Linear Algebra and differential systems (Sect. 5.4, 5.5, 5.6)

- **Eigenvalues, eigenvectors of a matrix (5.5).**
- Computing eigenvalues and eigenvectors (5.5).
- Diagonalizable matrices (5.5).
- $n \times n$ linear differential systems (5.4).
- Constant coefficients homogenoues systems (5.6).
- Examples: 2×2 linear systems (5.6).

Eigenvalues, eigenvectors of a matrix

Definition

A number λ and a non-zero n-vector $\mathbf{v}$ are respectively called an *eigenvalue* and *eigenvector* of an $n \times n$ matrix A iff the following equation holds,

$$A\mathbf{v} = \lambda \mathbf{v}\,.$$

Example

Verify that the pair $\lambda_1 = 4$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = -2$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvalue and eigenvector pairs of matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: $A\mathbf{v}_1 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \mathbf{v}_1$.

$A\mathbf{v}_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_2 \mathbf{v}_2$. $\triangleleft$

Eigenvalues, eigenvectors of a matrix

Remarks:

- If we interpret an $n \times n$ matrix A as a function $A : \mathbb{R}^n \to \mathbb{R}^n$, then the eigenvector $\mathbf{v}$ determines a particular *direction* on $\mathbb{R}^n$ where the action of A is *simple*: $A\mathbf{v}$ is proportional to $\mathbf{v}$.
- Matrices usually change the direction of the vector, like

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

- This is not the case for eigenvectors, like

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

Eigenvalues, eigenvectors of a matrix

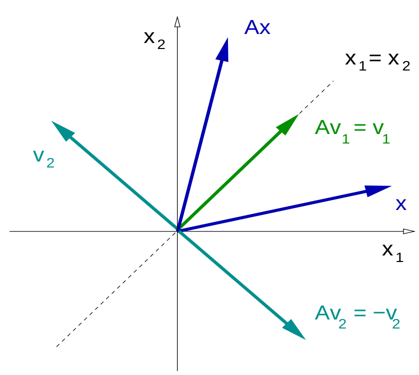
Example

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution:

The function $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a reflection along $x_1 = x_2$ axis.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$



The line $x_1 = x_2$ is invariant under A. Hence,

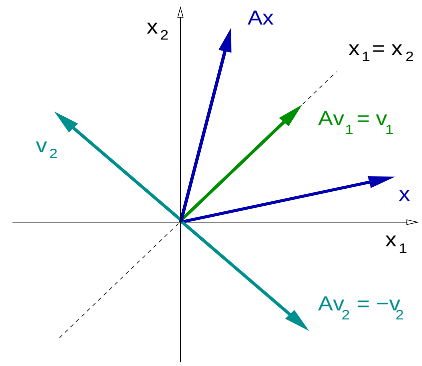
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = 1.$$

An eigenvalue eigenvector pair is: $\lambda_1 = 1$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Eigenvalues, eigenvectors of a matrix

Example

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.



Solution: Eigenvalue eigenvector pair:

$$\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

A second eigenvector eigenvalue pair is:

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \lambda_2 = -1.$$

A second eigenvalue eigenvector pair: $\lambda_2 = -1$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $\triangleleft$

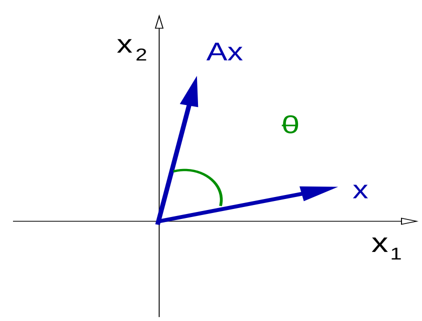
Eigenvalues, eigenvectors of a matrix

Remark: Not every $n \times n$ matrix has real eigenvalues.

Example

Fix $\theta \in (0, \pi)$ and define $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

Show that A has no real eigenvalues.



Solution: Matrix $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation by θ counterclockwise.

There is no direction left invariant by the function A.

We conclude: Matrix A has no eigenvalues eigenvector pairs. $\triangleleft$

Remark:

Matrix A has complex-values eigenvalues and eigenvectors.

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Computing eigenvalues and eigenvectors.

Problem:
Given an $n \times n$ matrix A, find, if possible, λ and $\mathbf{v} \neq \mathbf{0}$ solution of

$$A\mathbf{v} = \lambda\,\mathbf{v}.$$

Remark:
This is more complicated than solving a linear system $A\mathbf{v} = \mathbf{b}$, since in our case we do not know the source vector $\mathbf{b} = \lambda\mathbf{v}$.

