Second order linear ODE (Sect. 2.2).

- Idea: Solving constant coefficients equations.
- The characteristic equation.
- Solution formulas for constant coefficients equations.
Review: Second order linear ODE.

Definition
Given functions \( a_1, a_0, b : \mathbb{R} \rightarrow \mathbb{R} \), the differential equation in the unknown function \( y : \mathbb{R} \rightarrow \mathbb{R} \) given by

\[
y'' + a_1(t) y' + a_0(t) y = b(t)
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is called a second order linear differential equation.
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is called a **second order linear** differential equation. If $b = 0$, the equation is called **homogeneous**. If the coefficients $a_1, a_2 \in \mathbb{R}$ are constants, the equation is called of **constant coefficients**.
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is called a second order linear differential equation. If \(b = 0\), the equation is called homogeneous. If the coefficients \(a_1, a_2 \in \mathbb{R}\) are constants, the equation is called of constant coefficients.

Theorem (Superposition property)
If the functions \(y_1\) and \(y_2\) are solutions to the homogeneous linear equation

\[
y'' + a_1(t) y' + a_0(t) y = 0,
\]

then the linear combination \(c_1 y_1(t) + c_2 y_2(t)\) is also a solution for any constants \(c_1, c_2 \in \mathbb{R}\).
Second order linear ODE (Sect. 2.2).

- **Idea:** Soving constant coefficients equations.
- The characteristic equation.
- Solution formulas for constant coefficients equations.
Idea: Solving constant coefficients equations.

Remark: Just by trial and error one can find solutions to second order, constant coefficients, homogeneous, linear differential equations.

Example

Find solutions to the equation
\[ y'' + 5y' + 6y = 0. \]

Solution:

We look for solutions proportional to exponentials \( e^{rt} \), for an appropriate constant \( r \in \mathbb{R} \), since the exponential can be canceled out from the equation.

If \( y(t) = e^{rt} \), then \( y'(t) = re^{rt} \), and \( y''(t) = r^2e^{rt} \). Hence
\[
(r^2 + 5r + 6)e^{rt} = 0 \iff r^2 + 5r + 6 = 0.
\]

That is, \( r \) must be a root of the polynomial \( p(r) = r^2 + 5r + 6 \).

This polynomial is called the characteristic polynomial of the differential equation.
Idea: Solving constant coefficients equations.

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This polynomial is called the characteristic polynomial of the differential equation.
**Idea:** Solving constant coefficients equations.

**Example**
Find solutions to the equation $y'' + 5y' + 6y = 0$.

**Solution:** Recall: $p(r) = r^2 + 5r + 6$. 

\[ r = \frac{-5 \pm \sqrt{25 - 24}}{2} = \frac{-5 \pm 1}{2} \]

\[ r_1 = -3, \quad r_2 = -2 \]

Therefore, we have found two solutions to the ODE, $y_1(t) = e^{-2t}$, $y_2(t) = e^{-3t}$.

Their superposition provides infinitely many solutions, $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$, $c_1, c_2 \in \mathbb{R}$. 

\[ \triangleright \]
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Example
Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: Recall: $p(r) = r^2 + 5r + 6$.

The roots of the characteristic polynomial are

$$r = \frac{1}{2} \left( -5 \pm \sqrt{25 - 24} \right)$$
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y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \quad c_1, c_2 \in \mathbb{R}.
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Summary: The differential equation \( y'' + 5y' + 6y = 0 \) has infinitely many solutions,

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Remarks:

▶ There are two free constants in the solution found above.
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Remarks:

- There are **two free constants** in the solution found above.
- The ODE above is **second order**, so two integrations must be done to find the solution. This explain the origin of the two free constant in the solution.
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Remarks:

- There are two free constants in the solution found above.
- The ODE above is second order, so two integrations must be done to find the solution. This explain the origin of the two free constant in the solution.
- An IVP for a second order differential equation will have a unique solution if the IVP contains two initial conditions.
Second order linear ODE (Sect. 2.2).

- Idea: Soving constant coefficients equations.
- **The characteristic equation.**
- Solution formulas for constant coefficients equations.
The characteristic equation.

Definition
Given a second order linear homogeneous differential equation with constant coefficients

\[ y'' + a_1 y' + a_0 = 0, \tag{1} \]

the *characteristic polynomial* and the *characteristic equation* associated with the differential equation in (1) are, respectively,

\[ p(r) = r^2 + a_1 r + a_0, \quad p(r) = 0. \]
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Remark: If \( r_1, r_2 \) are the solutions of the characteristic equation and \( c_1, c_2 \) are constants, then we will show that the general solution of Eq. (1) is given by

\[ y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \]
Example
Find the solution $y$ of the initial value problem

$$y'' + 5y' + 6 = 0, \quad y(0) = 1, \quad y'(0) = -1.$$
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We now find the constants $c_1$ and $c_2$ that satisfy the initial conditions above:
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c_1 = 1 - c_2 \Rightarrow 1 = 2(1 - c_2) + 3c_2 \Rightarrow c_2 = -1 \Rightarrow c_1 = 2.
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Therefore, the unique solution to the initial value problem is
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y(t) = 2e^{-2t} - e^{-3t}.
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The characteristic equation.

**Example**

Find the general solution $y$ of the differential equation

$$2y'' - 3y' + y = 0.$$
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Therefore, the general solution of the equation above is

$$y(t) = c_1 e^t + c_2 e^{t/2},$$

where $c_1, c_2$ are arbitrary constants.
Second order linear ODE (Sect. 2.2).

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Theorem (Constant coefficients)

Given real constants $a_1$, $a_0$, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \to \mathbb{R}$ given by

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Let $r_+, r_-$ be the roots of the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$,

(a) If $r_+ \neq r_-$, real or complex, then $y(t) = c_0 e^{r_+ t} + c_1 e^{r_- t}$.

(b) If $r_+ = r_- = \hat{r} \in \mathbb{R}$, then $y(t) = c_0 e^{\hat{r} t} + c_1 t e^{\hat{r} t}$.

Furthermore, given real constants $t_0$, $y_0$ and $y_1$, there is a unique solution to the initial value problem

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(a) If $r_+ \neq r_-$, real or complex, then $y(t) = c_0 e^{r_+ t} + c_1 e^{r_- t}$.

(b) If $r_+ = r_- = \hat{r} \in \mathbb{R}$, then is $y(t) = c_0 e^{\hat{r} t} + c_1 te^{\hat{r} t}$.

Furthermore, given real constants $t_0, y_0$ and $y_1$, there is a unique solution to the initial value problem

$$y'' + a_1 y' + a_0 y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1.$$
Second order linear homogeneous ODE (Sect. 2.3).

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - Review of Complex numbers.
  - A real-valued fundamental and general solutions.
- Application: The RLC circuit.
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Theorem (Constant coefficients)

Given real constants \( a_1, a_0 \), consider the homogeneous, linear differential equation on the unknown \( y : \mathbb{R} \to \mathbb{R} \) given by
\[
y'' + a_1 y' + a_0 y = 0.
\]

Let \( r_+, r_- \) be the roots of the characteristic polynomial \( p(r) = r^2 + a_1 r + a_0 \), and let \( c_0, c_1 \) be arbitrary constants. Then, the general solution \( y \) of the differential equation is given by

(a) If \( r_+ \neq r_- \), real or complex, then \( y(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}. \)

(b) If \( r_+ = r_- = \hat{r} \in \mathbb{R} \), then \( y(t) = c_1 e^{\hat{r} t} + c_2 t e^{\hat{r} t}. \)

Furthermore, given real constants \( t_0, y_1 \) and \( y_2 \), there is a unique solution to the initial value problem
\[
y'' + a_1 y' + a_0 y = 0, \quad y(t_0) = y_1, \quad y'(t_0) = y_2.
\]
Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Example
Find the general solution of the equation $y'' - y' - 6y = 0$.
Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Example
Find the general solution of the equation $y'' - y' - 6y = 0$.

