Modeling with first order equations (Sect. 1.5).

- Radioactive decay.
  - Carbon-14 dating.
- Salt in a water tank.
  - The experimental device.
  - The main equations.
  - Analysis of the mathematical model.
  - Predictions for particular situations.
Radioactive decay

Remarks:

(a) Radioactive substances randomly emit protons, electors, radiation, and they are transformed in another substance.
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(b) It can be seen that the time rate of change of the amount $N$ of a radioactive substances is proportional to the negative amount of radioactive substance.

\[
N'(t) = -aN(t),
\]

$N(0) = N_0$, $a > 0$.

(c) The integrating factor method implies $N(t) = N_0 e^{-at}$.

(d) The half-life is the time $\tau$ needed to get $N(\tau) = N_0 / 2$.

\[
N_0 e^{-a\tau} = \frac{N_0}{2} \Rightarrow -a\tau = \ln\left(\frac{1}{2}\right) \Rightarrow \tau = \frac{\ln(2)}{a}.
\]

(e) Using the half-life, we get $N(t) = N_0 2^{-t/\tau}$.
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Radioactive decay

Example

Remains containing 14% of the original amount of Carbon-14 are found. Knowing that Carbon-14 half-life is $\tau = 5730$ years, date the remains.

Solution:

Set $t = 0$ when the organism dies. Since the amount $N$ of Carbon-14 only decays after the organism dies, $N(t) = N_0 \left(\frac{1}{2}\right)^{-t/\tau}$, $\tau = 5730$ years.

The remains contain 14% of the original amount at the time $t$, $N(t) = N_0 \frac{14}{100} \Rightarrow \frac{1}{2}^{-t/\tau} = \frac{14}{100} \Rightarrow t = \tau \log_2 \left(\frac{100}{14}\right)$.

The organism died $16,253$ years ago.
Radioactive decay

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$$\frac{N(t)}{N_0} = \frac{14}{100}$$
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The organism died 16,253 years ago.
Modeling with first order equations (Sect. 1.5).

- Radioactive decay.
  - Carbon-14 dating.
- **Salt in a water tank.**
  - The experimental device.
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Salt in a water tank.

**Problem:** Describe the salt concentration in a tank with water if salty water comes in and goes out of the tank.
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**Main ideas of the test:**

- Since the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
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**Problem:** Describe the salt concentration in a tank with water if salty water comes in and goes out of the tank.

**Main ideas of the test:**

- Since the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
- The amount of salt in the tank depends on the salt concentration coming in and going out of the tank.
Salt in a water tank.

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Main ideas of the test:

- Since the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
- The amount of salt in the tank depends on the salt concentration coming in and going out of the tank.
- The salt in the tank also depends on the water rates coming in and going out of the tank.
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- To construct a model means to find the differential equation that takes into account the above properties of the system.
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- To construct a model means to find the differential equation that takes into account the above properties of the system.
- Finding the solution to the differential equation with a particular initial condition means we can predict the evolution of the salt in the tank if we know the tank initial condition.
Modeling with first order equations (Sect. 1.5).

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The experimental device.

Diagram showing a tank with water and salt entering through a pipe labeled \( q_i(t) \). The water is instantaneously mixed as it enters the tank.
The experimental device.

Definitions:

- $r_i(t), r_o(t)$: Rates in and out of water entering and leaving the tank at the time $t$. 

Units:

- $\left[r_i(t)\right] = \left[r_o(t)\right] = \text{Volume/Time}$
- $\left[q_i(t)\right] = \left[q_o(t)\right] = \text{Mass/Volume}$
- $\left[V(t)\right] = \text{Volume}$
- $\left[Q(t)\right] = \text{Mass}$. 
The experimental device.

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Modeling with first order equations (Sect. 1.5).

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The main equations.

**Remark:** The mass conservation provides the main equations of the mathematical description for salt in water.
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Main equations:

\[
\frac{d}{dt} V(t) = r_i(t) - r_o(t), \quad \text{Volume conservation,} \quad (1)
\]
Remark: The mass conservation provides the main equations of the mathematical description for salt in water.

Main equations:

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\frac{d}{dt} V(t) = r_i(t) - r_o(t), \quad \text{Volume conservation,} \quad (1)
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\[
\frac{d}{dt} Q(t) = r_i(t) q_i(t) - r_o(t) q_o(t), \quad \text{Mass conservation,} \quad (2)
\]

\[
q_o(t) = \frac{Q(t)}{V(t)}, \quad \text{Instantaneously mixed}, \quad (3)
\]

\[r_i, r_o: \text{Constants.} \quad (4)\]
Remark: The mass conservation provides the main equations of the mathematical description for salt in water.

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r_i, \ r_o : \ \text{Constants.} \tag{4}
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The main equations.

Remarks:

\[
\frac{dV}{dt} = \frac{\text{Volume}}{\text{Time}} = \left[ r_i - r_o \right],
\]

\[
\frac{dQ}{dt} = \frac{\text{Mass}}{\text{Time}} = \left[ r_i q_i - r_o q_o \right],
\]

\[
\left[ r_i q_i - r_o q_o \right] = \frac{\text{Volume}}{\text{Time}} \cdot \frac{\text{Mass}}{\text{Volume}} = \frac{\text{Mass}}{\text{Time}}.
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Analysis of the mathematical model.

Eqs. (4) and (1) imply

\[ V(t) = (r_i - r_o) t + V_0, \]  \hspace{1cm} (5)

where \( V(0) = V_0 \) is the initial volume of water in the tank.
Analysis of the mathematical model.

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Eqs. (3) and (2) imply

$$\frac{d}{dt} Q(t) = r_i q_i(t) - r_o \frac{Q(t)}{V(t)}.$$  

(6)
Analysis of the mathematical model.

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(6)

Eqs. (5) and (6) imply

\[ \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \]

(7)
Analysis of the mathematical model.

Recall: \( \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \)
Analysis of the mathematical model.

Recall: \[
\frac{d}{dt} Q(t) = r_i \, q_i(t) - \frac{r_o}{(r_i - r_o) \, t + V_0} \, Q(t).
\]

Notation: \[
a(t) = -\frac{r_o}{(r_i - r_o) \, t + V_0},
\]
Analysis of the mathematical model.

Recall: \[
\frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t).
\]

Notation: \[a(t) = -\frac{r_o}{(r_i - r_o) t + V_0}, \quad \text{and} \quad b(t) = r_i q_i(t).\]
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\]

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The main equation of the description is given by

\[
Q'(t) = a(t) \cdot Q(t) + b(t).
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Analysis of the mathematical model.

