Review for Final Exam.

- Exam is cumulative.
- Heat equation not included.
- 15 problems.
- Two and half hours.

- Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx.$$

$$b_n = 2 \int_0^1 (-1)\, \sin(n\pi x)\, dx = (-2)\, \frac{(-1)}{n\pi}\, \cos(n\pi x)\Big|_0^1,$$

$$b_n = \frac{2}{n\pi} \big[\cos(n\pi) - 1\big] \quad \Rightarrow \quad b_n = \frac{2}{n\pi}\, \big[(-1)^n - 1\big].$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 1$ for $x \in (-1, 0)$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = \frac{2}{n\pi}\big[(-1)^n - 1\big]$.

If $n = 2k$, then $b_{2k} = \frac{2}{2k\pi}\big[(-1)^{2k} - 1\big] = 0$.

If $n = 2k-1$,
$b_{(2k-1)} = \frac{2}{(2k-1)\pi}\big[(-1)^{2k-1} - 1\big] = -\frac{4}{(2k-1)\pi}$.

We conclude: $f(x) = -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\sin[(2k-1)\pi x]$. $\triangleleft$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big[a_n\cos\Big(\frac{n\pi x}{L}\Big) + b_n\sin\Big(\frac{n\pi x}{L}\Big)\Big].$$

Since f is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx,\ \ L = 2,$$

$$b_n = \int_0^2 (2-x)\,\sin\Big(\frac{n\pi x}{2}\Big)\,dx.a$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $b_n = 2 \int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx.$

$$\int \sin\left(\frac{n\pi x}{2}\right) dx = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right),$$

The other integral is done by parts,

$$I = \int x \sin\left(\frac{n\pi x}{2}\right) dx, \quad \begin{cases} u = x, & v' = \sin\left(\frac{n\pi x}{2}\right) \\ u' = 1, & v = -\frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right) \end{cases}$$

$$I = \frac{-2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) - \int \left(\frac{-2}{n\pi}\right) \cos\left(\frac{n\pi x}{2}\right) dx.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the odd-periodic extension of $f(x) = 2 - x$ for $x \in (0, 2)$, and then find the Fourier Series of this extension.

Solution: $I = \frac{-2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) - \int \left(\frac{-2}{n\pi}\right) \cos\left(\frac{n\pi x}{2}\right) dx.$

$I = -\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right).$ So, we get

$$b_n = 2\left[\frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 + \left[\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 - \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)\Big|_0^2$$

$$b_n = \frac{-4}{n\pi}\big[\cos(n\pi) - 1\big] + \left[\frac{4}{n\pi} \cos(n\pi) - 0\right] \quad \Rightarrow \quad b_n = \frac{4}{n\pi}.$$

We conclude: $f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{2}\right).$ $\triangleleft$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

Since f is even and periodic, then the Fourier Series is a Cosine Series, that is, $b_n = 0$.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = \int_0^2 (2 - x)\, dx = \frac{\text{base x height}}{2} \;\Rightarrow\; a_0 = 2.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L} \int_0^L f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \;\; L = 2,$$

$$a_n = \int_0^2 (2 - x) \cos\Big(\frac{n\pi x}{2}\Big)\, dx.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: $a_n = 2 \int_0^2 \cos\Big(\frac{n\pi x}{2}\Big)\, dx - \int_0^2 x \cos\Big(\frac{n\pi x}{2}\Big)\, dx.$

$$\int \cos\Big(\frac{n\pi x}{2}\Big)\, dx = \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big),$$

The other integral is done by parts,

$$I = \int x \cos\Big(\frac{n\pi x}{2}\Big)\, dx, \qquad \begin{cases} u = x, & v' = \cos\Big(\frac{n\pi x}{2}\Big) \\ u' = 1, & v = \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) \end{cases}$$

$$I = \frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) - \int \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\, dx.$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $I = \frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) - \int \frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\, dx.$

