Second order linear ODE (Sect. 2.2).

- Review: Second order linear differential equations.
- Idea: Soving constant coefficients equations.
- The characteristic equation.
- Solution formulas for constant coefficients equations.

Review: Second order linear ODE.

Definition

Given functions a_1, a_0, $b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t)$$

is called a *second order linear* differential equation.

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Theorem (Superposition property)

If the functions y_1 and y_2 are solutions to the homogeneous linear equation

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0,$$

then the linear combination $c_1 y_1(t) + c_2 y_2(t)$ is also a solution for any constants c_1, $c_2 \in \mathbb{R}$.

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- Review: Second order linear differential equations.
- **Idea: Soving constant coefficients equations.**
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Idea: Soving constant coefficients equations.

Remark: Just by trial and error one can find solutions to second order, constant coefficients, homogeneous, linear differential equations.

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Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

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Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: We look for solutions proportional to exponentials e^{rt},

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Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

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Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: We look for solutions proportional to exponentials e^{rt}, for an appropriate constant $r \in \mathbb{R}$, since the exponential can be canceled out from the equation.

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$$(r^2 + 5r + 6)e^{rt} = 0$$

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That is, r must be a root of the polynomial $p(r) = r^2 + 5r + 6$.

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That is, r must be a root of the polynomial $p(r) = r^2 + 5r + 6$.

This polynomial is called the characteristic polynomial of the differential equation.

Idea: Soving constant coefficients equations.

Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: Recall: $p(r) = r^2 + 5r + 6$.

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Example

Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: Recall: $p(r) = r^2 + 5r + 6$.

The roots of the characteristic polynomial are

$$r = \frac{1}{2}\left(-5 \pm \sqrt{25 - 24}\right)$$

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Therefore, we have found two solutions to the ODE,

$$y_1(t) = e^{-2t}, \qquad y_2(t) = e^{-3t}.$$

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Their superposition provides infinitely many solutions,

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

$\triangleleft$

Idea: Soving constant coefficients equations.

Summary: The differential equation $y'' + 5y' + 6y = 0$ has infinitely many solutions,

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- There are two free constants in the solution found above.

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- The ODE above is second order, so two integrations must be done to find the solution. This explain the origin of the two free constant in the solution.

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- There are two free constants in the solution found above.
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- An IVP for a second order differential equation will have a unique solution if the IVP contains two initial conditions.

Second order linear ODE (Sect. 2.2).

- Review: Second order linear differential equations.
- Idea: Soving constant coefficients equations.
- **The characteristic equation.**
- Solution formulas for constant coefficients equations.

The characteristic equation.

Definition

Given a second order linear homogeneous differential equation with constant coefficients

$$y'' + a_1 y' + a_0 = 0, \tag{1}$$

the *characteristic polynomial* and the *characteristic equation* associated with the differential equation in (1) are, respectively,

$$p(r) = r^2 + a_1 r + a_0, \qquad p(r) = 0.$$

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Remark: If r_1, r_2 are the solutions of the characteristic equation and c_1, c_2 are constants, then we will show that the general solution of Eq. (1) is given by

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

The characteristic equation.

Example

Find the solution y of the initial value problem

$$y'' + 5y' + 6 = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

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We now find the constants c_1 and c_2 that satisfy the initial conditions above:

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$$c_1 = 1 - c_2 \Rightarrow 1 = 2(1 - c_2) + 3c_2$$

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Therefore, the unique solution to the initial value problem is

$$y(t) = 2e^{-2t} - e^{-3t}.$$

$\triangleleft$

The characteristic equation.

Example

Find the general solution y of the differential equation

$$2y'' - 3y' + y = 0.$$

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Find the general solution y of the differential equation

$$2y'' - 3y' + y = 0.$$

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Therefore, the general solution of the equation above is

$$y(t) = c_1 e^{t} + c_2 e^{t/2},$$

where c_1, c_2 are arbitrary constants. $\triangleleft$

Second order linear ODE (Sect. 2.2).

- Review: Second order linear differential equations.
- Idea: Soving constant coefficients equations.
- The characteristic equation.
- **Solution formulas for constant coefficients equations.**

Solution formulas for constant coefficients equations.

Theorem (Constant coefficients)

Given real constants a_1, a_0, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \to \mathbb{R}$ given by

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Solution formulas for constant coefficients equations.

Theorem (Constant coefficients)

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(a) *If $r_+ \neq r_-$, real or complex, then $y(t) = c_0\, e^{r_+ t} + c_1\, e^{r_- t}$.*

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Furthermore, given real constants t_0, y_0 and y_1, there is a unique solution to the initial value problem

$$y'' + a_1\, y' + a_0\, y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_1.$$

Second order linear homogeneous ODE (Sect. 2.3).