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The main equation of the description is given by

\[ Q'(t) = a(t) Q(t) + b(t). \]

Linear ODE for \( Q \).
Analysis of the mathematical model.

Recall: \[ \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \]

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Linear ODE for \( Q \). Solution: Integrating factor method.
Analysis of the mathematical model.

Recall: \( \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t) \).

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Linear ODE for \( Q \). Solution: Integrating factor method.

\[ Q(t) = e^{A(t)} \left[ Q_0 + \int_0^t e^{-A(s)} b(s) \, ds \right] \]
Analysis of the mathematical model.

Recall: \( \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t) \).

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\[ Q'(t) = a(t) Q(t) + b(t). \]

Linear ODE for \( Q \). Solution: Integrating factor method.

\[ Q(t) = e^{A(t)} \left[ Q_0 + \int_0^t e^{-A(s)} b(s) \, ds \right] \]

with \( Q(0) = Q_0 \), and \( A(t) = \int_0^t a(s) \, ds \).
Modeling with first order equations (Sect. 1.5).

- Radioactive decay.
  - Carbon-14 dating.
- **Main example: Salt in a water tank.**
  - The experimental device.
  - The main equations.
  - Analysis of the mathematical model.
  - Predictions for particular situations.
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Always holds \( Q'(t) = a(t) Q(t) + b(t) \).
Predictions for particular situations.

Example
Assume that $r_i = r_o = r$ and $q_i$ are constants. If $r$, $q_i$, $Q_0$ and $V_0$ are given, find $Q(t)$.

Solution: Always holds $Q'(t) = a(t)Q(t) + b(t)$. In this case:

$$a(t) = -\frac{r_o}{(r_i - r_o)t + V_0}$$
Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r$, $q_i$, $Q_0$ and $V_0$ are given, find $Q(t)$.

Solution: Always holds $Q'(t) = a(t)Q(t) + b(t)$.
In this case:

$$a(t) = -\frac{r_o}{(r_i - r_o) t + V_0} \Rightarrow a(t) = -\frac{r}{V_0} = -a_0,$$
Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and $q_i$ are constants. If $r, q_i, Q_0$ and $V_0$ are given, find $Q(t)$.

Solution: Always holds $Q'(t) = a(t) Q(t) + b(t)$. In this case:

$$a(t) = -\frac{r_o}{(r_i - r_o) t + V_0} \Rightarrow a(t) = -\frac{r}{V_0} = -a_0,$$

$$b(t) = r_i q_i(t)$$
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants. If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Always holds \( Q'(t) = a(t) Q(t) + b(t) \).

In this case:

\[
a(t) = -\frac{r_o}{(r_i - r_o) t + V_0} \quad \Rightarrow \quad a(t) = -\frac{r}{V_0} = -a_0,
\]

\[
b(t) = r_i q_i(t) \quad \Rightarrow \quad b(t) = r q_i = b_0.
\]
Predictions for particular situations.

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Assume that \( r_i = r_o = r \) and \( q_i \) are constants. If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

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\]

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b(t) = r_i q_i(t) \quad \Rightarrow \quad b(t) = rq_i = b_0.
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We need to solve the IVP:
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.

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\]

\[
b(t) = r_i q_i(t) \quad \Rightarrow \quad b(t) = r q_i = b_0.
\]

We need to solve the IVP:

\[
Q'(t) = -a_0 Q(t) + b_0, \quad Q(0) = Q_0.
\]
Predictions for particular situations.

Example
Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r$, $q_i$, $Q_0$ and $V_0$ are given, find $Q(t)$.

Solution: Recall the IVP: $Q'(t) + a_0 Q(t) = b_0$, $Q(0) = Q_0$. 
Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and $q_i$ are constants.

If $r$, $q_i$, $Q_0$ and $V_0$ are given, find $Q(t)$.

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Integrating factor method:
Predictions for particular situations.

Example
Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r$, $q_i$, $Q_0$ and $V_0$ are given, find $Q(t)$.

Solution: Recall the IVP: $Q'(t) + a_0 Q(t) = b_0$, $Q(0) = Q_0$.
Integrating factor method:

$$A(t) = a_0 t,$$
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants. If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) + a_0 Q(t) = b_0, \quad Q(0) = Q_0 \).

Integrating factor method:

\[
A(t) = a_0 t, \quad \mu(t) = e^{a_0 t},
\]
Predictions for particular situations.

Example
Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r$, $q_i$, $Q_0$ and $V_0$ are given, find $Q(t)$.

Solution: Recall the IVP: $Q'(t) + a_0 Q(t) = b_0$, $Q(0) = Q_0$.
Integrating factor method:

$$A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad e^{a_0 t} Q(t) = Q_0 + \int_0^t e^{a_0 s} b_0 \, ds.$$
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) + a_0 Q(t) = b_0, \quad Q(0) = Q_0. \)

Integrating factor method:

\[
A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad e^{a_0 t} Q(t) = Q_0 + \int_0^t e^{a_0 s} b_0 \, ds.
\]

\[
Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right].
\]
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) + a_0 Q(t) = b_0, \quad Q(0) = Q_0. \)

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\[
A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad e^{a_0 t} Q(t) = Q_0 + \int_0^t e^{a_0 s} b_0 ds.
\]

\[
Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right] = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0}.
\]
Predictions for particular situations.

Example
Assume that $r_i = r_o = r$ and $q_i$ are constants. If $r$, $q_i$, $Q_0$ and $V_0$ are given, find $Q(t)$.

Solution: Recall the IVP: $Q'(t) + a_0 Q(t) = b_0$, $Q(0) = Q_0$.
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\[ A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad e^{a_0 t} Q(t) = Q_0 + \int_0^t e^{a_0 s} b_0 \, ds. \]

\[ Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right] = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0}. \]

But \( \frac{b_0}{a_0} = r q_i \frac{V_0}{r} \)}
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants. If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) + a_0 Q(t) = b_0, \quad Q(0) = Q_0 \).

Integrating factor method:

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A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad e^{a_0 t} Q(t) = Q_0 + \int_0^t e^{a_0 s} b_0 \, ds.
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Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right]. = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0}.
\]

But \( \frac{b_0}{a_0} = rq_i \frac{V_0}{r} = q_i V_0 \),
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) + a_0 Q(t) = b_0, \quad Q(0) = Q_0. \)

Integrating factor method:

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A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad e^{a_0 t} Q(t) = Q_0 + \int_0^t e^{a_0 s} b_0 \, ds.
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\[
Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right]. = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0}.
\]

But \( \frac{b_0}{a_0} = rq_i \frac{V_0}{r} = q_i V_0, \) and \( a_0 = \frac{r}{V_0}. \)
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) + a_0 Q(t) = b_0, \quad Q(0) = Q_0 \).