$I = \frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big) + \Big(\frac{2}{n\pi}\Big)^2 \cos\Big(\frac{n\pi x}{2}\Big)$. So, we get

$$a_n = 2\Big[\frac{2}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big[\frac{2x}{n\pi} \sin\Big(\frac{n\pi x}{2}\Big)\Big]\Big|_0^2 - \Big(\frac{2}{n\pi}\Big)^2 \cos\Big(\frac{n\pi x}{2}\Big)\Big|_0^2$$

$$a_n = 0 - 0 - \frac{4}{n^2\pi^2}\big[\cos(n\pi) - 1\big] \quad \Rightarrow \quad a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n].$$

Fourier Series: Even/Odd-periodic extensions.

Example

Graph the even-periodic extension of $f(x) = 2 - x$ for $x \in [0, 2]$, and then find the Fourier Series of this extension.

Solution: Recall: $b_n = 0$, $a_0 = 2$, $a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n]$.

If $n = 2k$, then $a_{2k} = \frac{4}{(2k)^2\pi^2}\big[1 - (-1)^{2k}\big] = 0.$

If $n = 2k - 1$, then we obtain

$$a_{(2k-1)} = \frac{4}{(2k-1)^2\pi^2}\big[1 - (-1)^{2k-1}\big] = \frac{8}{(2k-1)^2\pi^2}.$$

We conclude: $f(x) = 1 + \frac{8}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos\Big(\frac{(2k-1)\pi x}{2}\Big).$ ◁

Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- **Eigenvalue-Eigenfunction BVP (Chptr. 6).**
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$\mu = \frac{n\pi}{8}, \quad \lambda = \left(\frac{n\pi}{8}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{8}\right), \quad n = 1, 2, \cdots \triangleleft$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

$$8\mu = (2n+1)\frac{\pi}{2}, \quad \Rightarrow \quad \mu = \frac{(2n+1)\pi}{16}.$$

Then, for $n = 1, 2, \cdots$ holds

$$\lambda = \Big[\frac{(2n+1)\pi}{16}\Big]^2, \quad y_n(x) = \sin\Big(\frac{(2n+1)\pi x}{16}\Big). \qquad \triangleleft$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: Case $\lambda > 0$. Then, $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Then, $y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$. The B.C. imply:

$$0 = y'(0) = c_2 \;\Rightarrow\; y(x) = c_1 \cos(\mu x), \; y'(x) = -c_1\mu \sin(\mu x).$$

$$0 = y'(8) = c_1 \mu \sin(\mu 8), \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$8\mu = n\pi, \quad \Rightarrow \quad \mu = \frac{n\pi}{8}.$$

Then, choosing $c_1 = 1$, for $n = 1, 2, \cdots$ holds

$$\lambda = \Big(\frac{n\pi}{8}\Big)^2, \quad y_n(x) = \cos\Big(\frac{n\pi x}{8}\Big).$$

Eigenvalue-Eigenfunction BVP.

Example

Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y'(0) = 0, \quad y'(8) = 0.$$

Solution: The case $\lambda = 0$. The general solution is

$$y(x) = c_1 + c_2 x.$$

The B.C. imply:

$$0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0.$$

$$0 = y'(8) = 0.$$

Then, choosing $c_1 = 1$, holds,

$$\lambda = 0, \qquad y_0(x) = 1. \qquad \triangleleft$$

A Boundary Value Problem.

Example

Find the solution of the BVP

$$y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0.$$

Solution: $y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm i.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

Then, $y'(x) = -c_1 \sin(x) + c_2 \cos(x)$. The B.C. imply:

$$1 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1 \cos(x) + \sin(x).$$

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \;\Rightarrow\; c_1 = -\frac{\sin(\pi/3)}{\cos(\pi/3)}.$$

$$c_1 = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3} \;\Rightarrow\; y(x) = -\sqrt{3}\cos(x) + \sin(x). \quad \triangleleft$$

Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- **Systems of linear Equations (Chptr. 5).**
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big] \pm i e^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Real-valued fundamental solutions are

$$\mathbf{x}^{(1)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big],$$

$$\mathbf{x}^{(2)} = e^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$, then find $\mathbf{w}$ solution of $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then fundamental solutions to the differential equation are given by

$$\mathbf{x}^{(1)} = \mathbf{v}\,e^{\lambda t}, \quad \mathbf{x}^{(2)} = (\mathbf{v}\,t + \mathbf{w})\,e^{\lambda t}.$$

Then, the general solution is

$$\mathbf{x} = c_1\,\mathbf{v}\,e^{\lambda t} + c_2\,(\mathbf{v}\,t + \mathbf{w})\,e^{\lambda t}.$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution:

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 4 \\ 2 & (-1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,$$

$$p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_\pm = \pm 3.$$

Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case $\lambda_- = -3$,

$$A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -v_2 \Rightarrow \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

Solution: Recall: $\lambda_\pm = \pm 3, \ \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

The general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$

The initial condition implies,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2+1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

We conclude: $\mathbf{x}(t) = \frac{5}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$ $\triangleleft$

Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- **Laplace transforms (Chptr. 4).**
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

Laplace transforms.

Summary:

- Main Properties:

$$\mathcal{L}\big[f^{(n)}(t)\big] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \qquad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \qquad (13)$$

$$\mathcal{L}[f(t)]\Big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]. \qquad (14)$$

- Convolutions:

$$\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)].$$

- Partial fraction decompositions, completing the squares.

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)] = \dfrac{e^{-5s}}{s}$, and recall,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad \mathcal{L}[y''] = s^2\,\mathcal{L}[y] - 3s - 2.$$

$$(s^2+9)\,\mathcal{L}[y] - 3s - 2 = \frac{e^{-5s}}{s}$$

$$\mathcal{L}[y] = \frac{(3s+2)}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

$$\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}$.

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,\frac{1}{s(s^2+9)}.$$

Partial fractions on

$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

$$1 = as^2 + 9a + bs^2 + cs = (a+b)\,s^2 + cs + 9a$$

$$a = \frac{1}{9}, \quad c = 0, \quad b = -a \quad \Rightarrow \quad b = -\frac{1}{9}.$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: So, $\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + e^{-5s}\,H(s)$, and

$$H(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\Big[\frac{1}{s} - \frac{s}{s^2+9}\Big] = \frac{1}{9}\Big(\mathcal{L}[u(t)] - \mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(e^{-5s}\,\mathcal{L}[u(t)] - e^{-5s}\,\mathcal{L}[\cos(3t)]\Big)$$

$$e^{-5s}\,H(s) = \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution:

$$\mathcal{L}[y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + \frac{1}{9}\Big(\mathcal{L}[u_5(t)] - \mathcal{L}\big[u_5(t)\cos(3(t-5))\big]\Big).$$

Therefore, we conclude that,

$$y(t) = 3\cos(3t) + \frac{2}{3}\sin(3t) + \frac{u_5(t)}{9}\Big[1 - \cos(3(t-5))\Big].$$

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Review for Final Exam.

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- Laplace transforms (Chptr. 4).
- **Second order linear equations (Chptr. 2).**
- First order differential equations (Chptr. 1).

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real, then the general solution is

$$y(t) = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}.$$

(b) If $r_1 \neq r_2$, complex, then denoting $r_\pm = \alpha \pm \beta i$, complex-valued fundamental solutions are

$$y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t}\left[\cos(\beta t) \pm i \sin(\beta t)\right],$$

and real-valued fundamental solutions are

$$y_1(t) = e^{\alpha t} \cos(\beta t), \qquad y_2(t) = e^{\alpha t} \sin(\beta t).$$

If $r_1 = r_2 = r$, real, then the general solution is

$$y(t) = (c_1 + c_2 t)\, e^{rt}.$$

Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

(ii) Variation of parameters: If y_1 and y_2 are fundamental solutions to the homogeneous equation, and W is their Wronskian, then $y_p = u_1 y_1 + u_2 y_2$, where

$$u_1' = -\frac{y_2 g}{W}, \qquad u_2' = \frac{y_1 g}{W}.$$

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\,y'' - 4x\,y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2\,(2) - 4x\,(2x) + 6x^2 = 0.$$