Solution:

(a) First solve for λ.

(b) Having λ, then solve for $\mathbf{v}$.

Computing eigenvalues and eigenvectors.

Theorem (Eigenvalues-eigenvectors)

(a) *The number λ is an eigenvalue of an $n \times n$ matrix A iff*

$$\det(A - \lambda I) = 0.$$

(b) *Given an eigenvalue λ of matrix A, the corresponding eigenvectors $\mathbf{v}$ are the non-zero solutions to the homogeneous linear system*

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Notation:
$p(\lambda) = \det(A - \lambda I)$ is called the *characteristic polynomial*.
If A is $n \times n$, then p is degree n.

Remark: An eigenvalue is a root of the characteristic polynomial.

Computing eigenvalues and eigenvectors.

Proof:
Find λ such that for a non-zero vector $\mathbf{v}$ holds,

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Leftrightarrow \quad (A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Recall, $\mathbf{v} \neq \mathbf{0}$.

This last condition implies that matrix $(A - \lambda I)$ is not invertible.

(Proof: If $(A - \lambda I)$ invertible, then $(A - \lambda I)^{-1}(A - \lambda I)\mathbf{v} = \mathbf{0}$, that is, $\mathbf{v} = \mathbf{0}$.)

Since $(A - \lambda I)$ is not invertible, then $\det(A - \lambda I) = 0$.

Once λ is known, the original eigenvalue-eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $(A - \lambda I)\mathbf{v} = \mathbf{0}$. $\square$

Computing eigenvalues and eigenvectors.

Example

Find the eigenvalues λ and eigenvectors $\mathbf{v}$ of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution:
The eigenvalues are the roots of the characteristic polynomial.

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{bmatrix}$$

The characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9$$

The roots are $\lambda_+ = 4$ and $\lambda_- = -2$.
Compute the eigenvector for $\lambda_+ = 4$. Solve $(A - 4I)\mathbf{v}_+ = \mathbf{0}$.

$$A - 4I = \begin{bmatrix} 1-4 & 3 \\ 3 & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}.$$

Computing eigenvalues and eigenvectors.

Example

Find the eigenvalues λ and eigenvectors $\mathbf{v}$ of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = 4$, $\lambda_- = -2$, $A - 4I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$.
We solve $(A - 4I)\mathbf{v}_+ = \mathbf{0}$, using Gauss elimination,

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^+ = v_2^+, \\ v_2^+ \quad \text{free.} \end{cases}$$

Al solutions to the equation above are then given by

$$\mathbf{v}_+ = \begin{bmatrix} v_2^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2^+ \quad \Rightarrow \quad \mathbf{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

The first eigenvalue eigenvector pair is $\lambda_+ = 4$, $\mathbf{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Computing eigenvalues and eigenvectors.

Example

Find the eigenvalues λ and eigenvectors $\mathbf{v}$ of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = 4$, $\mathbf{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda_- = -2$.

Solve $(A + 2I)\mathbf{v}_- = \mathbf{0}$, using Gauss operations on $A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^- = -v_2^-, \\ v_2^- \quad \text{free}. \end{cases}$$

Al solutions to the equation above are then given by

$$\mathbf{v}_- = \begin{bmatrix} -v_2^- \\ v_2^- \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2^- \quad \Rightarrow \quad \mathbf{v}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

The second eigenvalue eigenvector pair: $\lambda_- = -2$, $\mathbf{v}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $\triangleleft$

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Diagonalizable matrices.

Definition

An $n \times n$ matrix D is called *diagonal* iff $D = \begin{bmatrix} d_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_{nn} \end{bmatrix}$.

Definition

An $n \times n$ matrix A is called *diagonalizable* iff there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

Remark:

- Systems of linear *differential* equations are simple to solve in the case that the coefficient matrix A is diagonalizable.
- In such case, it is simple to *decouple* the differential equations.
- One solves the decoupled equations, and then transforms back to the original unknowns.

Diagonalizable matrices.

Theorem (Diagonalizability and eigenvectors)

An $n \times n$ matrix A is diagonalizable iff matrix A has a linearly independent set of n eigenvectors. Furthermore,

$$A = PDP^{-1}, \quad P = [\mathbf{v}_1, \cdots, \mathbf{v}_n], \quad D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix},$$

where $\lambda_i, \mathbf{v}_i$, for $i = 1, \cdots, n$, are eigenvalue-eigenvector pairs of A.

