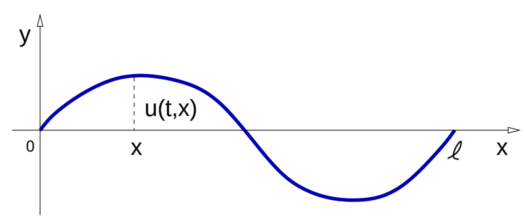
Overview of Fourier Series (Sect. 6.2).

- Origins of the Fourier Series.
- Periodic functions.
- Orthogonality of Sines and Cosines.
- Main result on Fourier Series.

Origins of the Fourier Series.

Summary:

Daniel Bernoulli ($\sim$ 1750) found solutions to the equation that describes waves propagating on a vibrating string.



The function u, measuring the vertical displacement of the string, is the solution to the wave equation,

$$\partial_t^2 u(t,x) = v^2\, \partial_x^2 u(t,x), \quad v \in \mathbb{R}, \quad x \in [0,L], \quad t \in [0,\infty),$$

with initial conditions,

$$u(0,x) = f(x), \qquad \partial_t u(0,x) = 0,$$

and boundary conditions,

$$u(t,0) = 0, \qquad u(t,L) = 0.$$

Origins of the Fourier Series.

Summary:

Bernoulli found particular solutions to the wave equation.

If the initial condition is $f_n(x) = \sin\Bigl(\dfrac{n\pi x}{L}\Bigr)$,

then the solution is $u_n(t,x) = \sin\Bigl(\dfrac{n\pi x}{L}\Bigr)\,\cos\Bigl(\dfrac{vn\pi t}{L}\Bigr)$.

Bernoulli also realized that

$$U_N(t,x) = \sum_{n=1}^{N} a_n \sin\Bigl(\frac{n\pi x}{L}\Bigr)\,\cos\Bigl(\frac{vn\pi t}{L}\Bigr), \qquad a_n \in \mathbb{R}$$

is also solution of the wave equation with initial condition

$$F_N(x) = \sum_{n=1}^{N} a_n \sin\Bigl(\frac{n\pi x}{L}\Bigr).$$

Remark: The wave equation and its solutions provide a mathematical description of music.

Origins of the Fourier Series.

Remarks:

- Bernoulli claimed he had obtained all solutions to the problem above for the wave equation.
- However, he did not prove that claim.
- A proof is: Given a function F with $F(0) = F(L) = 0$, but otherwise arbitrary, find N and the coefficients a_n such that F is approximated by an expansion F_N given in the previous slide.
- Joseph Fourier (~ 1800) provided such formula for the coefficients a_n, while studying a different problem:
  The heat transport in a solid material.
- Find the temperature function u solution of the heat equation

$$\partial_t u(t,x) = k\,\partial_x^2 u(t,x), \quad k > 0, \quad x \in [0,L], \quad t \in [0,\infty),$$

$$\text{I.C.} \quad u(0,x) = f(x),$$

$$\text{B.C.} \quad u(t,0) = 0, \quad u(t,L) = 0.$$

Origins of the Fourier Series.

Remarks:

Fourier found particular solutions to the heat equation.

If the initial condition is $f_n(x) = \sin\left(\frac{n\pi x}{L}\right)$,

then the solution is $u_n(t,x) = \sin\left(\frac{n\pi x}{L}\right) e^{-k(\frac{n\pi}{L})^2 t}$.

Fourier also realized that

$$U_N(t,x) = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(\frac{n\pi}{L})^2 t}, \qquad a_n \in \mathbb{R}$$

is also solution of the heat equation with initial condition

$$F_N(x) = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi x}{L}\right).$$

Remark: The heat equation and its solutions provide a mathematical description of heat transport in a solid material.

Origins of the Fourier Series.

Remarks:

- However, Fourier went farther than Bernoulli. Fourier found a formula for the coefficients a_n in terms of the function F.
- Given an initial data function F, satisfying $F(0) = F(L) = 0$, but otherwise arbitrary, Fourier proved that one can construct an expansion F_N as follows,

$$F_N(x) = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi x}{L}\right),$$

  for N any positive integer, where the a_n are given by

$$a_n = \frac{2}{L} \int_0^L F(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

- To find all solutions to the heat equation problem above one must prove one more thing: That F_N approximates F for large enough N. That is, $\lim_{N\to\infty} F_N = F$. Fourier didn't show this.

Origins of the Fourier Series.

