The Laplace Transform (Sect. 4.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
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Definition

The function $F : D_F \to \mathbb{R}$ is the *Laplace transform* of a function $f : [0, \infty) \to \mathbb{R}$ iff for all $s \in D_F$ holds,

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Recall: Improper integral are defined as a limit.

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Therefore $\mathcal{L}[1] = \dfrac{1}{s}$, for $s > 0$, and $\mathcal{L}[1]$ does not exists for $s \leqslant 0$.

In other words, $F(s) = \mathcal{L}[1]$ is the function $F : D_F \to \mathbb{R}$ given by

$$f(t) = 1, \qquad F(s) = \frac{1}{s}, \qquad D_F = (0, \infty). \qquad \triangleleft$$

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$$f(t) = e^{at}, \qquad F(s) = \frac{1}{(s-a)}, \qquad s > a.$$

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Compute $\mathcal{L}[\sin(at)]$, where $a \in \mathbb{R}$.

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Integrating by parts twice it is not difficult to obtain:

$$\int_0^N e^{-st} \sin(at)\, dt =$$

$$-\frac{1}{s}\left[e^{-st}\sin(at)\right]\Big|_0^N - \frac{a}{s^2}\left[e^{-st}\cos(at)\right]\Big|_0^N - \frac{a^2}{s^2}\int_0^N e^{-st}\sin(at)\, dt.$$

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This identity implies

$$\left(1 + \frac{a^2}{s^2}\right)\int_0^N e^{-st}\sin(at)\, dt = -\frac{1}{s}\left[e^{-st}\sin(at)\right]\Big|_0^N - \frac{a}{s^2}\left[e^{-st}\cos(at)\right]\Big|_0^N.$$

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Hence, it is not difficult to see that

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A table of Laplace Transforms.

$f(t) = 1$	$F(s) = \dfrac{1}{s}$	$s > 0,$
$f(t) = e^{at}$	$F(s) = \dfrac{1}{s-a}$	$s > \max\{a, 0\},$
$f(t) = t^n$	$F(s) = \dfrac{n!}{s^{(n+1)}}$	$s > 0,$
$f(t) = \sin(at)$	$F(s) = \dfrac{a}{s^2+a^2}$	$s > 0,$
$f(t) = \cos(at)$	$F(s) = \dfrac{s}{s^2+a^2}$	$s > 0,$
$f(t) = \sinh(at)$	$F(s) = \dfrac{a}{s^2-a^2}$	$s > 0,$
$f(t) = \cosh(at)$	$F(s) = \dfrac{s}{s^2-a^2}$	$s > 0,$
$f(t) = t^n e^{at}$	$F(s) = \dfrac{n!}{(s-a)^{(n+1)}}$	$s > \max\{a, 0\},$
$f(t) = e^{at}\sin(bt)$	$F(s) = \dfrac{b}{(s-a)^2+b^2}$	$s > \max\{a, 0\}.$

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Theorem (Sufficient conditions)

If the function $f : [0, \infty) \to \mathbb{R}$ *is piecewise continuous and there exist positive constants* k *and* a *such that*

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If the $\mathcal{L}[f]$ *and* $\mathcal{L}[g]$ *are well-defined and* a, b *are constants, then*

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$$\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\,f(0) - f'(0). \tag{2}$$

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$$\lim_{n\to\infty} \int_0^{n} e^{-st} f'(t)\, dt = \lim_{n\to\infty} \Big[\Big(e^{-st} f(t) \Big)\Big|_0^n - \int_0^n (-s) e^{-st} f(t)\, dt \Big]$$

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$$\lim_{n\to\infty} \int_0^n e^{-st} f'(t)\, dt = \lim_{n\to\infty} \left[\left(e^{-st} f(t) \right) \Big|_0^n - \int_0^n (-s) e^{-st} f(t)\, dt \right]$$

$$\mathcal{L}[f'] = \lim_{n\to\infty} \Big[e^{-sn} f(n) - f(0) \Big] + s \int_0^\infty e^{-st} f(t)\, dt$$

Properties of the Laplace Transform.

Proof of Eq (1): Recall the definition of the Laplace Transform,

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where we used that $\lim_{n\to\infty} e^{-sn} f(n) = 0$ for s big enough, and we also used that $\mathcal{L}[f]$ is well-defined.

We then conclude that $\mathcal{L}[f'] = s\, \mathcal{L}[f] - f(0)$.

The Laplace Transform (Sect. 4.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
- Properties of the Laplace Transform.
- **Laplace Transform and differential equations.**

Laplace Transform and differential equations.

Remark: Laplace Transforms can be used to find solutions to differential equations with constant coefficients.

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Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \qquad y(0) = 3.$$

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Find an algebraic equation for $\mathcal{L}[y]$.