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - Review of Complex numbers.
  - A real-valued fundamental and general solutions.
- Application: The RLC circuit.

Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.

Theorem (Constant coefficients)

Given real constants a_1, a_0, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \to \mathbb{R}$ given by

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Let r_+, r_- be the roots of the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$, and let c_0, c_1 be arbitrary constants. Then, the general solution y of the differential equation is given by

(a) *If $r_+ \neq r_-$, real or complex, then $y(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}$.*

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$$y'' + a_1\, y' + a_0\, y = 0, \qquad y(t_0) = y_1, \quad y'(t_0) = y_2.$$

Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.

Example

Find the general solution of the equation $y'' - y' - 6y = 0$.

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Example

Find the general solution of the equation $y'' - y' - 6y = 0$.

Solution: Since solutions have the form e^{rt}, we need to find the roots of the characteristic polynomial $p(r) = r^2 - r - 6$,

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Example

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Solution: Since solutions have the form e^{rt}, we need to find the roots of the characteristic polynomial $p(r) = r^2 - r - 6$, that is,

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Second order linear homogeneous ODE (Sect. 2.3).

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- **Characteristic polynomial with complex roots.**
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  - Review of Complex numbers.
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- In other words: It is not simple to see what values of $\tilde{c}_1$ and $\tilde{c}_2$ make the general solution above to be real-valued.
- One way to find the real-valued general solution is to find real-valued fundamental solutions.

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Two main sets of fundamental solutions.

Theorem (Complex roots)

If the constants a_1, $a_0 \in \mathbb{R}$ satisfy that $a_1^2 - 4a_0 < 0$, then the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$ of the equation

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has complex roots $r_+ = \alpha + i\beta$ and $r_- = \alpha - i\beta$, where

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Furthermore, a fundamental set of solutions to Eq. (2) is

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Idea of the Proof: Recall that the functions

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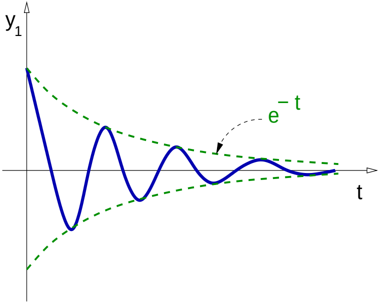

Differential equations like the one in this example describe physical processes related to damped oscillations. For example pendulums with friction.

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Remark: Equations like the one in this example describe oscillatory physical processes without dissipation, $\alpha = 0$.

Second order linear homogeneous ODE (Sect. 2.3).

- Review: On solutions of $y'' + a_1\,y' + a_0\,y = 0$.
- Characteristic polynomial with complex roots.
  - Two main sets of fundamental solutions.
  - Review of Complex numbers.
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- **Application: The RLC circuit.**

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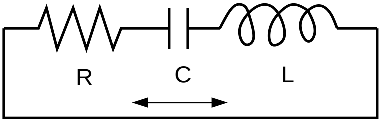
Consider an electric circuit with resistance R, non-zero capacitor C, and non-zero inductance L, as in the figure.



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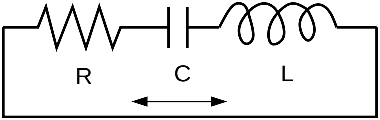

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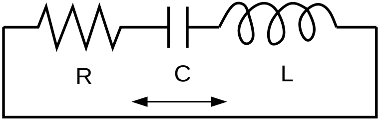

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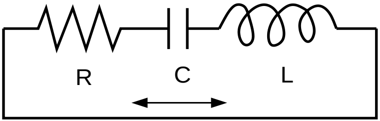

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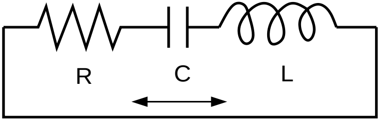
Divide by L: $I''(t) + 2\Big(\frac{R}{2L}\Big)\, I'(t) + \frac{1}{LC}\, I(t) = 0.$

Application: The RLC circuit.

Consider an electric circuit with resistance R, non-zero capacitor C, and non-zero inductance L, as in the figure.



I (t) : electric current.

The electric current flowing in such circuit satisfies:

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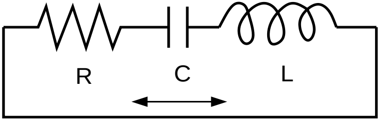

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Introduce $\alpha = \frac{R}{2L}$ and $\omega = \frac{1}{\sqrt{LC}}$, then $I'' + 2\alpha\,I' + \omega^2\,I = 0.$

Application: The RLC circuit.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha\, I' + \omega^2\, I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

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Application: The RLC circuit.