Solution: Since solutions have the form $e^{rt}$, we need to find the roots of the characteristic polynomial $p(r) = r^2 - r - 6$,  

$$r = \frac{1 \pm \sqrt{1 + 24}}{2} = \frac{1 \pm 5}{2} \Rightarrow r_1 = 3, \quad r_2 = -2.$$ 

So, $r_1, r_2$ are real-valued. A fundamental solution set is formed by $y_1(t) = e^{3t}, y_2(t) = e^{-2t}$. The general solution of the differential equations is an arbitrary linear combination of the fundamental solutions, that is, $y(t) = c_1 e^{3t} + c_2 e^{-2t}, c_1, c_2 \in \mathbb{R}$.  

Remark: Since $c_1, c_2 \in \mathbb{R}$, then $y$ is real-valued.
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r_{\pm} = \frac{1}{2} \left( 1 \pm \sqrt{1 + 24} \right)
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 y(t) = c_1 e^{3t} + c_2 e^{-2t}, \quad c_1, c_2 \in \mathbb{R}. \quad \triangleq
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Find the general solution of the equation \( y'' - 2y' + 6y = 0 \).

Solution: We first find the roots of the characteristic polynomial,

\[
r^2 - 2r + 6 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2}(2 \pm \sqrt{4 - 24}) \quad \Rightarrow \quad r_{\pm} = 1 \pm i\sqrt{5}.\]
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\tilde{y}_1(t) = e^{(1+i\sqrt{5})t}, \quad \tilde{y}_2(t) = e^{(1-i\sqrt{5})t}.
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These are complex-valued functions. The general solution is

\[ y(t) = \tilde{c}_1 e^{(1+i\sqrt{5})t} + \tilde{c}_2 e^{(1-i\sqrt{5})t}, \quad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}. \]
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- The solutions found above include real-valued and complex-valued solutions.
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- The solutions found above include real-valued and complex-valued solutions.
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- In other words: It is not simple to see what values of $\tilde{c}_1$ and $\tilde{c}_2$ make the general solution above to be real-valued.
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- The solutions found above include real-valued and complex-valued solutions.
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- In other words: It is not simple to see what values of $\tilde{c}_1$ and $\tilde{c}_2$ make the general solution above to be real-valued.
- One way to find the real-valued general solution is to find real-valued fundamental solutions.
Second order linear homogeneous ODE (Sect. 2.3).

- Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.
- **Characteristic polynomial with complex roots.**
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Review of complex numbers.

- Complex numbers have the form \( z = a + ib \), where \( i^2 = -1 \).
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- $\text{Re}(z) = a$, $\text{Im}(z) = b$ are the real and imaginary parts of $z$. 

Euler's formula:

$$e^{ib} = \cos(b) + i\sin(b).$$

From $e^{a+ib}$ and $e^{a-ib}$ we get the real numbers

$$\frac{1}{2}(e^{a+ib} + e^{a-ib}) = e^a \cos(b),$$

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Hence, a complex number of the form $e^{a + ib}$ can be written as $e^{a + ib} = e^{a} \left[ \cos(b) + i\sin(b) \right]$, $e^{a - ib} = e^{a} \left[ \cos(b) - i\sin(b) \right]$. 

From $e^{a + ib}$ and $e^{a - ib}$ we get the real numbers $\frac{1}{2}(e^{a + ib} + e^{a - ib}) = e^{a} \cos(b)$, $\frac{1}{2i}(e^{a + ib} - e^{a - ib}) = e^{a} \sin(b)$. 
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- $e^{a+ib} = \sum_{n=0}^{\infty} \frac{(a + ib)^n}{n!}$. In particular holds $e^{a+ib} = e^a e^{ib}$.
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- From $e^{a+ib}$ and $e^{a-ib}$ we get the real numbers.
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Two main sets of fundamental solutions.

Theorem (Complex roots)

If the constants $a_1, a_0 \in \mathbb{R}$ satisfy that $a_1^2 - 4a_0 < 0$, then the characteristic polynomial $p(r) = r^2 + a_1r + a_0$ of the equation

$$y'' + a_1 y' + a_0 y = 0$$

has complex roots $r_+ = \alpha + i\beta$ and $r_- = \alpha - i\beta$, where

$$\alpha = -\frac{a_1}{2}, \quad \beta = \frac{1}{2} \sqrt{4a_0 - a_1^2}.$$

Furthermore, a fundamental set of solutions to Eq. (2) is

$$\tilde{y}_1(t) = e^{(\alpha+i\beta)t}, \quad \tilde{y}_2(t) = e^{(\alpha-i\beta)t},$$
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Furthermore, a fundamental set of solutions to Eq. (2) is

$$\tilde{y}_1(t) = e^{(\alpha+i\beta)t}, \quad \tilde{y}_2(t) = e^{(\alpha-i\beta)t},$$

while another fundamental set of solutions to Eq. (2) is

$$y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).$$
Review of complex numbers.

Idea of the Proof: Recall that the functions

$$\tilde{y}_1(t) = e^{(\alpha+i\beta)t}, \quad \tilde{y}_2(t) = e^{(\alpha-i\beta)t},$$

are solutions to $y'' + a_1 y' + a_0 y = 0$. 
Idea of the Proof: Recall that the functions
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are solutions to \( y'' + a_1 y' + a_0 y = 0 \). Also recall that
\[ \tilde{y}_1(t) = e^{\alpha t} [\cos(\beta t) + i \sin(\beta t)], \]
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Then the functions

\[ y_1(t) = \frac{1}{2} (\tilde{y}_1(t) + \tilde{y}_2(t)) \]
Review of complex numbers.

Idea of the Proof: Recall that the functions
\[ \tilde{y}_1(t) = e^{(\alpha+i\beta)t}, \quad \tilde{y}_2(t) = e^{(\alpha-i\beta)t}, \]
are solutions to \( y'' + a_1 y' + a_0 y = 0 \). Also recall that
\[ \tilde{y}_1(t) = e^{\alpha t} [\cos(\beta t) + i \sin(\beta t)], \quad \tilde{y}_2(t) = e^{\alpha t} [\cos(\beta t) - i \sin(\beta t)]. \]
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Second order linear homogeneous ODE (Sect. 2.3).

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- **Characteristic polynomial with complex roots.**
  - Two main sets of fundamental solutions.
  - Review of Complex numbers.
    - **A real-valued fundamental and general solutions.**
- Application: The RLC circuit.
A real-valued fundamental and general solutions.

**Example**

Find the real-valued general solution of the equation

$$y'' - 2y' + 6y = 0.$$
A real-valued fundamental and general solutions.

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Solution: Recall: Complex valued solutions are
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The Euler formula and its complex-conjugate formula
\[ e^{i\sqrt{5}t} = \left[ \cos(\sqrt{5} \, t) + i \sin(\sqrt{5} \, t) \right], \]
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The calculation above says that a real-valued fundamental set is
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We just restricted the coefficients \( c_1, c_2 \) to be real-valued.
A real-valued fundamental and general solutions.

Example
Show that $y_1(t) = e^t \cos(\sqrt{5} t)$ and $y_2(t) = e^t \sin(\sqrt{5} t)$ are fundamental solutions to the equation $y'' - 2y' + 6y = 0$.

Solution: $y_1(t) = e^t \cos(\sqrt{5} t), \; y_2(t) = e^t \sin(\sqrt{5} t)$. 

Summary:
▶ These functions are solutions of the differential equation.
▶ They are not proportional to each other, hence they form a fundamental set.
▶ The general solution of the equation is $y(t) = [c_1 \cos(\sqrt{5} t) + c_2 \sin(\sqrt{5} t)] e^t$.
▶ $y$ is real-valued for $c_1, c_2 \in \mathbb{R}$.
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Real-valued fundamental solutions are

$$y_1(t) = e^{-t} \cos(\sqrt{5} t), \quad y_2(t) = e^{-t} \sin(\sqrt{5} t).$$

\triangle
Example
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Differential equations like the one in this example describe physical processes related to damped oscillations. For example, pendulums with friction.
A real-valued fundamental and general solutions.

Example
Find the real-valued general solution of \( y'' + 5y = 0 \).