Integrating factor method:

\[
A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad e^{a_0 t} Q(t) = Q_0 + \int_0^t e^{a_0 s} b_0 \, ds.
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\[
Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right] = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0}.
\]

But \( \frac{b_0}{a_0} = r q_i \frac{V_0}{r} = q_i V_0 \), and \( a_0 = \frac{r}{V_0} \). We conclude:

\[
Q(t) = \left( Q_0 - q_i V_0 \right) e^{-rt/V_0} + q_i V_0.
\]
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.

If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall: \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0. \)
Predictions for particular situations.

Example
Assume that $r_i = r_o = r$ and $q_i$ are constants. If $r$, $q_i$, $Q_0$ and $V_0$ are given, find $Q(t)$.

Solution: Recall: 
$$Q(t) = \left( Q_0 - q_i V_0 \right) e^{-rt/V_0} + q_i V_0.$$ 

Particular cases:

- $\frac{Q_0}{V_0} > q_i$;
- $\frac{Q_0}{V_0} = q_i$, so $Q(t) = Q_0$;
- $\frac{Q_0}{V_0} < q_i$. 
Predictions for particular situations.

Example
Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r$, $q_i$, $Q_0$ and $V_0$ are given, find $Q(t)$.

Solution: Recall: $Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0$.

Particular cases:

- $Q_0 / V_0 > q_i$;
- $Q_0 / V_0 = q_i$, so $Q(t) = Q_0$;
- $Q_0 / V_0 < q_i$. 

\[ Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0. \]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter,
find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1\% the initial value.
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter, find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example.
Predictions for particular situations.

Example
Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter,
find $t_1$ such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0$, 

...
Predictions for particular situations.

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Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter,
find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 e^{-rt/V_0}.
\]
Predictions for particular situations.

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Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter, find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 e^{-rt/V_0}.
\]

Since \( V(t) = (r_i - r_o) t + V_0 \)
Predictions for particular situations.

Example
Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find $t_1$ such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: This problem is a particular case $q_i = 0$ of the previous Example. Since $Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0$, we get

$$Q(t) = Q_0 e^{-rt/V_0}.$$  

Since $V(t) = (r_i - r_o) t + V_0$ and $r_i = r_o$.  

Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter, find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 e^{-rt/V_0}.
\]

Since \( V(t) = (r_i - r_o) t + V_0 \) and \( r_i = r_o \), we obtain \( V(t) = V_0 \).
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter,
find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 e^{-rt/V_0}.
\]

Since \( V(t) = (r_i - r_o) t + V_0 \) and \( r_i = r_o \), we obtain \( V(t) = V_0 \).
So \( q(t) = Q(t)/V(t) \) is given by \( q(t) = \frac{Q_0}{V_0} e^{-rt/V_0} \).
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter, find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1\% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 e^{-rt/V_0}.
\]

Since \( V(t) = (r_i - r_o) t + V_0 \) and \( r_i = r_o \), we obtain \( V(t) = V_0 \).
So \( q(t) = Q(t)/V(t) \) is given by \( q(t) = \frac{Q_0}{V_0} e^{-rt/V_0} \). Therefore,

\[
\frac{1}{100} \frac{Q_0}{V_0} = q(t_1)
\]
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter, find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 e^{-rt/V_0}.
\]

Since \( V(t) = (r_i - r_o) t + V_0 \) and \( r_i = r_o \), we obtain \( V(t) = V_0 \).
So \( q(t) = Q(t)/V(t) \) is given by \( q(t) = \frac{Q_0}{V_0} e^{-rt/V_0} \). Therefore,

\[
\frac{1}{100} \frac{Q_0}{V_0} = q(t_1) = \frac{Q_0}{V_0} e^{-rt_1/V_0}.
\]
Predictions for particular situations.

Example

Assume that \( r_i = r_o = r \) and \( q_i \) are constants.

If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter, find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since \( Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0 \), we get

\[
Q(t) = Q_0 e^{-rt/V_0}.
\]

Since \( V(t) = (r_i - r_o) t + V_0 \) and \( r_i = r_o \), we obtain \( V(t) = V_0 \).

So \( q(t) = Q(t)/V(t) \) is given by \( q(t) = Q_0 V_0 e^{-rt/V_0} \). Therefore,

\[
\frac{1}{100} \frac{Q_0}{V_0} = q(t_1) = \frac{Q_0}{V_0} e^{-rt_1/V_0} \quad \Rightarrow \quad e^{-rt_1/V_0} = \frac{1}{100}.
\]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter, find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1% the initial value.

Solution: Recall: \( e^{-rt_1}/V_0 = \frac{1}{100} \).
Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and $q_i$ are constants. If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find $t_1$ such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1}/V_0 = \frac{1}{100}$. Then,

$$-\frac{r}{V_0} t_1 = \ln\left(\frac{1}{100}\right)$$
Predictions for particular situations.

Example
Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find $t_1$ such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \frac{1}{100}$. Then,

$$-\frac{r}{V_0} t_1 = \ln\left(\frac{1}{100}\right) = -\ln(100)$$
Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and $q_i$ are constants.
If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter,
find $t_1$ such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1}/V_0 = \frac{1}{100}$. Then,

$$-\frac{r}{V_0} t_1 = \ln\left(\frac{1}{100}\right) = -\ln(100) \quad \Rightarrow \quad \frac{r}{V_0} t_1 = \ln(100).$$
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \text{ liters/min}, q_i = 0, V_0 = 200 \text{ liters}, Q_0/V_0 = 1 \text{ grams/liter} \), find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1\% the initial value.

Solution: Recall: \( e^{-rt_1/V_0} = \frac{1}{100} \). Then,

\[
- \frac{r}{V_0} t_1 = \ln \left( \frac{1}{100} \right) = - \ln(100) \quad \Rightarrow \quad \frac{r}{V_0} t_1 = \ln(100).
\]

We conclude that \( t_1 = \frac{V_0}{r} \ln(100) \).
Predictions for particular situations.

Example

Assume that $r_i = r_o = r$ and $q_i$ are constants. If $r = 2$ liters/min, $q_i = 0$, $V_0 = 200$ liters, $Q_0/V_0 = 1$ grams/liter, find $t_1$ such that $q(t_1) = Q(t_1)/V(t_1)$ is 1% the initial value.