Remarks:

- Based on Bernoulli and Fourier works, people have been able to prove that. Every continuous, τ-periodic function can be expressed as an infinite linear combination of sine and cosine functions.
- More precisely: Every continuous, τ-periodic function F, there exist constants a_0, a_n, b_n, for $n = 1, 2, \cdots$ such that

$$F_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \Big[a_n \cos\Big(\frac{2n\pi x}{\tau}\Big) + b_n \sin\Big(\frac{2n\pi x}{\tau}\Big) \Big],$$

satisfies $\lim_{N\to\infty} F_N(x) = F(x)$ for every $x \in \mathbb{R}$.

Notation: $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{2n\pi x}{\tau}\Big) + b_n \sin\Big(\frac{2n\pi x}{\tau}\Big) \Big].$

Origins of the Fourier Series.

The main problem in our class:
Given a continuous, τ-periodic function f, find the formulas for a_n and b_n such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{2n\pi x}{\tau}\Big) + b_n \sin\Big(\frac{2n\pi x}{\tau}\Big) \Big].$$

Remarks: We need to review two main concepts:

- The notion of periodic functions.
- The notion of orthogonal functions, in particular the orthogonality of Sines and Cosines.

Fourier Series (Sect. 6.2).

- Origins of the Fourier Series.
- **Periodic functions.**
- Orthogonality of Sines and Cosines.
- Main result on Fourier Series.

Periodic functions.

Definition
A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* iff there exists $\tau > 0$ such that for all $x \in \mathbb{R}$ holds

$$f(x + \tau) = f(x).$$

Remark: f is invariant under translations by τ.

Definition
A *period* T of a periodic function f is the smallest value of τ such that $f(x + \tau) = f(x)$ holds.

Notation:
A periodic function with period T is also called T-periodic.

Periodic functions.

Example

The following functions are periodic, with period T,

$$f(x) = \sin(x), \qquad T = 2\pi.$$

$$f(x) = \cos(x), \qquad T = 2\pi.$$

$$f(x) = \tan(x), \qquad T = \pi.$$

$$f(x) = \sin(ax), \qquad T = \frac{2\pi}{a}.$$

The proof of the latter statement is the following:

$$f\left(x + \frac{2\pi}{a}\right) = \sin\left(ax + a\frac{2\pi}{a}\right) = \sin(ax + 2\pi) = \sin(ax) = f(x).$$

$\triangleleft$

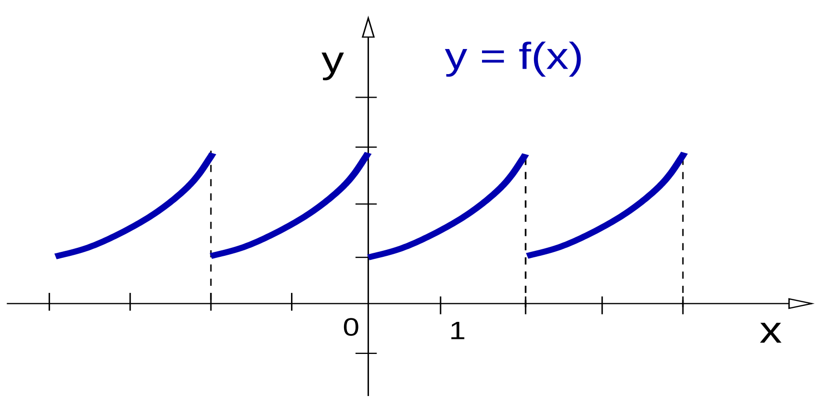
Periodic functions.

Example

Show that the function below is periodic, and find its period,

$$f(x) = e^x, \qquad x \in [0, 2), \qquad f(x-2) = f(x).$$

Solution: We just graph the function,



So the function is periodic with period $T = 2$. $\triangleleft$

Periodic functions.

Theorem

A linear combination of T-periodic functions is also T-periodic.

Proof: If $f(x+T) = f(x)$ and $g(x+T) = g(x)$, then

$$af(x+T) + bg(x+T) = af(x) + bg(x),$$

so $(af + bg)$ is also T-periodic. □

Example

$f(x) = 2\sin(3x) + 7\cos(3x)$ is periodic with period $T = 2\pi/3$. ◁

Remark: The functions below are periodic with period $T = \frac{\tau}{n}$,

$$f(x) = \cos\left(\frac{2\pi nx}{\tau}\right), \quad g(x) = \sin\left(\frac{2\pi nx}{\tau}\right),$$

Since f and g are invariant under translations by τ/n, they are also invariant under translations by τ.

Periodic functions.

Corollary

Any function f given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi x}{\tau}\right) + b_n \sin\left(\frac{2n\pi x}{\tau}\right)\right]$$

is periodic with period τ.

Remark: We will show that the converse statement is true.