The characteristic polynomial is \( p(r) = r^2 + 5 \). Its roots are \( r = \pm \sqrt{5}i \). This is the case \( \alpha = 0 \), and \( \beta = \sqrt{5} \). Real-valued fundamental solutions are \( y_1(t) = \cos(\sqrt{5}t) \), \( y_2(t) = \sin(\sqrt{5}t) \). The real-valued general solution is \( y(t) = c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t) \), \( c_1, c_2 \in \mathbb{R} \).

Remark: Equations like the one in this example describe oscillatory physical processes without dissipation, \( \alpha = 0 \).
A real-valued fundamental and general solutions.

Example
Find the real-valued general solution of \( y'' + 5y = 0 \).

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\]
A real-valued fundamental and general solutions.

Example
Find the real-valued general solution of $y'' + 5y = 0$.

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$$y_1(t) = \cos(\sqrt{5} t), \quad y_2(t) = \sin(\sqrt{5} t).$$

The real-valued general solution is

$$y(t) = c_1 \cos(\sqrt{5} t) + c_2 \sin(\sqrt{5} t), \quad c_1, c_2 \in \mathbb{R}.$$
Example
Find the real-valued general solution of $y'' + 5y = 0$.

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Remark: Equations like the one in this example describe oscillatory physical processes without dissipation, $\alpha = 0$. 
Second order linear homogeneous ODE (Sect. 2.3).

- Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - Review of Complex numbers.
  - A real-valued fundamental and general solutions.

- **Application:** The RLC circuit.
Consider an electric circuit with resistance $R$, non-zero capacitor $C$, and non-zero inductance $L$, as in the figure.
Application: The RLC circuit.

Consider an electric circuit with resistance $R$, non-zero capacitor $C$, and non-zero inductance $L$, as in the figure.

The electric current flowing in such circuit satisfies:

$$LL'(t) + RL(t) + \frac{1}{C} \int_{t_0}^{t} I(s) \, ds = 0.$$
Application: The RLC circuit.

Consider an electric circuit with resistance $R$, non-zero capacitor $C$, and non-zero inductance $L$, as in the figure.

The electric current flowing in such circuit satisfies:

$$L I'(t) + R I(t) + \frac{1}{C} \int_{t_0}^{t} I(s) \, ds = 0.$$  

Derivate both sides above:  
$$L I''(t) + R I'(t) + \frac{1}{C} I(t) = 0.$$
Application: The RLC circuit.

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Divide by $L$:

\[ I''(t) + 2 \left( \frac{R}{2L} \right) I'(t) + \frac{1}{LC} I(t) = 0. \]
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Introduce $\alpha = \frac{R}{2L}$ and $\omega = \frac{1}{\sqrt{LC}}$. 
Application: The RLC circuit.

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Derivate both sides above: $L I''(t) + R I'(t) + \frac{1}{C} I(t) = 0$.

Divide by $L$: $I''(t) + 2 \left( \frac{R}{2L} \right) I'(t) + \frac{1}{LC} I(t) = 0$.

Introduce $\alpha = \frac{R}{2L}$ and $\omega = \frac{1}{\sqrt{LC}}$, then $I'' + 2\alpha I' + \omega^2 I = 0$. 
Application: The RLC circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to \( I'' + 2\alpha I' + \omega^2 I = 0 \), where \( \alpha = R/(2L) \), \( \omega^2 = 1/(LC) \), in the cases (a) (b) below.

Solution: The characteristic polynomial is \( p(r) = r^2 + 2\alpha r + \omega^2 \).
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to $l'' + 2\alpha l' + \omega^2 l = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$. The roots are:

$$r_{\pm} = \frac{1}{2} \left[ -2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2} \right]$$

Remark: When the circuit has no resistance, the current oscillates without dissipation.
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = \frac{R}{2L}$, $\omega^2 = \frac{1}{LC}$, in the cases (a) (b) below.

Solution: The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$. The roots are:

$$r_\pm = \frac{1}{2} \left[ -2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2} \right] \Rightarrow r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.$$
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to \( l'' + 2\alpha l' + \omega^2 l = 0 \), where \( \alpha = R/(2L) \), \( \omega^2 = 1/(LC) \), in the cases (a) (b) below.

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\[
r_\pm = \frac{1}{2} \left[ -2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2} \right] \quad \Rightarrow \quad r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.
\]

Case (a) \( R = 0 \).
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

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Case (a) $R = 0$. This implies $\alpha = 0$,
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to \( I'' + 2\alpha I' + \omega^2 I = 0 \), where \( \alpha = \frac{R}{2L}, \ \omega^2 = \frac{1}{LC} \), in the cases (a) (b) below.

Solution: The characteristic polynomial is \( p(r) = r^2 + 2\alpha r + \omega^2 \). The roots are:

\[
r_\pm = \frac{1}{2} [-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}] \Rightarrow r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.
\]

Case (a) \( R = 0 \). This implies \( \alpha = 0 \), so \( r_\pm = \pm i\omega \).
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to \( l'' + 2\alpha l' + \omega^2 l = 0 \), where \( \alpha = R/(2L), \omega^2 = 1/(LC) \), in the cases (a) (b) below.

Solution: The characteristic polynomial is \( p(r) = r^2 + 2\alpha r + \omega^2 \). The roots are:

\[
\begin{align*}
    r_\pm &= \frac{1}{2} \left[ -2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2} \right] \\
    &= -\alpha \pm \sqrt{\alpha^2 - \omega^2}.
\end{align*}
\]

Case (a) \( R = 0 \). This implies \( \alpha = 0 \), so \( r_\pm = \pm i\omega \). Therefore,

\[
l_1(t) = \cos(\omega t),
\]
Application: The RLC circuit.

Example

Find real-valued fundamental solutions to \( I'' + 2\alpha I' + \omega^2 I = 0 \), where \( \alpha = \frac{R}{(2L)} \), \( \omega^2 = \frac{1}{(LC)} \), in the cases (a) (b) below.

Solution: The characteristic polynomial is \( p(r) = r^2 + 2\alpha r + \omega^2 \). The roots are:

\[
    r_{\pm} = \frac{1}{2} \left[ -2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2} \right] \quad \Rightarrow \quad r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.
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Case (a) \( R = 0 \). This implies \( \alpha = 0 \), so \( r_{\pm} = \pm i\omega \). Therefore,

\[
    I_1(t) = \cos(\omega t), \quad I_2(t) = \sin(\omega t).
\]
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to \( l'' + 2\alpha l' + \omega^2 l = 0 \), where \( \alpha = \frac{R}{2L} \), \( \omega^2 = \frac{1}{LC} \), in the cases (a) (b) below.

Solution: The characteristic polynomial is \( p(r) = r^2 + 2\alpha r + \omega^2 \). The roots are:

\[
r_{\pm} = \frac{1}{2} \left[ -2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2} \right] \quad \Rightarrow \quad r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.
\]

Case (a) \( R = 0 \). This implies \( \alpha = 0 \), so \( r_{\pm} = \pm i\omega \). Therefore,

\[
l_1(t) = \cos(\omega t), \quad l_2(t) = \sin(\omega t).
\]

Remark: When the circuit has no resistance, the current oscillates without dissipation.
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Example
Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

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Application: The RLC circuit.

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Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: Recall: $r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$.

Case (b) $R < \sqrt{4L/C}$.
Application: The RLC circuit.

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$$R^2 < \frac{4L}{C}$$
Application: The RLC circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: Recall: $r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$.

Case (b) $R < \sqrt{4L/C}$. This implies

$$R^2 < \frac{4L}{C} \quad \Leftrightarrow \quad \frac{R^2}{4L^2} < \frac{1}{LC}$$
Application: The RLC circuit.

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Find real-valued fundamental solutions to \( l'' + 2\alpha l' + \omega^2 l = 0 \), where \( \alpha = R/(2L) \), \( \omega^2 = 1/(LC) \), in the cases (a) (b) below.

Solution: Recall: \( r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2} \).

Case (b) \( R < \sqrt{4L/C} \). This implies
\[
R^2 < \frac{4L}{C} \iff \frac{R^2}{4L^2} < \frac{1}{LC} \iff \alpha^2 < \omega^2.
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Application: The RLC circuit.