Solution: Recall: $e^{-rt_1/V_0} = \frac{1}{100}$. Then,

$$-\frac{r}{V_0} t_1 = \ln\left(\frac{1}{100}\right) = -\ln(100) \quad \Rightarrow \quad \frac{r}{V_0} t_1 = \ln(100).$$

We conclude that $t_1 = \frac{V_0}{r} \ln(100)$.

In this case: $t_1 = 100 \ln(100)$. \(\triangleleft\)
Example

Assume that \( r_i = r_o = r \) are constants. If \( r = 5 \times 10^6 \) gal/year, \( q_i(t) = 2 + \sin(2t) \) grams/gal, \( V_0 = 10^6 \) gal, \( Q_0 = 0 \), find \( Q(t) \).
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Solution: Recall: \( Q'(t) = a(t) Q(t) + b(t) \).
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Solution: Recall: \( Q'(t) = a(t) Q(t) + b(t) \). In this case:

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a(t) = -\frac{r_o}{(r_i - r_o) t + V_0}
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Predictions for particular situations.

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Assume that $r_i = r_o = r$ are constants. If $r = 5 \times 10^6$ gal/year, $q_i(t) = 2 + \sin(2t)$ grams/gal, $V_0 = 10^6$ gal, $Q_0 = 0$, find $Q(t)$.

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Assume that \( r_i = r_o = r \) are constants. If \( r = 5 \times 10^6 \text{ gal/year}, \)
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We need to solve the IVP: \( Q'(t) = -a_0 Q(t) + b(t) \), \( Q(0) = 0 \).
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$$e^{a_0 t} Q(t) = \int_0^t e^{a_0 s} b(s) \, ds.$$
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We conclude: \( Q(t) = re^{-rt/V_0} \int_0^t e^{rs/V_0} [2 + \sin(2s)] \, ds. \)
Exact equations (Sect. 1.4).

- Exact differential equations.
- The Poincaré Lemma.
- Implicit solutions and the potential function.
- Generalization: The integrating factor method.
Exact differential equations.

Definition
Given an open rectangle \( R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2 \) and continuously differentiable functions \( M, N : R \rightarrow \mathbb{R} \),
Exact differential equations.

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$$N(t, y(t)) y'(t) + M(t, y(t)) = 0$$

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Recall: we use the notation: \( \partial_t N = \frac{\partial N}{\partial t} \), and \( \partial_u M = \frac{\partial M}{\partial u} \).
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Show whether the differential equation below is exact,

\[ 2ty(t)y'(t) + 2t + y^2(t) = 0. \]
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$$N(t,u) = 2tu, 
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The equation is exact iff

$$\frac{\partial}{\partial t}N(t,u) = \frac{\partial}{\partial u}M(t,u).$$ 

We conclude:

$$\frac{\partial}{\partial t}N(t,u) = 2u, 
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Remark: The ODE above is not separable and non-linear.
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Show whether the differential equation below is exact,

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Show whether the differential equation below is exact,

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  M(t, u) = a(t)u - b(t) \quad &\Rightarrow \quad \partial_u M(t, u) = a(t).
\end{align*} \]

This implies that \( \partial_t N(t, u) \neq \partial_u M(t, u) \).

\[ \triangle \]
Exact equations (Sect. 1.4).

- Exact differential equations.
- **The Poincaré Lemma.**
- Implicit solutions and the potential function.
- Generalization: The integrating factor method.
The Poincaré Lemma.

**Remark:** The coefficients $N$ and $M$ of an exact equations are the derivatives of a potential function $\psi$. 

**Proof:**

$(\Leftarrow)$ Simple: $\partial_t N = \partial_u \partial_t \psi$, $\partial_u M = \partial_t \partial_u \psi$, \{⇒\} $\partial_t N = \partial_u M$.

$(\Rightarrow)$ Difficult: Poincaré, 1880.
The Poincaré Lemma.

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Lemma (Poincaré)

Given an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, the continuously differentiable functions $M, N : R \to \mathbb{R}$ satisfy the equation

$$\partial_t N(t, u) = \partial_u M(t, u)$$

iff there exists a twice continuously differentiable function $\psi : R \to \mathbb{R}$, called potential function, such that for all $(t, u) \in R$ holds

$$\partial_u \psi(t, u) = N(t, u), \quad \partial_t \psi(t, u) = M(t, u).$$
The Poincaré Lemma.

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\begin{aligned}
\partial_t N &= \partial_t \partial_u \psi, \\
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Example
Show that the function $\psi(t, u) = t^2 + tu^2$ is the potential function for the exact differential equation

$$2ty(t) y'(t) + 2t + y^2(t) = 0.$$
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$$\partial_t \psi = 2t + u^2 = M, \quad \partial_u \psi = 2tu = N.$$ 

Remark: The Poincaré Lemma only states necessary and sufficient conditions on $N$ and $M$ for the existence of $\psi$. 

\[\triangle\]
Exact equations (Sect. 1.4).

- Exact differential equations.
- The Poincaré Lemma.
- **Implicit solutions and the potential function.**
- Generalization: The integrating factor method.
Implicit solutions and the potential function.

Theorem (Exact differential equations)

Let $M, N : \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions on an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$. If the differential equation

$$N(t, y(t)) y'(t) + M(t, y(t)) = 0$$

is exact, then every solution $y : (t_1, t_2) \to \mathbb{R}$ must satisfy the algebraic equation

$$\psi(t, y(t)) = c,$$

where $c \in \mathbb{R}$ and $\psi : R \to \mathbb{R}$ is a potential function for Eq. (8).
Implicit solutions and the potential function.

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Proof: \( 0 = N(t, y) y' + M(t, y) = \partial_y \psi(t, y) \frac{dy}{dt} + \partial_t \psi(t, y) \).
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**Theorem (Exact differential equations)**

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$$0 = \frac{d}{dt} \psi(t, y(t)) \iff \psi(t, y(t)) = c.$$
Implicit solutions and the potential function.

Example

Find all solutions $y$ to the equation

$$[\sin(t) + t^2 e^{y(t)} - 1] y'(t) + y(t) \cos(t) + 2te^{y(t)} = 0.$$
Implicit solutions and the potential function.

Example
Find all solutions $y$ to the equation

$$\left[ \sin(t) + t^2 e^{y(t)} - 1 \right] y'(t) + y(t) \cos(t) + 2te^{y(t)} = 0.$$ 

Solution: Recall: The equation is exact,
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hence, $\partial_t N = \partial_u M$. 