Look for a solution $y_2(x) = v(x)\,y_1(x)$, and find an equation for v.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x\,(x^2 v' + 2xv) + 6\,(x^2 v) = 0.$$

$$x^4 v'' + (4x^3 - 4x^3)\,v' + (2x^2 - 8x^2 + 6x^2)\,v = 0.$$

$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2 x\,y_1.$$

Choose $c_1 = 0$, $c_2 = 1$. Hence $y_2(x) = x^3$, and $y_1(x) = x^2$. $\lhd$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$

$$r_\pm = \frac{1}{2}\big[2 \pm \sqrt{4+12}\big] = \frac{1}{2}\big[2 \pm \sqrt{16}\big] = 1 \pm 2 \;\Rightarrow\; \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess y_p. Since $g(t) = 3\,e^{-t} \quad \Rightarrow \quad y_p(t) = k\,e^{-t}$.

But this $y_p = k\,e^{-t}$ is solution of the homogeneous equation.

Then propose $y_p(t) = kt\,e^{-t}$.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

(3) Find the undetermined coefficient k.

$$y_p' = k\,e^{-t} - kt\,e^{-t}, \quad y_p'' = -2k\,e^{-t} + kt\,e^{-t}.$$

$$(-2k\,e^{-t} + kt\,e^{-t}) - 2(k\,e^{-t} - kt\,e^{-t}) - 3(kt\,e^{-t}) = 3\,e^{-t}$$

$$(-2 + t - 2 + 2t - 3t)\,k\,e^{-t} = 3\,e^{-t} \;\Rightarrow\; -4k = 3 \;\Rightarrow\; k = -\frac{3}{4}.$$

We obtain: $y_p(t) = -\dfrac{3}{4}t\,e^{-t}$.

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\dfrac{3}{4}t\,e^{-t}$.

(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \dfrac{3}{4}t\,e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1\,e^{3t} - c_2\,e^{-t} - \frac{3}{4}(e^{-t} - t\,e^{-t}).$$

$$1 = y(0) = c_1 + c_2, \qquad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}.$$

$$\left.\begin{aligned} c_1 + c_2 &= 1, \\ 3_1 - c_2 &= 1 \end{aligned}\right\} \;\Rightarrow\; \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}t\,e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Since $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$, we obtain,

$$y(t) = \frac{1}{2}\left(e^{3t} + e^{-t}\right) - \frac{3}{4}t\,e^{-t}.$$

$\triangleleft$

Review for Final Exam.

- Fourier Series expansions (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- **First order differential equations (Chptr. 1).**

First order differential equations.

Summary:

- Linear, first order equations: $y' + p(t)\,y = q(t)$.

  Use the integrating factor method: $\mu(t) = e^{\int p(t)\,dt}$.

- Separable, non-linear equations: $h(y)\,y' = g(t)$.

  Integrate with the substitution: $u = y(t)$, $du = y'(t)\,dt$, that is,

$$\int h(u)\,du = \int g(t)\,dt + c.$$

  The solution can be found in implicit of explicit form.

- Homogeneous equations can be converted into separable equations.

  Read page 49 in the textbook.

- No modeling problems from Sect. 2.3.

First order differential equations.

Summary:

- Bernoulli equations: $y' + p(t)\,y = q(t)\,y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for y can be converted into a linear equation for $v = \dfrac{1}{y^{n-1}}$.

- Exact equations and integrating factors.

$$N(x,y)\,y' + M(x,y) = 0.$$

  The equation is exact iff $\partial_x N = \partial_y M$.

  If the equation is exact, then there is a potential function ψ, such that $N = \partial_y \psi$ and $M = \partial_x \psi$.