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Therefore, \( r_{\pm} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2} \).
Application: The RLC circuit.

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$$R^2 < \frac{4L}{C} \iff \frac{R^2}{4L^2} < \frac{1}{LC} \iff \alpha^2 < \omega^2.$$ 

Therefore, $r_\pm = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$. The fundamental solutions are

$$I_1(t) = e^{-\alpha t} \cos(\sqrt{\omega^2 - \alpha^2} t),$$
Application: The RLC circuit.

Example

Find real-valued fundamental solutions to \( I'' + 2\alpha I' + \omega^2 I = 0 \), where \( \alpha = R/(2L) \), \( \omega^2 = 1/(LC) \), in the cases (a) (b) below.

Solution: Recall: \( r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2} \).

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\[
R^2 < \frac{4L}{C} \quad \Leftrightarrow \quad \frac{R^2}{4L^2} < \frac{1}{LC} \quad \Leftrightarrow \quad \alpha^2 < \omega^2.
\]

Therefore, \( r_\pm = -\alpha \pm i\sqrt{\omega^2 - \alpha^2} \). The fundamental solutions are

\[
l_1(t) = e^{-\alpha t} \cos(\sqrt{\omega^2 - \alpha^2} t), \quad l_2(t) = e^{-\alpha t} \sin(\sqrt{\omega^2 - \alpha^2} t).
\]
Application: The RLC circuit.

Example
Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: Recall: $r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$.

Case (b) $R < \sqrt{4L/C}$. This implies

$$R^2 < \frac{4L}{C} \iff \frac{R^2}{4L^2} < \frac{1}{LC} \iff \alpha^2 < \omega^2.$$ 

Therefore, $r_{\pm} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$. The fundamental solutions are

$$I_1(t) = e^{-\alpha t} \cos(\sqrt{\omega^2 - \alpha^2} t), \quad I_2(t) = e^{-\alpha t} \sin(\sqrt{\omega^2 - \alpha^2} t).$$
Application: The RLC circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: Recall: $r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$.

Case (b) $R < \sqrt{4L/C}$. This implies

$$R^2 < \frac{4L}{C} \iff \frac{R^2}{4L^2} < \frac{1}{LC} \iff \alpha^2 < \omega^2.$$ 

Therefore, $r_{\pm} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$. The fundamental solutions are

$$I_1(t) = e^{-\alpha t} \cos(\sqrt{\omega^2 - \alpha^2} t), \quad I_2(t) = e^{-\alpha t} \sin(\sqrt{\omega^2 - \alpha^2} t).$$
Application: The RLC circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: Recall: $r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$.

Case (b) $R < \sqrt{4L/C}$. This implies

$$R^2 < \frac{4L}{C} \iff \frac{R^2}{4L^2} < \frac{1}{LC} \iff \alpha^2 < \omega^2.$$

Therefore, $r_{\pm} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$. The fundamental solutions are

$$I_1(t) = e^{-\alpha t} \cos(\sqrt{\omega^2 - \alpha^2} t), \quad I_2(t) = e^{-\alpha t} \sin(\sqrt{\omega^2 - \alpha^2} t).$$

The resistance $R$ damps the current oscillations.
Second order linear homogeneous ODE (Sect. 2.4).

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- Repeated roots as a limit case.
- Main result for repeated roots.
- Reduction of the order method:
  - Constant coefficients equations.
  - Variable coefficients equations.
Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Summary:
Given constants $a_1, a_0 \in \mathbb{R}$, consider the differential equation

$$y'' + a_1 y' + a_0 y = 0$$

with characteristic polynomial having roots

$$r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_0}.$$
Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Summary:
Given constants $a_1, a_0 \in \mathbb{R}$, consider the differential equation

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$$r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_0}.$$

(1) If $a_1^2 - 4a_0 > 0$, 

(2) If $a_1^2 - 4a_0 < 0$, introducing

$$\alpha = -\frac{a_1}{2}, \quad \beta = \frac{1}{2} \sqrt{4a_0 - a_1^2},$$

$$y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).$$

(3) If $a_1^2 - 4a_0 = 0$, then

$$y_1(t) = e^{-\frac{a_1}{2} t}.$$
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Summary:
Given constants \( a_1, a_0 \in \mathbb{R} \), consider the differential equation

\[
y'' + a_1 y' + a_0 y = 0
\]

with characteristic polynomial having roots

\[
r_\pm = -\frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_0}.
\]

(1) If \( a_1^2 - 4a_0 > 0 \), then \( y_1(t) = e^{r_+ t} \) and \( y_2(t) = e^{r_- t} \).
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Summary:
Given constants \( a_1, a_0 \in \mathbb{R} \), consider the differential equation
\[
y'' + a_1 y' + a_0 y = 0
\]
with characteristic polynomial having roots
\[
r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_0}.
\]

(1) If \( a_1^2 - 4a_0 > 0 \), then \( y_1(t) = e^{r_+ t} \) and \( y_2(t) = e^{r_- t} \).

(2) If \( a_1^2 - 4a_0 < 0 \),
Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

**Summary:**
Given constants $a_1, a_0 \in \mathbb{R}$, consider the differential equation

$$y'' + a_1 y' + a_0 y = 0$$

with characteristic polynomial having roots

$$r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_0}.$$

**1.** If $a_1^2 - 4a_0 > 0$, then $y_1(t) = e^{r_+ t}$ and $y_2(t) = e^{r_- t}$.

**2.** If $a_1^2 - 4a_0 < 0$, then introducing $\alpha = -\frac{a_1}{2}, \beta = \frac{1}{2} \sqrt{4a_0 - a_1^2}$,
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Summary:
Given constants \( a_1, a_0 \in \mathbb{R} \), consider the differential equation
\[
y'' + a_1 y' + a_0 y = 0
\]
with characteristic polynomial having roots
\[
r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_0}.
\]

(1) If \( a_1^2 - 4a_0 > 0 \), then \( y_1(t) = e^{r_+ t} \) and \( y_2(t) = e^{r_- t} \).

(2) If \( a_1^2 - 4a_0 < 0 \), then introducing \( \alpha = -\frac{a_1}{2} \), \( \beta = \frac{1}{2} \sqrt{4a_0 - a_1^2} \),
\[
y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).
\]
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0. \)

Summary:
Given constants \( a_1, a_0 \in \mathbb{R} \), consider the differential equation
\[
y'' + a_1 y' + a_0 y = 0
\]
with characteristic polynomial having roots
\[
r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_0}.
\]

(1) If \( a_1^2 - 4a_0 > 0 \), then \( y_1(t) = e^{r_+ t} \) and \( y_2(t) = e^{r_- t} \).

(2) If \( a_1^2 - 4a_0 < 0 \), then introducing \( \alpha = -\frac{a_1}{2}, \beta = \frac{1}{2} \sqrt{4a_0 - a_1^2} \),
\[
y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).
\]

(3) If \( a_1^2 - 4a_0 = 0 \),
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Summary:
Given constants \( a_1, a_0 \in \mathbb{R} \), consider the differential equation

\[
y'' + a_1 y' + a_0 y = 0
\]

with characteristic polynomial having roots

\[
r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_0}.
\]

(1) If \( a_1^2 - 4a_0 > 0 \), then \( y_1(t) = e^{r_+ t} \) and \( y_2(t) = e^{r_- t} \).

(2) If \( a_1^2 - 4a_0 < 0 \), then introducing \( \alpha = -\frac{a_1}{2} \), \( \beta = \frac{1}{2} \sqrt{4a_0 - a_1^2} \),

\[
y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).
\]

(3) If \( a_1^2 - 4a_0 = 0 \), then \( y_1(t) = e^{-\frac{a_1}{2} t} \).
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Question:
Consider the case (3), with \( a_1^2 - 4a_0 = 0 \), that is, \( a_0 = \frac{a_1^2}{4} \).
Question:

Consider the case (3), with $a_1^2 - 4a_0 = 0$, that is, $a_0 = \frac{a_1^2}{4}$.

Does the equation

$$y'' + a_1 y' + \frac{a_1^2}{4} y = 0$$

have two linearly independent solutions?
Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).