Implicit solutions and the potential function.

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hence, $\partial_t N = \partial_u M$. Poincaré Lemma says the exists $\psi$,

$$\partial_u \psi(t, u) = N(t, u), \quad \partial_t \psi(t, u) = M(t, u).$$
Implicit solutions and the potential function.

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Find all solutions $y$ to the equation
\[ \sin(t) + t^2 e^{y(t)} - 1 \] \[ y'(t) + y(t) \cos(t) + 2te^{y(t)} = 0. \]

Solution: Recall: The equation is exact,
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\[ \partial_u \psi(t, u) = N(t, u), \quad \partial_t \psi(t, u) = M(t, u). \]

These are actually equations for $\psi$. 
Implicit solutions and the potential function.

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Find all solutions $y$ to the equation
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[sin(t) + t^2e^{y(t)} - 1] y'(t) + y(t) \cos(t) + 2te^{y(t)} = 0.
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hence, $\partial_t N = \partial_u M$. Poincaré Lemma says the exists $\psi$,
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\partial_u \psi(t, u) = N(t, u), \quad \partial_t \psi(t, u) = M(t, u).
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These are actually equations for $\psi$. From the first one,
\[
\psi(t, u) = \int [\sin(t) + t^2e^u - 1] \, du + g(t).
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Implicit solutions and the potential function.

Example
Find all solutions \( y \) to the equation

\[
\left[ \sin(t) + t^2 e^{y(t)} - 1 \right] y'(t) + y(t) \cos(t) + 2te^{y(t)} = 0.
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Solution: \( \psi(t, u) = \int \left[ \sin(t) + t^2 e^u - 1 \right] du + g(t) \).
Implicit solutions and the potential function.

Example
Find all solutions $y$ to the equation

$$\left[\sin(t) + t^2 e^{y(t)} - 1\right] y'(t) + y(t) \cos(t) + 2 te^{y(t)} = 0.$$  

Solution: $\psi(t, u) = \int \left[\sin(t) + t^2 e^u - 1\right] du + g(t)$. Integrating,

$$\psi(t, u) = u \sin(t) + t^2 e^u - u + g(t).$$
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Find all solutions $y$ to the equation

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Introduce this expression into $\partial_t \psi(t, u) = M(t, u)$,
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Implicit solutions and the potential function.

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Find all solutions $y$ to the equation

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$$\partial_t \psi(t, u) = u \cos(t) + 2te^u + g'(t) = M(t, u) = u \cos(t) + 2te^u,$$

Therefore, $g'(t) = 0$, so we choose $g(t) = 0$.

We obtain,

$$\psi(t, u) = u \sin(t) + t^2 e^u - u.$$  

So the solution $y$ satisfies $y(t) \sin(t) + t^2 e^y(t) = c.$
Implicit solutions and the potential function.

Example

Find all solutions $y$ to the equation

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\[ \int \]
Implicit solutions and the potential function.

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Find all solutions $y$ to the equation

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Solution: $\psi(t, u) = \int [\sin(t) + t^2 e^u - 1] \, du + g(t)$. Integrating,

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$$\partial_t \psi(t, u) = u \cos(t) + 2te^u + g'(t) = M(t, u) = u \cos(t) + 2te^u,$$

Therefore, $g'(t) = 0$, so we choose $g(t) = 0$. We obtain,

$$\psi(t, u) = u \sin(t) + t^2 e^u - u.$$

So the solution $y$ satisfies $y(t) \sin(t) + t^2 e^{y(t)} - y(t) = c$.  

\[\triangle\]
Exact equations (Sect. 1.4).

- Exact differential equations.
- The Poincaré Lemma.
- Implicit solutions and the potential function.
- **Generalization: The integrating factor method.**

**Remark:**
Sometimes a non-exact equation can be transformed into an exact equation multiplying the equation by an integrating factor. Just like in the case of linear differential equations.
Generalization: The integrating factor method.

Theorem (Integrating factor)

Let $M, N : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable functions on $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, with $N \neq 0$. If the equation

$$N(t, y(t)) y'(t) + M(t, y(t)) = 0$$

is not exact, that is, $\partial_t N(t, u) \neq \partial_u M(t, u)$,
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is not exact, that is, $\partial_t N(t, u) \neq \partial_u M(t, u)$, and if the function

$$\frac{1}{N(t, u)} \left[ \partial_u M(t, u) - \partial_t N(t, u) \right]$$

does not depend on the variable $u$,
Generalization: The integrating factor method.

Theorem (Integrating factor)

Let $M, N : \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions on $\mathbb{R} = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, with $N \neq 0$. If the equation

$$N(t, y(t)) y'(t) + M(t, y(t)) = 0$$

is not exact, that is, $\partial_t N(t, u) \neq \partial_u M(t, u)$, and if the function

$$\frac{1}{N(t, u)} \left[ \partial_u M(t, u) - \partial_t N(t, u) \right]$$

does not depend on the variable $u$, then the equation

$$\mu(t) \left[ N(t, y(t)) y'(t) + M(t, y(t)) \right] = 0$$

is exact, where

$$\frac{\mu'(t)}{\mu(t)} = \frac{1}{N(t, u)} \left[ \partial_u M(t, u) - \partial_t N(t, u) \right].$$
Generalization: The integrating factor method.

Example

Find all solutions $y$ to the differential equation

$$\left[t^2 + t y(t)\right] y'(t) + \left[3t y(t) + y^2(t)\right] = 0.$$
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[
[t^2 + ty(t)]' + [3ty(t) + y^2(t)] = 0.
\]

Solution: The equation is not exact:
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[
\left[ t^2 + t \, y(t) \right] y'(t) + \left[ 3t \, y(t) + y^2(t) \right] = 0.
\]

Solution: The equation is not exact:
\[ N(t, u) = t^2 + tu \]
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[
[t^2 + t \, y(t)] \, y'(t) + [3t \, y(t) + y^2(t)] = 0.
\]

Solution: The equation is not exact:
\[
N(t, u) = t^2 + tu \quad \Rightarrow \quad \partial_t N(t, u) = 2t + u,
\]
Generalization: The integrating factor method.