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$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = 0.$$

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Also recall the property: $\mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0)$, that is,

$$\Big[s\,\mathcal{L}[y] - y(0)\Big] + 2\,\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s+2)\mathcal{L}[y] = y(0).$$

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$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

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Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}]$

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Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}] \quad \Rightarrow \quad y(t) = 3e^{-2t}.$ $\lhd$

The Laplace Transform and the IVP (Sect. 4.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - Homogeneous IVP.
  - First, second, higher order equations.
  - Non-homogeneous IVP.

Solving differential equations using $\mathcal{L}[\ ]$.

Remark: The method works with:

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$$\mathcal{L}\begin{bmatrix} \text{Differential Eq.} \\ \text{for } y(t). \end{bmatrix}$$

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Recall:

(a) $\mathcal{L}[a\,f(t) + b\,g(t)] = a\,\mathcal{L}[f(t)] + b\,\mathcal{L}[g(t)]$;

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Recall:

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(b) $\mathcal{L}\left[y^{(n)}\right] = s^n\,\mathcal{L}[y] - s^{(n-1)}\,y(0) - s^{(n-2)}\,y'(0) - \cdots - y^{(n-1)}(0)$.

The Laplace Transform and the IVP (Sect. 4.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - **Homogeneous IVP.**
  - First, second, higher order equations.
  - Non-homogeneous IVP.

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

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Solution: Compute the $\mathcal{L}[\;]$ of the differential equation,

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$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0] \quad \Rightarrow \quad \mathcal{L}[y'' - y' - 2y] = 0.$$

The $\mathcal{L}[\ ]$ is a linear function,

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The $\mathcal{L}[\ ]$ is a linear function, so

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\,\mathcal{L}[y] = 0.$$

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The $\mathcal{L}[\ ]$ is a linear function, so

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\,\mathcal{L}[y] = 0.$$

Derivatives are transformed into power functions,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - \Big[s\,\mathcal{L}[y] - y(0)\Big] - 2\,\mathcal{L}[y] = 0,$$

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Derivatives are transformed into power functions,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - \Big[s\,\mathcal{L}[y] - y(0)\Big] - 2\,\mathcal{L}[y] = 0,$$

We the obtain $(s^2 - s - 2)\,\mathcal{L}[y] = (s-1)\,y(0) + y'(0).$

Homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Differential equation for y $\xrightarrow{\mathcal{L}[\ ]}$ Algebraic equation for $\mathcal{L}[y]$.

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

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Differential equation for y $\xrightarrow{\mathcal{L}[\ ]}$ Algebraic equation for $\mathcal{L}[y]$.

Introduce the initial condition,

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Recall: $(s^2 - s - 2)\,\mathcal{L}[y] = (s-1)\,y(0) + y'(0)$.

Differential equation for y $\xrightarrow{\mathcal{L}[\ ]}$ Algebraic equation for $\mathcal{L}[y]$.

Introduce the initial condition,

$$(s^2 - s - 2)\,\mathcal{L}[y] = (s-1).$$

Homogeneous IVP.

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$$\mathcal{L}[y] = \frac{(s-1)}{(s^2 - s - 2)}.$$

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$$s^2 - s - 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\left[1 \pm \sqrt{1+8}\right]$$

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Therefore, we rewrite: $\mathcal{L}[y] = \dfrac{(s-1)}{(s-2)(s+1)}$.

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Therefore, we rewrite: $\mathcal{L}[y] = \dfrac{(s-1)}{(s-2)(s+1)}$.

Find constants a and b such that

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}.$$

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$$(s-1) = s(a+b) + (a-2b)$$

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Hence, $a = \dfrac{1}{3}$ and $b = \dfrac{2}{3}$.

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$$(s-1) = s(a+b) + (a-2b) \quad \Rightarrow \quad \begin{cases} a + b = 1, \\ a - 2b = -1 \end{cases}$$

Hence, $a = \dfrac{1}{3}$ and $b = \dfrac{2}{3}$. Then, $\mathcal{L}[y] = \dfrac{1}{3}\,\dfrac{1}{(s-2)} + \dfrac{2}{3}\,\dfrac{1}{(s+1)}.$

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$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

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So we arrive at the equation

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We conclude that: $y(t) = \frac{1}{3}\big(e^{2t} + 2e^{-t}\big)$. $\lhd$

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Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

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$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,

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$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[0] = 0.$$

The $\mathcal{L}[\ ]$ is a linear function,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = 0.$$

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Derivatives are transformed into power functions,

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$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = 0,$$

Therefore, $(s^2 - 4s + 4)\,\mathcal{L}[y] = (s - 4)\,y(0) + y'(0).$

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Introduce the initial conditions,

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Solution: Recall: $(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0)$.

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Solve for $\mathcal{L}[y]$ as follows: $\mathcal{L}[y] = \dfrac{(s-3)}{(s^2 - 4s + 4)}$.

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We obtain: $\mathcal{L}[y] = \dfrac{(s-3)}{(s-2)^2}$.