Question:
Consider the case (3), with \( a_1^2 - 4a_0 = 0 \), that is, \( a_0 = \frac{a_1^2}{4} \).

- Does the equation
  \[
y'' + a_1 y' + \frac{a_1^2}{4} y = 0
  \]
  have two linearly independent solutions?

- Or, is every solution to the equation above proportional to
  \[
y_1(t) = e^{-\frac{a_1}{2} t}
  \]
Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Repeated roots as a limit case.

Main result for repeated roots.

Reduction of the order method:

- Constant coefficients equations.
- Variable coefficients equations.
Repeated roots as a limit case.

Remark:

- Case (3), where $4a_0 - a_1^2 = 0$ can be obtained as the limit $\beta \to 0$ in case (2).
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- Case (3), where \( 4a_0 - a_1^2 = 0 \) can be obtained as the limit \( \beta \to 0 \) in case (2).
- Let us study the solutions of the differential equation in the case (2) as \( \beta \to 0 \) for fixed \( t \).
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Remark:

- Case (3), where \(4a_0 - a_1^2 = 0\) can be obtained as the limit \(\beta \to 0\) in case (2).
- Let us study the solutions of the differential equation in the case (2) as \(\beta \to 0\) for fixed \(t\).
- Since \(\cos(\beta t) \to 1\) as \(\beta \to 0\),
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- Case (3), where $4a_0 - a_1^2 = 0$ can be obtained as the limit $\beta \to 0$ in case (2).
- Let us study the solutions of the differential equation in the case (2) as $\beta \to 0$ for fixed $t$.
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  $$y_{1\beta}(t) = e^{-\frac{a_1}{2}t} \cos(\beta t)$$
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$$y_{1\beta}(t) = e^{-\frac{a_1}{2} t} \cos(\beta t) \to e^{-\frac{a_1}{2} t} = y_1(t).$$
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  \[ y_{2\beta}(t) = e^{-\frac{a_1}{2} t} \sin(\beta t). \]
Repeated roots as a limit case.

Remark:

► Case (3), where \(4a_0 - a_1^2 = 0\) can be obtained as the limit \(\beta \to 0\) in case (2).

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y_{1\beta}(t) = e^{-\frac{a_1}{2} t} \cos(\beta t) \to e^{-\frac{a_1}{2} t} = y_1(t).
\]

► Since \(\frac{\sin(\beta t)}{\beta t} \to 1\) as \(\beta \to 0\), that is, \(\sin(\beta t) \to \beta t\),

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- Is $y_2(t) = t y_1(t)$ solution of the differential equation?
Repeated roots as a limit case.

Remark:

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- Is \(y_2(t) = ty_1(t)\) solution of the differential equation? Introducing \(y_2\) in the differential equation one obtains: Yes.
Repeated roots as a limit case.

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  \]
- Is \(y_2(t) = t \, y_1(t)\) solution of the differential equation? Introducing \(y_2\) in the differential equation one obtains: Yes.
- Since \(y_2\) is not proportional to \(y_1\), the functions \(y_1, y_2\) are a fundamental set for the differential equation in case (3).
Second order linear homogeneous ODE (Sect. 2.4).

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- Repeated roots as a limit case.
- **Main result for repeated roots.**
- Reduction of the order method:
  - Constant coefficients equations.
  - Variable coefficients equations.
Main result for repeated roots.

Theorem

If \( a_1, a_0 \in \mathbb{R} \) satisfy that \( a_1^2 = 4a_0 \), then the functions

\[
y_1(t) = e^{-\frac{a_1}{2} t}, \quad y_2(t) = t e^{-\frac{a_1}{2} t},
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are a fundamental solution set for the differential equation

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Main result for repeated roots.

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If $a_1, a_0 \in R$ satisfy that $a_1^2 = 4a_0$, then the functions

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are a fundamental solution set for the differential equation

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**Example**

Find the general solution of $9y'' + 6y' + y = 0$. 

Solution:

The characteristic equation is $9r^2 + 6r + 1 = 0$, so

$$r \pm = \frac{-6 \pm \sqrt{36 - 36}}{2} \Rightarrow r \pm = \frac{-1}{3}.$$ 

The Theorem above implies that the general solution is

$$y(t) = c_1 e^{-t/3} + c_2 t e^{-t/3}.$$ 

$\triangleright$
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Find the general solution of \(9y'' + 6y' + y = 0\).

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r_{\pm} = \frac{1}{(2)(9)} \left[-6 \pm \sqrt{36 - 36}\right]
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- Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.
- Repeated roots as a limit case.
- Main result for repeated roots.
- **Reduction of the order method:**
  - Constant coefficients equations.
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Reduction of the order method: Constant coefficients.

Proof case $a_1^2 - 4a_0 = 0$:
Recall: The characteristic equation is $r^2 + a_1 r + a_0 = 0$,
Reduction of the order method: Constant coefficients.

Proof case $a_1^2 - 4a_0 = 0$: 
Recall: The characteristic equation is $r^2 + a_1 r + a_0 = 0$, and its solutions are $r_{\pm} = (1/2) \left[ -a_1 \pm \sqrt{a_1^2 - 4a_0} \right]$. 

A second solution $y_2(t)$ not proportional to $y_1(t)$ can be found as follows: (D'Alembert ∼ 1750.) 
Express: $y_2(t) = v(t)y_1(t)$, and find the equation that function $v$ satisfies from the condition $y_2'' + a_1 y_2' + a_0 y_2 = 0$. 


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The hypothesis $a_1^2 = 4a_0$ implies $r_+ = r_- = -a_1/2$. 
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Proof case \( a_1^2 - 4a_0 = 0 \):
Recall: The characteristic equation is \( r^2 + a_1 r + a_0 = 0 \), and its solutions are \( r_\pm = (1/2)\left[-a_1 \pm \sqrt{a_1^2 - 4a_0}\right] \).

The hypothesis \( a_1^2 = 4a_0 \) implies \( r_+ = r_- = -a_1/2 \).

So, the solution \( r_+ \) of the characteristic equation satisfies both

\[
    r_+^2 + a_1 r_+ + a_0 = 0, \quad 2r_+ + a_1 = 0.
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Reduction of the order method: Constant coefficients.

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It is clear that $y_1(t) = e^{r_+ t}$ is solutions of the differential equation.
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A second solution $y_2$ not proportional to $y_1$ can be found as follows: (D’Alembert $\sim$ 1750.)
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It is clear that $y_1(t) = e^{r_+ t}$ is solutions of the differential equation.

A second solution $y_2$ not proportional to $y_1$ can be found as follows: (D’Alembert ~ 1750.)

Express: $y_2(t) = \nu(t) y_1(t)$, and find the equation that function $\nu$ satisfies from the condition $y_2'' + a_1 y_2' + a_0 y_2 = 0$. 
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = vy_1 \) and \( y_2'' + a_1 y_2' + a_0 y_2 = 0 \).
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = vy_1 \) and \( y''_2 + a_1y'_2 + a_0y_2 = 0 \). So, \( y_2 = ve^{r_1t} \)
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = vy_1 \) and \( y_2'' + a_1y_2' + a_0y_2 = 0 \). So, \( y_2 = ve^{r_+t} \) and

\[
y_2' = v'e^{r_+t} + r_+ve^{r_+t},
\]

Recall that \( r_+ \) satisfies:

\[
r_+^2 + a_1r_+ + a_0 = 0 \quad \text{and} \quad 2r_+ + a_1 = 0.
\]

\[
y_2'' = 0 \quad \Rightarrow \quad y_2 = \left( c_1 + c_2t \right)e^{r_+t}.
\]
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = vy_1 \) and \( y_2'' + a_1y_2' + a_0y_2 = 0 \). So, \( y_2 = ve^{r_+ t} \) and

\[
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Recall: $y_2 = vy_1$ and $y''_2 + a_1 y'_2 + a_0 y_2 = 0$. So, $y_2 = ve^{r_+ t}$ and

$$y'_2 = v' e^{r_+ t} + r_+ v e^{r_+ t}, \quad y''_2 = v'' e^{r_+ t} + 2r_+ v' e^{r_+ t} + r_+^2 v e^{r_+ t}.$$

Introducing this information into the differential equation

$$[v'' + 2r_+ v' + r_+^2 v] e^{r_+ t} + a_1 [v' + r_+ v] e^{r_+ t} + a_0 v e^{r_+ t} = 0.$$
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = v y_1 \) and \( y_2'' + a_1 y_2' + a_0 y_2 = 0 \). So, \( y_2 = ve^{r_+ t} \) and

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\]

Introducing this information into the differential equation

\[
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\]

\[
\left[ v'' + 2r_+ v' + r_+^2 v \right] + a_1 \left[ v' + r_+ v \right] + a_0 v = 0
\]

\[
v'' + (2r_+ + a_1) v' + \left( r_+^2 + a_1 r_+ + a_0 \right) v = 0
\]
Reduction of the order method: Constant coefficients.