Example

Find all solutions $y$ to the differential equation

$$\left[ t^2 + t \, y(t) \right] \, y'(t) + \left[ 3t \, y(t) + y^2(t) \right] = 0.$$ 

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Find all solutions \( y \) to the differential equation
\[
[t^2 + ty(t)] y'(t) + [3t y(t) + y^2(t)] = 0.
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Solution: The equation is not exact:
\[
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\]
\[
M(t, u) = 3tu + u^2 \quad \Rightarrow \quad \partial_u M(t, u) = 3t + 2u,
\]
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation

$$\left[t^2 + ty(t)\right] y'(t) + \left[3ty(t) + y^2(t)\right] = 0.$$

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hence $\partial_t N \neq \partial_u M$. 
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation

$[t^2 + t y(t)] y'(t) + [3t y(t) + y^2(t)] = 0$.

Solution: The equation is not exact:

$N(t, u) = t^2 + tu \quad \Rightarrow \quad \partial_t N(t, u) = 2t + u,$

$M(t, u) = 3tu + u^2 \quad \Rightarrow \quad \partial_u M(t, u) = 3t + 2u,$

hence $\partial_t N \neq \partial_u M$. We now verify whether the extra condition in Theorem above holds:
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[
[t^2 + t \, y(t)] \, y'(t) + [3t \, y(t) + y^2(t)] = 0.
\]

Solution: The equation is not exact:
\[
N(t, u) = t^2 + tu \quad \Rightarrow \quad \partial_t N(t, u) = 2t + u,
\]
\[
M(t, u) = 3tu + u^2 \quad \Rightarrow \quad \partial_u M(t, u) = 3t + 2u,
\]
hence $\partial_t N \neq \partial_u M$. We now verify whether the extra condition in Theorem above holds:
\[
\frac{\partial_u M(t, u) - \partial_t N(t, u)}{N(t, u)}
\]
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation

$$\left[ t^2 + ty(t) \right] y'(t) + \left[ 3ty(t) + y^2(t) \right] = 0.$$ 

Solution: The equation is not exact:

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hence $\partial_t N \neq \partial_u M$. We now verify whether the extra condition in Theorem above holds:

$$\frac{\partial_u M(t, u) - \partial_t N(t, u)}{N(t, u)} = \frac{1}{(t^2 + tu)} \left[ (3t + 2u) - (2t + u) \right]$$
Generalization: The integrating factor method.

Example
Find all solutions \( y \) to the differential equation
\[
[t^2 + t y(t)] y'(t) + [3t y(t) + y^2(t)] = 0.
\]

Solution: The equation is not exact:
\[
N(t, u) = t^2 + tu \quad \Rightarrow \quad \partial_t N(t, u) = 2t + u,
\]
\[
M(t, u) = 3tu + u^2 \quad \Rightarrow \quad \partial_u M(t, u) = 3t + 2u,
\]
hence \( \partial_t N \neq \partial_u M \). We now verify whether the extra condition in Theorem above holds:
\[
\frac{\left[\partial_u M(t, u) - \partial_t N(t, u)\right]}{N(t, u)} = \frac{1}{(t^2 + tu)}[(3t + 2u) - (2t + u)]
\]
\[
= \frac{1}{t(t + u)}(t + u)
\]
Generalization: The integrating factor method.

Example

Find all solutions $y$ to the differential equation

$$[t^2 + ty(t)] y'(t) + [3t y(t) + y^2(t)] = 0.$$

Solution: The equation is not exact:

$$N(t, u) = t^2 + tu \quad \Rightarrow \quad \partial_t N(t, u) = 2t + u,$$

$$M(t, u) = 3tu + u^2 \quad \Rightarrow \quad \partial_u M(t, u) = 3t + 2u,$$

hence $\partial_t N \neq \partial_u M$. We now verify whether the extra condition in Theorem above holds:

$$\frac{[\partial_u M(t, u) - \partial_t N(t, u)]}{N(t, u)} = \frac{1}{(t^2 + tu)} [(3t + 2u) - (2t + u)]$$

$$\frac{[\partial_u M(t, u) - \partial_t N(t, u)]}{N(t, u)} = \frac{1}{t(t + u)} (t + u) = \frac{1}{t}.$$
Generalization: The integrating factor method.

Example

Find all solutions $y$ to the differential equation

$$
\left[ t^2 + t y(t) \right] y'(t) + \left[ 3t y(t) + y^2(t) \right] = 0.
$$

Solution: \[
\frac{\partial_u M(t, u) - \partial_t N(t, u)}{N(t, u)} = \frac{1}{t}.
\]
Generalization: The integrating factor method.

Example

Find all solutions $y$ to the differential equation

$$[t^2 + t y(t)] y'(t) + [3t y(t) + y^2(t)] = 0.$$ 

Solution:

$$\frac{\partial_u M(t, u) - \partial_t N(t, u)}{N(t, u)} = \frac{1}{t}.$$ 

We find a function $\mu$ solution of

$$\frac{\mu'}{\mu} = \frac{\partial_u M - \partial_t N}{N}.$$
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation

$$[t^2 + t y(t)] y'(t) + [3t y(t) + y^2(t)] = 0.$$ 

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We find a function $\mu$ solution of

$$\frac{\mu'}{\mu} = \frac{[\partial_u M - \partial_t N]}{N},$$ 

that is

$$\frac{\mu'(t)}{\mu(t)} = \frac{1}{t}$$
Generalization: The integrating factor method.

Example

Find all solutions $y$ to the differential equation

$$[t^2 + ty(t)] y'(t) + [3ty(t) + y^2(t)] = 0.$$ 

Solution: \[
\frac{\partial_u M(t, u) - \partial_t N(t, u)}{N(t, u)} = \frac{1}{t}.
\]

We find a function $\mu$ solution of \[
\frac{\mu'}{\mu} = \frac{\partial_u M - \partial_t N}{N},
\]
that is

$$\frac{\mu'(t)}{\mu(t)} = \frac{1}{t} \implies \ln(\mu(t)) = \ln(t)$$
Example

Find all solutions $y$ to the differential equation

$$[t^2 + t y(t)] y'(t) + [3t y(t) + y^2(t)] = 0.$$ 

Solution:

$$\frac{\partial_u M(t, u) - \partial_t N(t, u)}{N(t, u)} = \frac{1}{t}.$$ 

We find a function $\mu$ solution of

$$\frac{\mu'}{\mu} = \frac{\partial_u M - \partial_t N}{N},$$

that is

$$\frac{\mu'(t)}{\mu(t)} = \frac{1}{t} \Rightarrow \ln(\mu(t)) = \ln(t) \Rightarrow \mu(t) = t.$$
Generalization: The integrating factor method.