Theorem

A function f is τ-periodic iff holds

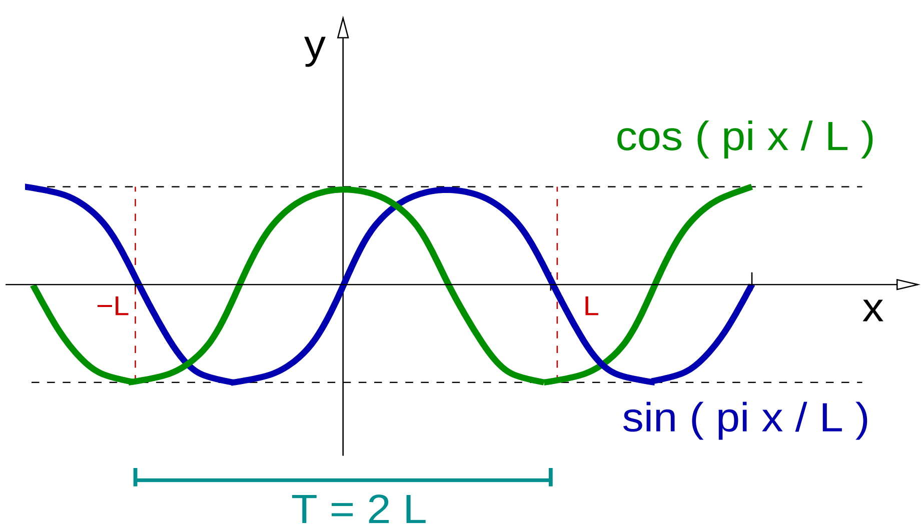
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi x}{\tau}\right) + b_n \sin\left(\frac{2n\pi x}{\tau}\right)\right].$$

Fourier Series (Sect. 6.2).

- Origins of the Fourier Series.
- Periodic functions.
- **Orthogonality of Sines and Cosines.**
- Main result on Fourier Series.

Orthogonality of Sines and Cosines.

Remark:
From now on we work on the following domain: $[-L, L]$.



Orthogonality of Sines and Cosines.

Theorem (Orthogonality)

The following relations hold for all n, $m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big)\,\sin\Big(\frac{m\pi x}{L}\Big)\,dx = \begin{cases} 0 & n \neq m, \\ L & n = m, \end{cases}$$

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\sin\Big(\frac{m\pi x}{L}\Big)\,dx = 0.$$

Remark:

- The operation $f \cdot g = \int_{-L}^{L} f(x)\,g(x)\,dx$ is an inner product in the vector space of functions. Like the dot product is in $\mathbb{R}^2$.
- Two functions f, g, are orthogonal iff $f \cdot g = 0$.

Orthogonality of Sines and Cosines.

Recall:
$$\cos(\theta)\,\cos(\phi) = \frac{1}{2}\big[\cos(\theta+\phi) + \cos(\theta-\phi)\big];$$
$$\sin(\theta)\,\sin(\phi) = \frac{1}{2}\big[\cos(\theta-\phi) - \cos(\theta+\phi)\big];$$
$$\sin(\theta)\,\cos(\phi) = \frac{1}{2}\big[\sin(\theta+\phi) + \sin(\theta-\phi)\big].$$

Proof: First formula: If $n = m = 0$, it is simple to see that

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \int_{-L}^{L} dx = 2L.$$

In the case where one of n or m is non-zero, use the relation

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx$$
$$+\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

Orthogonality of Sines and Cosines.

Proof: Since one of n or m is non-zero, holds

$$\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\,dx = \frac{L}{2(n+m)\pi}\,\sin\Big[\frac{(n+m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

We obtain that

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\,\cos\Big(\frac{m\pi x}{L}\Big)\,dx = \frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx.$$

If we further restrict $n \neq m$, then

$$\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx = \frac{L}{2(n-m)\pi}\,\sin\Big[\frac{(n-m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

If $n = m \neq 0$, we have that

$$\frac{1}{2}\int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\,dx = \frac{1}{2}\int_{-L}^{L} dx = L.$$

This establishes the first equation in the Theorem. The remaining equations are proven in a similar way. □

Overview of Fourier Series (Sect. 6.2).

- Origins of the Fourier Series.
- Periodic functions.
- Orthogonality of Sines and Cosines.
- **Main result on Fourier Series.**

Main result on Fourier Series.

Theorem (Fourier Series)

If the function $f : [-L, L] \subset \mathbb{R} \to \mathbb{R}$ is continuous, then f can be expressed as an infinite series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big] \qquad (1)$$

with the constants a_n and b_n given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n \geqslant 1.$$

*Furthermore, the Fourier series in Eq. (**??**) provides a $2L$-periodic extension of f from the domain $[-L, L] \subset \mathbb{R}$ to $\mathbb{R}$.*