Recall: $y_2 = vy_1$ and $y''_2 + a_1y'_2 + a_0y_2 = 0$. So, $y_2 = ve^{r_+t}$ and

$$y'_2 = v'e^{r_+t} + r_+ve^{r_+t}, \quad y''_2 = v''e^{r_+t} + 2r_+v'e^{r_+t} + r_+^2ve^{r_+t}.$$

Introducing this information into the differential equation

$$[v'' + 2r_+v' + r_+^2v]e^{r_+t} + a_1[v' + r_+v]e^{r_+t} + a_0ve^{r_+t} = 0.$$  

$$[v'' + 2r_+v' + r_+^2v] + a_1[v' + r_+v] + a_0v = 0$$

$$v'' + (2r_+ + a_1)v' + (r_+^2 + a_1r_+ + a_0)v = 0$$

Recall that $r_+$ satisfies: $r_+^2 + a_1r_+ + a_0 = 0$
**Reduction of the order method: Constant coefficients.**

Recall: \( y_2 = vy_1 \) and \( y_2'' + a_1 y_2' + a_0 y_2 = 0 \). So, \( y_2 = ve^{r_+ t} \) and

\[
y_2' = v' e^{r_+ t} + r_+ ve^{r_+ t}, \quad y_2'' = v'' e^{r_+ t} + 2r_+ v' e^{r_+ t} + r_+^2 ve^{r_+ t}.
\]

Introducing this information into the differential equation

\[
\left[ v'' + 2r_+ v' + r_+^2 v \right] e^{r_+ t} + a_1 \left[ v' + r_+ v \right] e^{r_+ t} + a_0 v e^{r_+ t} = 0.
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Recall that \( r_+ \) satisfies: \( r_+^2 + a_1 r_+ + a_0 = 0 \) and \( 2r_+ + a_1 = 0 \).
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = v y_1 \) and \( y''_2 + a_1 y'_2 + a_0 y_2 = 0 \). So, \( y_2 = v e^{r_+ t} \) and

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y'_2 = v' e^{r_+ t} + r_+ v e^{r_+ t}, \quad y''_2 = v'' e^{r_+ t} + 2 r_+ v' e^{r_+ t} + r_+^2 v e^{r_+ t}.
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v'' + (2 r_+ + a_1) v' + (r_+^2 + a_1 r_+ + a_0) v = 0
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Recall that \( r_+ \) satisfies: \( r_+^2 + a_1 r_+ + a_0 = 0 \) and \( 2 r_+ + a_1 = 0 \).

\( v'' = 0 \)
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = vy_1 \) and \( y_2'' + a_1 y_2' + a_0 y_2 = 0 \). So, \( y_2 = ve^{r_+ t} \) and

\[
y_2' = v' e^{r_+ t} + r_+ ve^{r_+ t}, \quad y_2'' = v'' e^{r_+ t} + 2r_+ v' e^{r_+ t} + r_+^2 ve^{r_+ t}.
\]

Introducing this information into the differential equation

\[
\left[ v'' + 2r_+ v' + r_+^2 v \right] e^{r_+ t} + a_1 \left[ v' + r_+ v \right] e^{r_+ t} + a_0 v e^{r_+ t} = 0.
\]

\[
\left[ v'' + 2r_+ v' + r_+^2 v \right] + a_1 \left[ v' + r_+ v \right] + a_0 v = 0
\]

\[
v'' + (2r_+ + a_1) v' + (r_+^2 + a_1 r_+ + a_0) v = 0
\]

Recall that \( r_+ \) satisfies: \( r_+^2 + a_1 r_+ + a_0 = 0 \) and \( 2r_+ + a_1 = 0 \).

\[
v'' = 0 \quad \Rightarrow \quad v = (c_1 + c_2 t)
\]
Reduction of the order method: Constant coefficients.

Recall: \( y_2 = vy_1 \) and \( y_2'' + a_1 y_2' + a_0 y_2 = 0 \). So, \( y_2 = ve^{r_+ t} \) and

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Introducing this information into the differential equation

\[
\left[ v'' + 2r_+ v' + r_+^2 v \right] e^{r_+ t} + a_1 \left[ v' + r_+ v \right] e^{r_+ t} + a_0 v e^{r_+ t} = 0.
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v'' + (2r_+ + a_1) v' + (r_+^2 + a_1 r_+ + a_0) v = 0
\]

Recall that \( r_+ \) satisfies: \( r_+^2 + a_1 r_+ + a_0 = 0 \) and \( 2r_+ + a_1 = 0 \).

\[
v'' = 0 \quad \Rightarrow \quad v = (c_1 + c_2 t) \quad \Rightarrow \quad y_2 = (c_1 + c_2 t) e^{r_+ t}.
\]
Reduction of the order method: Constant coefficients.

Recall: We have obtained that \( y_2(t) = (c_1 + c_2 t) e^{rt} \).
Reduction of the order method: Constant coefficients.

Recall: We have obtained that $y_2(t) = (c_1 + c_2 t) e^{r_t}$.

If $c_2 = 0$, then $y_2 = c_1 e^{r_t}$ and $y_1 = e^{r_t}$ are linearly dependent functions.
Reduction of the order method: Constant coefficients.

Recall: We have obtained that $y_2(t) = (c_1 + c_2 t) e^{r_+ t}$.

If $c_2 = 0$, then $y_2 = c_1 e^{r_+ t}$ and $y_1 = e^{r_+ t}$ are linearly dependent functions.

If $c_2 \neq 0$, then $y_2 = (c_1 + c_2 t) e^{r_+ t}$ and $y_1 = e^{r_+ t}$ are linearly independent functions.
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Recall: We have obtained that $y_2(t) = (c_1 + c_2 t) e^{r+t}$.

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Simplest choice: $c_1 = 0$ and $c_2 = 1$. 
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Simplest choice: \( c_1 = 0 \) and \( c_2 = 1 \). Then, a fundamental solution set to the differential equation is

\[
\begin{align*}
y_1(t) &= e^{r+t}, \\
y_2(t) &= t e^{r+t}
\end{align*}
\]
Reduction of the order method: Constant coefficients.

Recall: We have obtained that $y_2(t) = (c_1 + c_2 t) e^{r_1 t}$.

If $c_2 = 0$, then $y_2 = c_1 e^{r_1 t}$ and $y_1 = e^{r_1 t}$ are linearly dependent functions.

If $c_2 \neq 0$, then $y_2 = (c_1 + c_2 t) e^{r_1 t}$ and $y_1 = e^{r_1 t}$ are linearly independent functions.

Simplest choice: $c_1 = 0$ and $c_2 = 1$. Then, a fundamental solution set to the differential equation is

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = t e^{r_1 t}$$

The general solution to the differential equation is

$$y(t) = \tilde{c}_1 e^{r_1 t} + \tilde{c}_2 t e^{r_1 t}.$$
Reduction of the order method: Constant coefficients.

Example
Find the solution to the initial value problem

\[ 9y'' + 6y' + y = 0, \quad y(0) = 1, \quad y'(0) = \frac{5}{3}. \]
Reduction of the order method: Constant coefficients.

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Find the solution to the initial value problem

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Solution: The solutions of \( 9r^2 + 6r + 1 = 0 \), are \( r_+ = r_- = -\frac{1}{3} \).
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The Theorem above says that the general solution is

\[ y(t) = c_1 e^{-t/3} + c_2 te^{-t/3} \]
Reduction of the order method: Constant coefficients.