  The solution of the differential equation is

$$\psi\big(x, y(x)\big) = c.$$

First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)\, y^n$.)
3. Separable equations.
   (Few manipulations: $h(y)\, y' = g(t)$.)
4. Homogeneous equations.
   (Several manipulations: $y' = F(y/t)$.)
5. Exact equations.
   (Check one equation: $N\, y' + M = 0$, and $\partial_t N = \partial_y M$.)
6. Exact equation with integrating factor.
   (Very complicated to check.)

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy}\,\frac{(1/x^2)}{(1/x^2)} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.$$

$$v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.$$

$$y = x\, v, \quad y' = x\, v' + v \quad x\, v' + v = \frac{1 + v + v^2}{v}.$$

$$x\, v' = \frac{1 + v + v^2}{v} - v = \frac{1 + v + v^2 - v^2}{v} \quad \Rightarrow \quad x\, v' = \frac{1 + v}{v}.$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \dfrac{1+v}{v}$. This is a separable equation.

$$\frac{v(x)}{1+v(x)}\, v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1+v(x)}\, v'(x)\, dx = \int \frac{dx}{x} + c.$$

Use the substitution $u = 1 + v$, hence $du = v'(x)\, dx$.

$$\int \frac{(u-1)}{u}\, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \Big(1 - \frac{1}{u}\Big)\, du = \int \frac{dx}{x} + c$$

$$u - \ln|u| = \ln|x| + c \quad \Rightarrow \quad 1 + v - \ln|1+v| = \ln|x| + c.$$

$$v = \frac{y}{x} \quad \Rightarrow \quad 1 + \frac{y(x)}{x} - \ln\Big|1 + \frac{y(x)}{x}\Big| = \ln|x| + c. \qquad \lhd$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n$, $n = 3$.

Divide by y^3. That is, $\dfrac{y'}{y^3} + \dfrac{1}{y^2} = -e^{2x}$.

Let $v = \dfrac{1}{y^2}$. Since $v' = -2\dfrac{y'}{y^3}$, we obtain $-\dfrac{1}{2}\, v' + v = -e^{2x}$.

We obtain the linear equation $v' - 2v = 2e^{2x}$.

Use the integrating factor method. $\mu(x) = e^{-2x}$.

$$e^{-2x}\, v' - 2\, e^{-2x}\, v = 2 \quad \Rightarrow \quad \big(e^{-2x}\, v\big)' = 2.$$

First order differential equations.

Example

Find the solution y to the initial value problem

$$y' + y + e^{2x}\,y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \dfrac{1}{y^2}$ and $\left(e^{-2x}\,v\right)' = 2.$

$$e^{-2x}\,v = 2x + c \;\Rightarrow\; v(x) = (2x + c)\,e^{2x} \;\Rightarrow\; \frac{1}{y^2} = (2x + c)\,e^{2x}.$$

$$y^2 = \frac{1}{e^2x\,(2x + c)} \quad\Rightarrow\quad y_{\pm}(x) = \pm\frac{e^{-x}}{\sqrt{2x + c}}.$$

The initial condition $y(0) = 1/3 > 0$ implies: Choose y_+.

$$\frac{1}{3} = y_+(0) = \frac{1}{\sqrt{c}} \quad\Rightarrow\quad c = 9 \quad\Rightarrow\quad y(x) = \frac{e^{-x}}{\sqrt{2x + 9}}. \quad \lhd$$

First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

$$\left.\begin{aligned} N = [2x^2y + 2x] \quad &\Rightarrow \quad \partial_x N = 4xy + 2. \\ M = [2xy^2 + 2y] \quad &\Rightarrow \quad \partial_y M = 4xy + 2. \end{aligned}\right\} \;\Rightarrow\; \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function ψ with

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

$$\partial_y \psi = 2x^2y + 2x \quad\Rightarrow\quad \psi(x, y) = x^2y^2 + 2xy + g(x).$$

$$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y \;\Rightarrow\; g'(x) = 0.$$

$$\psi(x, y) = x^2y^2 + 2xy + c, \quad x^2\,y^2(x) + 2x\,y(x) + c = 0. \quad \lhd$$