Remark: It is not simple to know whether an $n \times n$ matrix A has a linearly independent set of n eigenvectors. One simple case is given in the following result.

Theorem (n different eigenvalues)

If an $n \times n$ matrix A has n different eigenvalues, then A is diagonalizable.

Diagonalizable matrices.

Example

Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Solution: We known that the eigenvalue eigenvector pairs are

$$\lambda_1 = 4, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -2, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Introduce P and D as follows,

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

Then

$$PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Diagonalizable matrices.

Example

Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Solution: Recall:

$$PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$PDP^{-1} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

We conclude,

$$PDP^{-1} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = A,$$

that is, A is diagonalizable. $\triangleleft$

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$n \times n$ linear differential systems (5.4).

Definition

An *$n \times n$ linear differential system* is a the following: Given an $n \times n$ matrix-valued function A, and an n-vector-valued function $\mathbf{b}$, find an n-vector-valued function $\mathbf{x}$ solution of

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t).$$

The system above is called *homogeneous* iff holds $\mathbf{b} = 0$.

Recall:

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad \mathbf{b}(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix}, \quad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t) \;\Leftrightarrow\; \begin{aligned} x_1' &= a_{11}(t)\,x_1 + \cdots + a_{1n}(t)\,x_n + b_1(t) \\ &\;\vdots \\ x_n' &= a_{n1}(t)\,x_1 + \cdots + a_{nn}(t)\,x_n + b_n(t). \end{aligned}$$

$n \times n$ linear differential systems (5.4).

Example

Find the explicit expression for the linear system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ in the case that

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad \mathbf{b}(t) = \begin{bmatrix} e^t \\ 2e^{3t} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: The 2×2 linear system is given by

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^t \\ 2e^{3t} \end{bmatrix}.$$

That is,

$$x_1'(t) = x_1(t) + 3x_2(t) + e^t,$$
$$x_2'(t) = 3x_1(t) + x_2(t) + 2e^{3t}.$$

◁

$n \times n$ linear differential systems (5.4).

Remark: Derivatives of vector-valued functions are computed component-wise.

$$\mathbf{x}'(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}' = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

Example

Compute $\mathbf{x}'$ for $\mathbf{x}(t) = \begin{bmatrix} e^{2t} \\ \sin(t) \\ \cos(t) \end{bmatrix}$.

Solution:

$$\mathbf{x}'(t) \begin{bmatrix} e^{2t} \\ \sin(t) \\ \cos(t) \end{bmatrix}' = \begin{bmatrix} 2e^{2t} \\ \cos(t) \\ -\sin(t) \end{bmatrix}.$$

◁

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- Examples: 2×2 linear systems (5.6).

Constant coefficients homogenoues systems (5.6).

Summary:

- Given an $n \times n$ matrix $A(t)$, n-vector $\mathbf{b}(t)$, find $\mathbf{x}(t)$ solution

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{b}(t).$$

- The system is *homogeneous* iff $\mathbf{b} = 0$, that is,

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t).$$

- The system has *constant coefficients* iff matrix A does not depend on t, that is,

$$\mathbf{x}'(t) = A\,\mathbf{x}(t) + \mathbf{b}(t).$$

- We study homogeneous, constant coefficient systems, that is,

$$\mathbf{x}'(t) = A\,\mathbf{x}(t).$$

Constant coefficients homogenoues systems (5.6).

Theorem (Diagonalizable matrix)

If $n \times n$ matrix A is diagonalizable, with a linearly independent eigenvectors set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution $\mathbf{x}$ to the homogeneous, constant coefficients, linear system

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is given by the expression below, where $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n\, e^{\lambda_n t}.$$

Remark:

- The differential system for the variable $\mathbf{x}$ is coupled, that is, A is not diagonal.
- We transform the system into a system for a variable $\mathbf{y}$ such that the system for $\mathbf{y}$ is decoupled, that is, $\mathbf{y}'(t) = D\,\mathbf{y}(t)$, where D is a diagonal matrix.
- We solve for $\mathbf{y}(t)$ and we transform back to $\mathbf{x}(t)$.

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Examples: 2×2 linear systems (5.6).