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y(t) = c_1 e^{-t/3} + c_2 te^{-t/3} \quad \Rightarrow \quad y'(t) = -\frac{c_1}{3} e^{-t/3} + c_2 \left(1 - \frac{t}{3}\right) e^{-t/3}.
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The initial conditions imply that

\[ 1 = y(0) \]
Reduction of the order method: Constant coefficients.

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\[1 = y(0) = c_1,\]
Reduction of the order method: Constant coefficients.

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Find the solution to the initial value problem

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\[ \frac{5}{3} = y'(0) \]
Reduction of the order method: Constant coefficients.

Example

Find the solution to the initial value problem

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\[
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\frac{5}{3} = y'(0) = -\frac{c_1}{3} + c_2
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\begin{align*}
1 &= y(0) = c_1, \\
\frac{5}{3} &= y'(0) = -\frac{c_1}{3} + c_2
\end{align*}
\]

\[ \Rightarrow \quad c_1 = 1, \quad c_2 = 2. \]
Reduction of the order method: Constant coefficients.

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Find the solution to the initial value problem

\[ 9y'' + 6y' + y = 0, \quad y(0) = 1, \quad y'(0) = \frac{5}{3}. \]

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The Theorem above says that the general solution is

\[ y(t) = c_1 e^{-t/3} + c_2 t e^{-t/3} \quad \Rightarrow \quad y'(t) = -\frac{c_1}{3} e^{-t/3} + c_2 \left(1 - \frac{t}{3}\right) e^{-t/3}. \]

The initial conditions imply that

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\frac{5}{3} &= y'(0) = -\frac{c_1}{3} + c_2
\end{aligned}
\quad \Rightarrow \quad c_1 = 1, \quad c_2 = 2.
\]

We conclude that \( y(t) = (1 + 2t) e^{-t/3} \). \( \triangleright \)
Second order linear homogeneous ODE (Sect. 2.4).

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- Repeated roots as a limit case.
- Main result for repeated roots.
- **Reduction of the order method:**
  - Constant coefficients equations.
  - **Variable coefficients equations.**
Reduction of the order method: Variable coefficients.

Remark: The same idea used to prove the constant coefficients Theorem above can be used in variable coefficients equations.
Reduction of the order method: Variable coefficients.

**Remark:** The same idea used to prove the constant coefficients Theorem above can be used in variable coefficients equations.

**Theorem**

*Given continuous functions* $p, q : (t_1, t_2) \rightarrow \mathbb{R}$, *let* $y_1 : (t_1, t_2) \rightarrow \mathbb{R}$ *be a solution of*

$$y'' + p(t) y' + q(t) y = 0,$$

*If the function* $v : (t_1, t_2) \rightarrow \mathbb{R}$ *is solution of*

$$y_1(t) v'' + [2y'(t) + p(t)y_1(t)] v' = 0. \quad (3)$$

*then the functions* $y_1$ *and* $y_2 = v y_1$ *are fundamental solutions to the differential equation above.*
Reduction of the order method: Variable coefficients.

**Remark:** The same idea used to prove the constant coefficients Theorem above can be used in variable coefficients equations.

**Theorem**

Given continuous functions $p, q : (t_1, t_2) \to \mathbb{R}$, let $y_1 : (t_1, t_2) \to \mathbb{R}$ be a solution of

$$y'' + p(t) y' + q(t) y = 0,$$

If the function $v : (t_1, t_2) \to \mathbb{R}$ is solution of

$$y_1(t) v'' + [2y'(t) + p(t)y_1(t)] v' = 0. \quad (3)$$

then the functions $y_1$ and $y_2 = v y_1$ are fundamental solutions to the differential equation above.

**Remark:** The reason for the name Reduction of order method is that the function $v$ does not appear in Eq. (3). This is a first order equation in $v'$. 
Reduction of the order method: Variable coefficients.

Example

Find a fundamental set of solutions to

\[ t^2 y'' + 2ty' - 2y = 0, \]

knowing that \( y_1(t) = t \) is a solution.
Reduction of the order method: Variable coefficients.

Example
Find a fundamental set of solutions to

\[ t^2 y'' + 2ty' - 2y = 0, \]

knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = v(t) y_1(t) \).
Example
Find a fundamental set of solutions to

\[ t^2 y'' + 2ty' - 2y = 0, \]

knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = \nu(t)y_1(t) \). The equation for \( \nu \) comes from \( t^2y_2'' + 2ty_2' - 2y_2 = 0. \)
Example
Find a fundamental set of solutions to
\[ t^2 y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = v(t) y_1(t) \). The equation for \( v \) comes from \( t^2 y_2'' + 2ty_2' - 2y_2 = 0 \). We need to compute
\[ y_2 = v \cdot t, \]
Example
Find a fundamental set of solutions to
\[ t^2 y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = \nu(t) y_1(t) \). The equation for \( \nu \) comes from \( t^2 y_2'' + 2ty_2' - 2y_2 = 0 \). We need to compute
\[ y_2 = \nu t, \quad y_2' = t \nu' + \nu, \]
Example
Find a fundamental set of solutions to
\[ t^2 y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = v(t)y_1(t) \). The equation for \( v \) comes from \( t^2 y_2'' + 2ty_2' - 2y_2 = 0 \). We need to compute
\[
\begin{align*}
y_2 &= v \cdot t, \\
y_2' &= t \cdot v' + v, \\
y_2'' &= t \cdot v'' + 2v'.
\end{align*}
\]
Example
Find a fundamental set of solutions to
\[ t^2 y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = v(t) y_1(t) \). The equation for \( v \) comes from \( t^2 y''_2 + 2ty'_2 - 2y_2 = 0 \). We need to compute
\[
\begin{align*}
y_2 &= v t, \\
y'_2 &= t v' + v, \\
y''_2 &= t v'' + 2v'.
\end{align*}
\]
So, the equation for \( v \) is given by
\[
t^2 \left(t v'' + 2v'\right) + 2t \left(t v' + v\right) - 2t v = 0
\]
Reduction of the order method: Variable coefficients.

Example
Find a fundamental set of solutions to
\[ t^2 y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = v(t) y_1(t) \). The equation for \( v \) comes from \( t^2 y_2'' + 2ty_2' - 2y_2 = 0 \). We need to compute
\[
\begin{align*}
 y_2 &= vt, & y_2' &= tv' + v, & y_2'' &= tv'' + 2v'. \\
\end{align*}
\]
So, the equation for \( v \) is given by
\[
\begin{align*}
 t^2 (tv'' + 2v') + 2t (tv' + v) - 2tv &= 0 \\
 t^3 v'' + (2t^2 + 2t^2) v' + (2t - 2t) v &= 0
\end{align*}
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Example
Find a fundamental set of solutions to
\[ t^2 y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

Solution: Express \( y_2(t) = v(t) y_1(t) \). The equation for \( v \) comes from \( t^2 y_2'' + 2ty_2' - 2y_2 = 0 \). We need to compute
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y_2 &= v t, \\
y_2' &= t v' + v, \\
y_2'' &= t v'' + 2v'.
\end{align*}
\]
So, the equation for \( v \) is given by
\[
\begin{align*}
t^2 (t v'' + 2v') + 2t (t v' + v) - 2t v &= 0 \\
t^3 v'' + (2t^2 + 2t^2) v' + (2t - 2t) v &= 0 \\
t^3 v'' + (4t^2) v' &= 0
\end{align*}
\]
Reduction of the order method: Variable coefficients.

**Example**
Find a fundamental set of solutions to
\[ t^2 y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

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y_2' &= t v' + v, \\
y_2'' &= t v'' + 2v'.
\end{align*}
\]
So, the equation for \( v \) is given by
\[
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t^2(t v'' + 2v') + 2t(t v' + v) - 2t v &= 0 \\
t^3 v'' + (2t^2 + 2t^2) v' + (2t - 2t) v &= 0 \\
t^3 v'' + (4t^2) v' &= 0 \\
&\Rightarrow v'' + \frac{4}{t} v' = 0.
\end{align*}
\]
Reduction of the order method: Variable coefficients.