Example

Find all solutions $y$ to the differential equation

$$[t^2 + t y(t)] y'(t) + [3 t y(t) + y^2(t)] = 0.$$  

Solution:  

$$\frac{\partial_u M(t, u) - \partial_t N(t, u)}{N(t, u)} = \frac{1}{t}.$$  

We find a function $\mu$ solution of  

$$\frac{\mu'}{\mu} = \frac{\partial_u M - \partial_t N}{N},$$  

that is

$$\frac{\mu'(t)}{\mu(t)} = \frac{1}{t} \quad \Rightarrow \quad \ln(\mu(t)) = \ln(t) \quad \Rightarrow \quad \mu(t) = t.$$  

Therefore, the equation below is exact:

$$[t^3 + t^2 y(t)] y'(t) + [3 t^2 y(t) + t y^2(t)] = 0.$$
Generalization: The integrating factor method.

Example

Find all solutions $y$ to the differential equation

$$\left[t^2 + t \, y(t)\right] y'(t) + \left[3t \, y(t) + y^2(t)\right] = 0.$$ 

Solution:  

$$\left[t^3 + t^2 \, y(t)\right] y'(t) + \left[3t^2 \, y(t) + t \, y^2(t)\right] = 0.$$
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[
\left[ t^2 + t y(t) \right] y'(t) + \left[ 3t y(t) + y^2(t) \right] = 0.
\]

Solution: 
\[
\left[ t^3 + t^2 y(t) \right] y'(t) + \left[ 3t^2 y(t) + t y^2(t) \right] = 0.
\]

This equation is exact:
Generalization: The integrating factor method.

Example

Find all solutions $y$ to the differential equation

$$\left[t^2 + t \, y(t)\right] y'(t) + \left[3t \, y(t) + y^2(t)\right] = 0.$$ 

Solution: $\left[t^3 + t^2 \, y(t)\right] y'(t) + \left[3t^2 \, y(t) + t \, y^2(t)\right] = 0$.

This equation is exact:

$$\tilde{N}(t, u) = t^3 + t^2 u.$$
Example
Find all solutions $y$ to the differential equation

$$
[t^2 + t y(t)] y'(t) + [3t y(t) + y^2(t)] = 0.
$$

Solution: $[t^3 + t^2 y(t)] y'(t) + [3t^2 y(t) + t y^2(t)] = 0$.

This equation is exact:

$$
\tilde{N}(t, u) = t^3 + t^2 u \quad \Rightarrow \quad \partial_t \tilde{N}(t, u) = 3t^2 + 2tu,
$$
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation

$$\left[ t^2 + ty(t) \right] y'(t) + \left[ 3ty(t) + y^2(t) \right] = 0.$$ 

Solution: $$\left[ t^3 + t^2 y(t) \right] y'(t) + \left[ 3t^2 y(t) + ty^2(t) \right] = 0.$$ 

This equation is exact:

$$\tilde{N}(t, u) = t^3 + t^2 u \quad \Rightarrow \quad \partial_t \tilde{N}(t, u) = 3t^2 + 2tu,$$

$$\tilde{M}(t, u) = 3t^2 u + tu^2.$$
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation

$$[t^2 + ty(t)]y'(t) + [3ty(t) + y^2(t)] = 0.$$ 

Solution: $[t^3 + t^2y(t)]y'(t) + [3t^2y(t) + ty^2(t)] = 0.$

This equation is exact:

$$\tilde{N}(t, u) = t^3 + t^2u \quad \Rightarrow \quad \partial_t \tilde{N}(t, u) = 3t^2 + 2tu,$$

$$\tilde{M}(t, u) = 3t^2u + tu^2 \quad \Rightarrow \quad \partial_u \tilde{M}(t, u) = 3t^2 + 2tu,$$
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[
[t^2 + t \, y(t)] \, y'(t) + [3t \, y(t) + y^2(t)] = 0.
\]

Solution: $[t^3 + t^2 \, y(t)] \, y'(t) + [3t^2 \, y(t) + t \, y^2(t)] = 0$. This equation is exact:
\[
\tilde{N}(t, u) = t^3 + t^2 \, u \quad \Rightarrow \quad \partial_t \tilde{N}(t, u) = 3t^2 + 2tu,
\]
\[
\tilde{M}(t, u) = 3t^2 \, u + tu^2 \quad \Rightarrow \quad \partial_u \tilde{M}(t, u) = 3t^2 + 2tu,
\]
that is, $\partial_t \tilde{N} = \partial_u \tilde{M}$. 
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[
[t^2 + t y(t)] y'(t) + [3 t y(t) + y^2(t)] = 0.
\]

Solution: \[
[t^3 + t^2 y(t)] y'(t) + [3 t^2 y(t) + t y^2(t)] = 0.
\]
This equation is exact:
\[
\tilde{N}(t, u) = t^3 + t^2 u \quad \Rightarrow \quad \partial_t \tilde{N}(t, u) = 3 t^2 + 2tu,
\]
\[
\tilde{M}(t, u) = 3 t^2 u + tu^2 \quad \Rightarrow \quad \partial_u \tilde{M}(t, u) = 3 t^2 + 2tu,
\]
that is, \(\partial_t \tilde{N} = \partial_u \tilde{M}\). Therefore, there exists \(\psi\) such that
\[
\partial_u \psi(t, u) = \tilde{N}(t, u), \quad \partial_t \psi(t, u) = \tilde{M}(t, u).
\]
Generalization: The integrating factor method.

Example

Find all solutions $y$ to the differential equation

$[t^2 + t \, y(t)] \, y'(t) + [3t \, y(t) + y^2(t)] = 0.$

Solution: $[t^3 + t^2 \, y(t)] \, y'(t) + [3t^2 \, y(t) + t \, y^2(t)] = 0.$

This equation is exact:

$\tilde{N}(t, u) = t^3 + t^2 \, u \quad \Rightarrow \quad \partial_t \tilde{N}(t, u) = 3t^2 + 2tu,$

$\tilde{M}(t, u) = 3t^2 \, u + tu^2 \quad \Rightarrow \quad \partial_u \tilde{M}(t, u) = 3t^2 + 2tu,$

that is, $\partial_t \tilde{N} = \partial_u \tilde{M}$. Therefore, there exists $\psi$ such that

$\partial_u \psi(t, u) = \tilde{N}(t, u), \quad \partial_t \psi(t, u) = \tilde{M}(t, u).$

From the first equation above we obtain

$\partial_u \psi = t^3 + t^2 \, u$
Generalization: The integrating factor method.