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: Find eigenvalues and eigenvectors of A. We found that:

$$\lambda_1 = 4, \quad \mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_2 = -2, \quad \mathbf{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Fundamental solutions are

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

The general solution is $\mathbf{x}(t) = c_1\, \mathbf{x}^{(1)}(t) + c_2\, \mathbf{x}^{(2)}(t)$, that is,

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}, \qquad c_1, c_2 \in \mathbb{R}. \qquad \triangleleft$$

Examples: 2×2 linear systems (5.6).

Example

Verify that $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$, and $\mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$ are solutions to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: We compute $\mathbf{x}^{(1)\prime}$ and then we compare it with $A\mathbf{x}^{(1)}$,

$$\mathbf{x}^{(1)\prime}(t) = \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix}' = \begin{bmatrix} 4e^{4t} \\ 4e^{4t} \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \quad \Rightarrow \quad \mathbf{x}^{(1)\prime} = 4\mathbf{x}^{(1)}.$$

$$A\mathbf{x}^{(1)} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} e^{4t} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \Rightarrow A\mathbf{x}^{(1)} = 4\mathbf{x}^{(1)}.$$

We conclude that $\mathbf{x}^{(1)\prime} = A\mathbf{x}^{(1)}$.

Examples: 2×2 linear systems (5.6).

Example

Verify that $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$, and $\mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$ are solutions to $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: We compute $\mathbf{x}^{(2)\prime}$ and then we compare it with $A\mathbf{x}^{(2)}$,

$$\mathbf{x}^{(2)\prime} = \begin{bmatrix} -e^{-2t} \\ e^{-2t} \end{bmatrix}' = \begin{bmatrix} 2e^{-2t} \\ -2e^{-2t} \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} \Rightarrow \mathbf{x}^{(2)\prime} = -2\mathbf{x}^{(2)}.$$

$$A\mathbf{x}^{(2)} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-2t} = -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t},$$

So, $A\mathbf{x}^{(2)} = -2\mathbf{x}^{(2)}$. Hence, $\mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)}$. $\triangleleft$

Examples: 2×2 linear systems (5.6).

Example

Solve the IVP $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: The general solution: $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$.

The initial condition is,

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We need to solve the linear system

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, hence $\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$. $\triangleleft$

Constant coefficients homogenoues systems (5.6).

Proof: Since A is diagonalizable, we know that $A = PDP^{-1}$, with

$$P = \big[\mathbf{v}_1, \cdots, \mathbf{v}_n\big], \qquad D = \operatorname{diag}\big[\lambda_1, \cdots, \lambda_n\big].$$

Equivalently, $P^{-1}AP = D$. Multiply $\mathbf{x}' = A\,\mathbf{x}$ by P^{-1} on the left

$$P^{-1}\mathbf{x}'(t) = P^{-1}A\,\mathbf{x}(t) \quad \Leftrightarrow \quad \big(P^{-1}\mathbf{x}\big)' = \big(P^{-1}AP\big)\,\big(P^{-1}\mathbf{x}\big).$$

Introduce the new unknown $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, then

$$\mathbf{y}'(t) = D\,\mathbf{y}(t) \;\Leftrightarrow\; \begin{cases} y_1'(t) = \lambda_1\, y_1(t), \\ \quad \vdots \\ y_n'(t) = \lambda_n\, y_n(t), \end{cases} \quad \Rightarrow \; \mathbf{y}(t) = \begin{bmatrix} c_1\, e^{\lambda_1 t} \\ \vdots \\ c_n\, e^{\lambda_n t} \end{bmatrix}.$$

Constant coefficients homogenoues systems (5.6).

Proof: Recall: $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$, and $\mathbf{y}(t) = \begin{bmatrix} c_1\, e^{\lambda_1 t} \\ \vdots \\ c_n\, e^{\lambda_n t} \end{bmatrix}$.

Transform back to $\mathbf{x}(t)$, that is,

$$\mathbf{x}(t) = P\,\mathbf{y}(t) = \big[\mathbf{v}_1, \cdots, \mathbf{v}_n\big] \begin{bmatrix} c_1\, e^{\lambda_1 t} \\ \vdots \\ c_n\, e^{\lambda_n t} \end{bmatrix}$$

We conclude: $\mathbf{x}(t) = c_1\mathbf{v}_1\, e^{\lambda_1 t} + \cdots + c_n\mathbf{v}_n\, e^{\lambda_n t}$. □

Remark:

- $A\,\mathbf{v}_i = \lambda_i \mathbf{v}_i$.
- The eigenvalues and eigenvectors of A are crucial to solve the differential linear system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$.