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Remark: When the circuit has no resistance, the current oscillates without dissipation.

Application: The RLC circuit.

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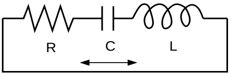

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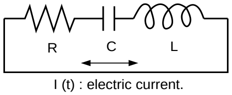

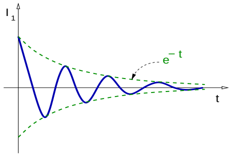

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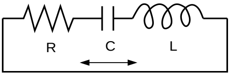

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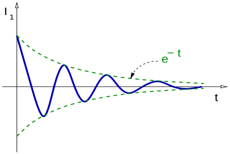

The resistance R damps the current oscillations.

Second order linear homogeneous ODE (Sect. 2.4).

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Repeated roots as a limit case.
- Main result for repeated roots.
- Reduction of the order method:
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Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.

Summary:

Given constants $a_1,\ a_0 \in \mathbb{R}$, consider the differential equation

$$y'' + a_1 y' + a_0 y = 0$$

with characteristic polynomial having roots

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Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Question:

Consider the case (3), with $a_1^2 - 4a_0 = 0$, that is, $a_0 = \frac{a_1^2}{4}$.

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

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Consider the case (3), with $a_1^2 - 4a_0 = 0$, that is, $a_0 = \frac{a_1^2}{4}$.

- Does the equation

$$y'' + a_1 y' + \frac{a_1^2}{4} y = 0$$

have two linearly independent solutions?

Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.

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- Or, is every solution to the equation above proportional to

$$y_1(t) = e^{-\frac{a_1}{2}\, t} \qquad ?$$

Second order linear homogeneous ODE (Sect. 2.4).

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
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Repeated roots as a limit case.

Remark:

- Case (3), where $4a_0 - a_1^2 = 0$ can be obtained as the limit $\beta \to 0$ in case (2).

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- Let us study the solutions of the differential equation in the case (2) as $\beta \to 0$ for fixed t.
- Since $\cos(\beta t) \to 1$ as $\beta \to 0$, we conclude that

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- Is $y_2(t) = t\, y_1(t)$ solution of the differential equation? Introducing y_2 in the differential equation one obtains: Yes.
- Since y_2 is not proportional to y_1, the functions y_1, y_2 are a fundamental set for the differential equation in case (3).

Second order linear homogeneous ODE (Sect. 2.4).

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Repeated roots as a limit case.
- **Main result for repeated roots.**
- Reduction of the order method:
  - Constant coefficients equations.
  - Variable coefficients equations.

Main result for repeated roots.

Theorem

If a_1, $a_0 \in R$ satisfy that $a_1^2 = 4a_0$, then the functions

$$y_1(t) = e^{-\frac{a_1}{2}\,t}, \qquad y_2(t) = t\,e^{-\frac{a_1}{2}\,t},$$

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Find the general solution of $9y'' + 6y' + y = 0$.

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The Theorem above implies that the general solution is

$$y(t) = c_1\, e^{-t/3} + c_2\, te^{-t/3}.$$

◁

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Recall: The characteristic equation is $r^2 + a_1 r + a_0 = 0$,

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So, the solution r_+ of the characteristic equation satisfies both

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Express: $y_2(t) = v(t)\, y_1(t)$, and find the equation that function v satisfies from the condition $y_2'' + a_1 y_2' + a_0 y_2 = 0$.

Reduction of the order method: Constant coefficients.

Recall: $y_2 = v y_1$ and $y_2'' + a_1 y_2' + a_0 y_2 = 0$.

Reduction of the order method: Constant coefficients.

Recall: $y_2 = vy_1$ and $y_2'' + a_1 y_2' + a_0 y_2 = 0$. So, $y_2 = ve^{r_+ t}$

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Recall: $y_2 = v y_1$ and $y_2'' + a_1 y_2' + a_0 y_2 = 0$. So, $y_2 = v e^{r_+ t}$ and

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Introducing this information into the differential equation

$$\left[v'' + 2r_+ v' + r_+^2 v\right] e^{r_+ t} + a_1 \left[v' + r_+ v\right] e^{r_+ t} + a_0 v \, e^{r_+ t} = 0.$$

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$v'' = 0$

Reduction of the order method: Constant coefficients.