Example
Find a fundamental set of solutions to

\[ t^2 y'' + 2ty' - 2y = 0, \]

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Find a fundamental set of solutions to
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knowing that \( y_1(t) = t \) is a solution.

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This is a first order equation for \( w = v' \),
Reduction of the order method: Variable coefficients.

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Find a fundamental set of solutions to
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Solution: Recall: \( v'' + \frac{4}{t}v' = 0 \).

This is a first order equation for \( w = v' \), given by \( w' + \frac{4}{t}w = 0 \),
Reduction of the order method: Variable coefficients.

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This is a first order equation for \( w = v' \), given by \( w' + \frac{4}{t}w = 0 \), so

\[
\frac{w'}{w} = -\frac{4}{t}
\]
Reduction of the order method: Variable coefficients.

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\]
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\frac{w'}{w} = -\frac{4}{t} \quad \Rightarrow \quad \ln(w) = -4\ln(t) + c_0 \quad \Rightarrow \quad w(t) = c_1 t^{-4}, \quad c_1 \in \mathbb{R}.
\]
**Reduction of the order method: Variable coefficients.**

**Example**

Find a fundamental set of solutions to

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\]

Integrating \( w \) we obtain \( v \),

\[
\int w(t) \, dt = \int c_1 t^{-4} \, dt = c_2 \int t^{-4} \, dt = c_2 \left( -\frac{t^{-3}}{3} \right) + c_3,
\]

where \( c_2 \) and \( c_3 \) are constants.
Reduction of the order method: Variable coefficients.

Example

Find a fundamental set of solutions to

\[ t^2 y'' + 2ty' - 2y = 0, \]

knowing that \( y_1(t) = t \) is a solution.

Solution: Recall: \( v'' + \frac{4}{t}v' = 0 \).

This is a first order equation for \( w = v' \), given by \( w' + \frac{4}{t}w = 0 \), so

\[ \frac{w'}{w} = -\frac{4}{t} \implies \ln(w) = -4\ln(t) + c_0 \implies w(t) = c_1 t^{-4}, \ c_1 \in \mathbb{R}. \]

Integrating \( w \) we obtain \( v \), that is, \( v = c_2 t^{-3} + c_3 \), with \( c_2, c_3 \in \mathbb{R} \).
Reduction of the order method: Variable coefficients.

Example

Find a fundamental set of solutions to
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\frac{w'}{w} = -\frac{4}{t} \quad \Rightarrow \quad \ln(w) = -4 \ln(t) + c_0 \quad \Rightarrow \quad w(t) = c_1 t^{-4}, \quad c_1 \in \mathbb{R}.
\]

Integrating \( w \) we obtain \( v \), that is, \( v = c_2 t^{-3} + c_3 \), with \( c_2, c_3 \in \mathbb{R} \).

Recalling that \( y_2 = t \cdot v \).
Reduction of the order method: Variable coefficients.

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Find a fundamental set of solutions to
\[ t^2 y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

Solution: Recall: \( v'' + \frac{4}{t}v' = 0. \)

This is a first order equation for \( w = v' \), given by \( w' + \frac{4}{t}w = 0 \), so
\[
\frac{w'}{w} = -\frac{4}{t} \quad \Rightarrow \quad \ln(w) = -4\ln(t) + c_0 \quad \Rightarrow \quad w(t) = c_1 t^{-4}, \quad c_1 \in \mathbb{R}.
\]

Integrating \( w \) we obtain \( v \), that is, \( v = c_2 t^{-3} + c_3 \), with \( c_2, c_3 \in \mathbb{R} \).

Recalling that \( y_2 = t \cdot v \) we then conclude that \( y_2 = c_2 t^{-2} + c_3 t \).
Reduction of the order method: Variable coefficients.

Example
Find a fundamental set of solutions to
\[ t^2 y'' + 2ty' - 2y = 0, \]
knowing that \( y_1(t) = t \) is a solution.

Solution: Recall:\n\[ v'' + \frac{4}{t}v' = 0. \]
This is a first order equation for \( w = v' \), given by \( w' + \frac{4}{t}w = 0 \), so
\[ \frac{w'}{w} = -\frac{4}{t} \quad \Rightarrow \quad \ln(w) = -4\ln(t) + c_0 \quad \Rightarrow \quad w(t) = c_1 t^{-4}, \quad c_1 \in \mathbb{R}. \]
Integrating \( w \) we obtain \( v \), that is, \( v = c_2 t^{-3} + c_3 \), with \( c_2, c_3 \in \mathbb{R} \). Recalling that \( y_2 = t \ v \) we then conclude that \( y_2 = c_2 t^{-2} + c_3 t \).
Choosing \( c_2 = 1 \) and \( c_3 = 0 \) we obtain the fundamental solutions \( y_1(t) = t \) and \( y_2(t) = \frac{1}{t^2} \). \( \triangleq \)
Reduction of the order method: Variable coefficients.

Proof of the Theorem: The choice of $y_2 = vy_1$ implies

$$y_2' = v' y_1 + v y_1', \quad y_2'' = v'' y_1 + 2v' y_1' + v y_1''.$$
Reduction of the order method: Variable coefficients.

**Proof of the Theorem:** The choice of \( y_2 = vy_1 \) implies

\[
y'_2 = v' y_1 + v y'_1,
\]

\[
y''_2 = v'' y_1 + 2v' y'_1 + v y''_1.
\]

This information introduced into the differential equation says that

\[
(v'' y_1 + 2v' y'_1 + v y''_1) + p (v' y_1 + v y'_1) + qv y_1 = 0
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y_1 v'' + (2y_1' + p y_1) v' + (y_1'' + p y_1' + q y_1) v = 0.
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The function $y_1$ is solution of $y''_1 + p y'_1 + q y_1 = 0$. 

Reduction of the order method: Variable coefficients.

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The function $y_1$ is solution of $y_1'' + p y_1' + q y_1 = 0$. Then, the equation for $v$ is given by Eq. (3), that is,

$$y_1 v'' + (2y_1' + p y_1) v' = 0.$$
Reduction of the order method: Variable coefficients.

Proof: Recall $y_1 v'' + (2y'_1 + p y_1) v' = 0$. 

We now need to show that $y_1$ and $y_2 = vy_1$ are linearly independent. 

We obtain $Wy_1 y_2 = v' y_2$. 

We need to find $v'$. 

Denote $w = v'$, so $y_1 w'' + (2y'_1 + p y_1) w' = 0 \Rightarrow w''w = -2y'_1 y_1 - p$. 

Let $P$ be a primitive of $p$, that is, $P'(t) = p(t)$, then $\ln(|w|) = -2 \ln(y_1) - P \Rightarrow w = y_1 e^{-2 \ln(y_1) - P} \Rightarrow w = y_1 e^{-P}$. 

We obtain $v' y_2 = e^{-P}$, hence $Wy_1 y_2 = e^{-P}$, which is non-zero. 

We conclude that $y_1$ and $y_2 = vy_1$ are linearly independent.
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\[
W_{y_1y_2}
\]
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Proof: Recall $y_1 v'' + (2y'_1 + p y_1) v' = 0$. We now need to show that $y_1$ and $y_2 = vy_1$ are linearly independent.

$$\begin{vmatrix} y_1 & vy_1 \\ y'_1 & (v' y_1 + vy'_1) \end{vmatrix}$$
Reduction of the order method: Variable coefficients.

**Proof:** Recall $y_1 \, v'' + (2y_1' + p \, y_1) \, v' = 0$. We now need to show that $y_1$ and $y_2 = vy_1$ are linearly independent.

$$W_{y_1y_2} = \begin{vmatrix} y_1 & vy_1 \\ y_1' & (v' \, y_1 + vy_1') \end{vmatrix} = y_1(v' \, y_1 + vy_1') - vy_1y_1'.$$
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y_1 \, w' + (2y_1' + p \, y_1) \, w = 0 \quad \Rightarrow \quad \frac{w'}{w} = -2 \frac{y_1'}{y_1} - p.
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