, higher order equations.
 - ▶ **Non-homogeneous IVP.**

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

Non-homogeneous IVP.

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Solution: Compute the Laplace transform of the equation,

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Introduce this source term in the differential equation,

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$$\mathcal{L}[y''] - 4 \mathcal{L}[y'] + 4 \mathcal{L}[y] = \frac{6}{s^2 + 4}.$$

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Derivatives are transformed into power functions,

$$\left[s^2 \mathcal{L}[y] - s y(0) - y'(0) \right] - 4 \left[s \mathcal{L}[y] - y(0) \right] + 4 \mathcal{L}[y] = \frac{6}{s^2 + 4}.$$

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Rewrite the above equation,

$$(s^2 - 4s + 4) \mathcal{L}[y] = (s - 4) y(0) + y'(0) + \frac{6}{s^2 + 4}.$$

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Rewrite the above equation,

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Non-homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

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Solution: Recall: $(s^2 - 4s + 4) \mathcal{L}[y] = s - 3 + \frac{6}{s^2 + 4}$.

Therefore, $\mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)} + \frac{6}{(s^2 - 4s + 4)(s^2 + 4)}$.

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From an Example above: $s^2 - 4s + 4 = (s - 2)^2$,

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From an Example above: $s^2 - 4s + 4 = (s - 2)^2$,

$$\mathcal{L}[y] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2} + \frac{6}{(s - 2)^2(s^2 + 4)}.$$

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From an Example above: $s^2 - 4s + 4 = (s - 2)^2$,

$$\mathcal{L}[y] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2} + \frac{6}{(s - 2)^2(s^2 + 4)}.$$

From an Example above we know that

$$\mathcal{L}[e^{2t} - te^{2t}] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2}.$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Use Partial fractions to simplify the last term above.

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Use Partial fractions to simplify the last term above.

Find constants a, b, c, d , such that

$$\frac{6}{(s-2)^2(s^2+4)} = \frac{as+b}{s^2+4} + \frac{c}{s-2} + \frac{d}{(s-2)^2}$$

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$$\frac{6}{(s-2)^2(s^2+4)} = \frac{(as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4)}{(s^2+4)(s-2)^2}$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

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Find constants a, b, c, d , such that

$$\frac{6}{(s-2)^2(s^2+4)} = \frac{as+b}{s^2+4} + \frac{c}{s-2} + \frac{d}{(s-2)^2}$$

$$\frac{6}{(s-2)^2(s^2+4)} = \frac{(as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4)}{(s^2+4)(s-2)^2}$$

$$6 = (as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4).$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: $6 = (as + b)(s - 2)^2 + c(s - 2)(s^2 + 4) + d(s^2 + 4).$

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$$6 = (as + b)(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2 + 4)$$

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$$6 = (a + c)s^3 + (-4a + b - 2c + d)s^2 \\ + (4a - 4b + 4c)s + (4b - 8c + 4d).$$

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We obtain the system

$$\begin{aligned} a + c &= 0, & -4a + b - 2c + d &= 0, \\ 4a - 4b + 4c &= 0, & 4b - 8c + 4d &= 6. \end{aligned}$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3 \sin(2t), \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: The solution for this linear system is

$$a = \frac{3}{8}, \quad b = 0, \quad c = -\frac{3}{8}, \quad d = \frac{3}{4}.$$

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Use the table of Laplace Transforms

$$\frac{6}{(s-2)^2(s^2+4)} = \frac{3}{8} \mathcal{L}[\cos(2t)] - \frac{3}{8} \mathcal{L}[e^{2t}] + \frac{3}{4} \mathcal{L}[te^{2t}].$$

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$$\mathcal{L}[y(t)] = \mathcal{L}\left[(1-t)e^{2t} + \frac{3}{8}(-1+2t)e^{2t} + \frac{3}{8} \cos(2t)\right].$$

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We conclude that

$$y(t) = (1-t)e^{2t} + \frac{3}{8}(2t-1)e^{2t} + \frac{3}{8} \cos(2t). \quad \triangleleft$$