Example

Find all solutions $y$ to the differential equation

$$[t^2 + t y(t)] y'(t) + [3t y(t) + y^2(t)] = 0.$$ 

Solution: $[t^3 + t^2 y(t)] y'(t) + [3t^2 y(t) + t y^2(t)] = 0.$

This equation is exact:

$$\tilde{N}(t, u) = t^3 + t^2 u \quad \Rightarrow \quad \partial_t \tilde{N}(t, u) = 3t^2 + 2tu,$$

$$\tilde{M}(t, u) = 3t^2 u + tu^2 \quad \Rightarrow \quad \partial_u \tilde{M}(t, u) = 3t^2 + 2tu,$$

that is, $\partial_t \tilde{N} = \partial_u \tilde{M}$. Therefore, there exists $\psi$ such that

$$\partial_u \psi(t, u) = \tilde{N}(t, u), \quad \partial_t \psi(t, u) = \tilde{M}(t, u).$$

From the first equation above we obtain

$$\partial_u \psi = t^3 + t^2 u \quad \Rightarrow \quad \psi(t, u) = \int (t^3 + t^2 u) \, du + g(t).$$
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation

$$[t^2 + t y(t)] y'(t) + [3t y(t) + y^2(t)] = 0.$$ 

Solution: $\psi(t, u) = \int (t^3 + t^2 u) \, du + g(t)$.
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[
\left[t^2 + t \, y(t)\right] y'(t) + \left[3t \, y(t) + y^2(t)\right] = 0.
\]

Solution: $\psi(t, u) = \int \left(t^3 + t^2 u\right) du + g(t)$.

Integrating, $\psi(t, u) = t^3 u + \frac{1}{2} t^2 u^2 + g(t)$.\)
Example

Find all solutions $y$ to the differential equation

$$[t^2 + ty(t)] y'(t) + [3ty(t) + y^2(t)] = 0.$$ 

Solution: $\psi(t, u) = \int (t^3 + t^2 u) ~ du + g(t).$

Integrating, $\psi(t, u) = t^3 u + \frac{1}{2} t^2 u^2 + g(t).$

Introduce $\psi$ in $\partial_t \psi = \tilde{M}$, where $\tilde{M} = 3t^2 u + tu^2.$
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[
\left[ t^2 + ty(t) \right] y'(t) + \left[ 3ty(t) + y^2(t) \right] = 0.
\]

Solution: $\psi(t, u) = \int (t^3 + t^2u) \, du + g(t)$.

Integrating, $\psi(t, u) = t^3u + \frac{1}{2}t^2u^2 + g(t)$.

Introduce $\psi$ in $\partial_t \psi = \tilde{M}$, where $\tilde{M} = 3t^2u + tu^2$. So,
\[
\partial_t \psi(t, u) = 3t^2u + tu^2 + g'(t)
\]
Generalization: The integrating factor method.

Example
Find all solutions \( y \) to the differential equation

\[
[t^2 + ty(t)] y'(t) + [3ty(t) + y^2(t)] = 0.
\]

Solution: \( \psi(t, u) = \int (t^3 + t^2u) \, du + g(t). \)

Integrating, \( \psi(t, u) = t^3u + \frac{1}{2} t^2u^2 + g(t). \)

Introduce \( \psi \) in \( \partial_t \psi = \tilde{M} \), where \( \tilde{M} = 3t^2u + tu^2 \). So,

\[
\partial_t \psi(t, u) = 3t^2u + tu^2 + g'(t) = \tilde{M}(t, u)
\]
Example
Find all solutions $y$ to the differential equation
\[ [t^2 + ty(t)] y'(t) + [3ty(t) + y^2(t)] = 0. \]

Solution: $\psi(t, u) = \int (t^3 + t^2u) \, du + g(t)$.

Integrating, $\psi(t, u) = t^3u + \frac{1}{2}t^2u^2 + g(t)$.
Introduce $\psi$ in $\partial_t \psi = \tilde{M}$, where $\tilde{M} = 3t^2u + tu^2$. So,
\[ \partial_t \psi(t, u) = 3t^2u + tu^2 + g'(t) = \tilde{M}(t, u) = 3t^2u + tu^2, \]
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[
[t^2 + t \, y(t)] \, y'(t) + [3t \, y(t) + y^2(t)] = 0.
\]

Solution: $\psi(t, u) = \int (t^3 + t^2 u) \, du + g(t)$.

Integrating, $\psi(t, u) = t^3 u + \frac{1}{2} t^2 u^2 + g(t)$.

Introduce $\psi$ in $\partial_t \psi = \tilde{M}$, where $\tilde{M} = 3t^2 u + tu^2$. So,
\[
\partial_t \psi(t, u) = 3t^2 u + tu^2 + g'(t) = \tilde{M}(t, u) = 3t^2 u + tu^2,
\]
So $g'(t) = 0$ and we choose $g(t) = 0$. 
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[
[t^2 + t \cdot y(t)] \cdot y'(t) + [3t \cdot y(t) + y^2(t)] = 0.
\]

Solution: $\psi(t, u) = \int (t^3 + t^2 u) \, du + g(t)$.

Integrating, $\psi(t, u) = t^3 u + \frac{1}{2} t^2 u^2 + g(t)$.

Introduce $\psi$ in $\partial_t \psi = \tilde{M}$, where $\tilde{M} = 3t^2 u + tu^2$. So,
\[
\partial_t \psi(t, u) = 3t^2 u + tu^2 + g'(t) = \tilde{M}(t, u) = 3t^2 u + tu^2,
\]

So $g'(t) = 0$ and we choose $g(t) = 0$. We conclude that a potential function is $\psi(t, u) = t^3 u + \frac{1}{2} t^2 u^2$.
Generalization: The integrating factor method.

Example

Find all solutions $y$ to the differential equation

$$[t^2 + ty(t)] y'(t) + [3t y(t) + y^2(t)] = 0.$$ 

Solution: $\psi(t, u) = \int (t^3 + t^2 u) \, du + g(t)$. 

Integrating, $\psi(t, u) = t^3 u + \frac{1}{2} t^2 u^2 + g(t)$. Introduce $\psi$ in $\partial_t \psi = \tilde{M}$, where $\tilde{M} = 3t^2 u + tu^2$. So,

$$\partial_t \psi(t, u) = 3t^2 u + tu^2 + g'(t) = \tilde{M}(t, u) = 3t^2 u + tu^2,$$

So $g'(t) = 0$ and we choose $g(t) = 0$. We conclude that a potential function is $\psi(t, u) = t^3 u + \frac{1}{2} t^2 u^2$. And every solution $y$ satisfies $t^3 y(t) + \frac{1}{2} t^2 [y(t)]^2 = c$. \triangle