Recall: $y_2 = v y_1$ and $y_2'' + a_1 y_2' + a_0 y_2 = 0$. So, $y_2 = v e^{r_+ t}$ and

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Recall that r_+ satisfies: $r_+^2 + a_1 r_+ + a_0 = 0$ and $2 r_+ + a_1 = 0$.

$$v'' = 0 \quad \Rightarrow \quad v = (c_1 + c_2 t)$$

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Recall: $y_2 = v y_1$ and $y_2'' + a_1 y_2' + a_0 y_2 = 0$. So, $y_2 = v e^{r_+ t}$ and

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$$v'' = 0 \quad \Rightarrow \quad v = (c_1 + c_2 t) \quad \Rightarrow \quad y_2 = (c_1 + c_2 t)\, e^{r_+ t}.$$

Reduction of the order method: Constant coefficients.

Recall: We have obtained that $y_2(t) = (c_1 + c_2 t)\, e^{r_+ t}$.

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If $c_2 = 0$, then $y_2 = c_1 e^{r_+ t}$ and $y_1 = e^{r_+ t}$ are linearly dependent functions.

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The general solution to the differential equation is

$$y(t) = \tilde{c}_1\, e^{r_+ t} + \tilde{c}_2\, t e^{r_+ t}.$$

Reduction of the order method: Constant coefficients.

Example

Find the solution to the initial value problem

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We conclude that $y(t) = (1 + 2t)\, e^{-t/3}$. $\triangleleft$

Second order linear homogeneous ODE (Sect. 2.4).

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Repeated roots as a limit case.
- Main result for repeated roots.
- **Reduction of the order method:**
  - Constant coefficients equations.
  - **Variable coefficients equations.**

Reduction of the order method: Variable coefficients.

Remark: The same idea used to prove the constant coefficients Theorem above can be used in variable coefficients equations.

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Theorem

Given continuous functions $p,\, q : (t_1, t_2) \to \mathbb{R}$, *let* $y_1 : (t_1, t_2) \to \mathbb{R}$ *be a solution of*

$$y'' + p(t)\, y' + q(t)\, y = 0,$$

If the function $v : (t_1, t_2) \to \mathbb{R}$ *is solution of*

$$y_1(t)\, v'' + \big[2y'(t) + p(t) y_1(t)\big]\, v' = 0. \tag{3}$$

then the functions y_1 *and* $y_2 = v\, y_1$ *are fundamental solutions to the differential equation above.*

Reduction of the order method: Variable coefficients.

Remark: The same idea used to prove the constant coefficients Theorem above can be used in variable coefficients equations.

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Remark: The reason for the name Reduction of order method is that the function v does not appear in Eq. (3). This is a first order equation in v'.

Reduction of the order method: Variable coefficients.

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

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Example

Find a fundamental set of solutions to

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Solution: Express $y_2(t) = v(t)\, y_1(t)$.

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Integrating w we obtain v, that is, $v = c_2 t^{-3} + c_3$, with $c_2, c_3 \in \mathbb{R}$. Recalling that $y_2 = t\,v$ we then conclude that $y_2 = c_2 t^{-2} + c_3 t$. Choosing $c_2 = 1$ and $c_3 = 0$ we obtain the fundamental solutions $y_1(t) = t$ and $y_2(t) = \frac{1}{t^2}$. $\triangleleft$

Reduction of the order method: Variable coefficients.

Proof of the Theorem: The choice of $y_2 = vy_1$ implies

$$y_2' = v'\,y_1 + v\,y_1', \qquad y_2'' = v''\,y_1 + 2v'\,y_1' + v\,y_1''.$$

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This information introduced into the differential equation says that

$$(v''\,y_1 + 2v'\,y_1' + v\,y_1'') + p\,(v'\,y_1 + v\,y_1') + qv\,y_1 = 0$$

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The function y_1 is solution of $y_1'' + p \, y_1' + q \, y_1 = 0$.

Then, the equation for v is given by Eq. (3), that is,

$$y_1 \, v'' + (2y_1' + p \, y_1) \, v' = 0.$$

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Proof: Recall $y_1 \, v'' + (2y_1' + p \, y_1) \, v' = 0$.

Reduction of the order method: Variable coefficients.

Proof: Recall $y_1\, v'' + (2y_1' + p\, y_1)\, v' = 0$. We now need to show that y_1 and $y_2 = v y_1$ are linearly independent.

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$$W_{y_1 y_2}$$

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Proof: Recall $y_1\, v'' + (2y_1' + p\, y_1)\, v' = 0$. We now need to show that y_1 and $y_2 = vy_1$ are linearly independent.

$$W_{y_1 y_2} = \begin{vmatrix} y_1 & vy_1 \\ y_1' & (v' y_1 + v y_1') \end{vmatrix}$$

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