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$$\frac{(s-3)}{(s-2)^2} = \frac{a}{(s-2)} + \frac{b}{(s-2)^2}$$

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If $s = 2$,

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$$\frac{(s-3)}{(s-2)^2} = \frac{a}{(s-2)} + \frac{b}{(s-2)^2} \;\Rightarrow\; s - 3 = a(s-2) + b$$

If $s = 2$, then $b = -1$. If $s = 3$, then $a = 1$. Hence

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s-3)}{(s-2)^2}$. We find the partial fraction,

$$\frac{(s-3)}{(s-2)^2} = \frac{a}{(s-2)} + \frac{b}{(s-2)^2} \;\Rightarrow\; s - 3 = a(s-2) + b$$

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$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

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Homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

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We conclude that $y(t) = e^{2t} - te^{2t}$. $\triangleleft$

The Laplace Transform and the IVP (Sect. 4.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - Homogeneous IVP.
  - **First, second, higher order equations.**
  - Non-homogeneous IVP.

First, second, higher order equations.

Example

Use the Laplace Transform to find the solution of $y^{(4)} - 4y = 0$,

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$$

First, second, higher order equations.

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$$\mathcal{L}\big[y^{(4)}\big] - 4\,\mathcal{L}[y] = 0.$$

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$$\left[s^4\,\mathcal{L}[y] - s^3 + 2s\right] - 4\,\mathcal{L}[y] = 0$$

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We obtain, $\mathcal{L}[y] = \dfrac{s^3 - 2s}{(s^4 - 4)}$.

First, second, higher order equations.

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Solution: Recall: $\mathcal{L}[y] = \dfrac{s^3 - 2s}{(s^4 - 4)}$.

$$\mathcal{L}[y] = \frac{s(s^2 - 2)}{(s^2 - 2)(s^2 + 2)}$$

First, second, higher order equations.

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The last expression is in the table of Laplace Transforms,

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$$\mathcal{L}[y] = \frac{s}{\left(s^2 + \left[\sqrt{2}\right]^2\right)}$$

First, second, higher order equations.

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We conclude that $y(t) = \cos(\sqrt{2}\, t)$. $\triangleleft$

The Laplace Transform and the IVP (Sect. 4.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - Homogeneous IVP.
  - First, second, higher order equations.
  - **Non-homogeneous IVP.**

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Non-homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Compute the Laplace transform of the equation,

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$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3\sin(2t)].$$

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$$\mathcal{L}[3\sin(2t)] = 3\,\mathcal{L}[\sin(2t)] = 3\,\frac{2}{s^2 + 2^2} = \frac{6}{s^2 + 4}.$$

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Introduce this source term in the differential equation,

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Introduce this source term in the differential equation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \frac{6}{s^2 + 4}.$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \dfrac{6}{s^2+4}$.

Derivatives are transformed into power functions,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = \frac{6}{s^2+4}.$$

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Rewrite the above equation,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0) + \frac{6}{s^2+4}.$$

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Rewrite the above equation,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0) + \frac{6}{s^2+4}.$$

Introduce the initial conditions,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \frac{6}{s^2+4}.$$

Non-homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Solution: Recall: $(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \dfrac{6}{s^2 + 4}$.

Therefore, $\mathcal{L}[y] = \dfrac{(s-3)}{(s^2 - 4s + 4)} + \dfrac{6}{(s^2 - 4 + 4)(s^2 + 4)}$.

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From an Example above: $s^2 - 4s + 4 = (s-2)^2$,

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From an Example above: $s^2 - 4s + 4 = (s-2)^2$,

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2} + \frac{6}{(s-2)^2(s^2+4)}.$$

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From an Example above: $s^2 - 4s + 4 = (s-2)^2$,

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From an Example above we know that

$$\mathcal{L}[e^{2t} - te^{2t}] = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Solution: Recall: $\mathcal{L}[y] = \mathcal{L}[e^{2t} - te^{2t}] + \dfrac{6}{(s-2)^2(s^2+4)}$.

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Use Partial fractions to simplify the last term above.

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Use Partial fractions to simplify the last term above.

Find constants a, b, c, d, such that

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{as+b}{s^2+4} + \frac{c}{(s-2)} + \frac{d}{(s-2)^2}$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Solution: Recall: $\mathcal{L}[y] = \mathcal{L}[e^{2t} - te^{2t}] + \dfrac{6}{(s-2)^2(s^2+4)}$.

Use Partial fractions to simplify the last term above.

Find constants a, b, c, d, such that

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{as+b}{s^2+4} + \frac{c}{(s-2)} + \frac{d}{(s-2)^2}$$

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Non-homogeneous IVP.

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We obtain the system

$$a + c = 0, \qquad -4a + b - 2c + d = 0,$$

$$4a - 4b + 4c = 0, \qquad 4b - 8c + 4d = 6.$$

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$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: The solution for this linear system is

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We conclude that

$$y(t) = (1-t)\,e^{2t} + \frac{3}{8}\,(2t-1)\,e^{2t} + \frac{3}{8}\,\cos(2t).$$

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