Convolution solutions (Sect. 4.5).

- Convolution of two functions.
- Properties of convolutions.
- Laplace Transform of a convolution.
- Impulse response solution.
- Solution decomposition theorem.
Convolution of two functions.

Definition
The convolution of piecewise continuous functions \( f, g : \mathbb{R} \to \mathbb{R} \) is the function \( f \ast g : \mathbb{R} \to \mathbb{R} \) given by

\[
(f \ast g)(t) = \int_0^t f(\tau)g(t-\tau) \, d\tau.
\]

Remarks:

\( f \ast g \) is also called the generalized product of \( f \) and \( g \).

The definition of convolution of two functions also holds in the case that one of the functions is a generalized function, like Dirac’s delta.

Example
Find the convolution of \( f(t) = e^{-t} \) and \( g(t) = \sin(t) \).

Solution: By definition: \( (f \ast g)(t) = \int_0^t e^{-\tau} \sin(t-\tau) \, d\tau \).

Integrate by parts twice: \[
\int_0^t e^{-\tau} \sin(t-\tau) \, d\tau = \left[ e^{-\tau} \cos(t-\tau) \right]_0^t - \left[ e^{-\tau} \sin(t-\tau) \right]_0^t - \int_0^t e^{-\tau} \sin(t-\tau) \, d\tau,
\]

\[
2 \int_0^t e^{-\tau} \sin(t-\tau) \, d\tau = \left[ e^{-\tau} \cos(t-\tau) \right]_0^t - \left[ e^{-\tau} \sin(t-\tau) \right]_0^t,
\]

\[
2(f \ast g)(t) = e^{-t} - \cos(t) - 0 + \sin(t).
\]

We conclude: \( (f \ast g)(t) = \frac{1}{2} [e^{-t} + \sin(t) - \cos(t)] \).  

\[\triangle\]
Properties of convolutions.

Theorem (Properties)

For every piecewise continuous functions $f$, $g$, and $h$, hold:

(i) Commutativity: $f * g = g * f$;

(ii) Associativity: $f * (g * h) = (f * g) * h$;

(iii) Distributivity: $f * (g + h) = f * g + f * h$;

(iv) Neutral element: $f * 0 = 0$;

(v) Identity element: $f * \delta = f$.

Proof:

(v):

$$(f * \delta)(t) = \int_{0}^{t} f(\tau) \delta(t - \tau) \, d\tau = \int_{0}^{t} f(\tau) \delta(\tau - t) \, d\tau = f(t).$$
Properties of convolutions.

Proof:
(1): Commutativity: \( f \ast g = g \ast f \).

The definition of convolution is,
\[
(f \ast g)(t) = \int_0^t f(\tau) g(t - \tau) \, d\tau.
\]

Change the integration variable: \( \hat{\tau} = t - \tau \), hence \( d\hat{\tau} = -d\tau \),
\[
(f \ast g)(t) = \int_t^0 f(t - \hat{\tau}) g(\hat{\tau})(-1) \, d\hat{\tau}
\]
\[
(f \ast g)(t) = \int_0^t g(\hat{\tau}) f(t - \hat{\tau}) \, d\hat{\tau}
\]

We conclude: \( (f \ast g)(t) = (g \ast f)(t) \). \( \square \)

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Laplace Transform of a convolution.

Theorem (Laplace Transform)
If \( f, g \) have well-defined Laplace Transforms \( \mathcal{L}[f], \mathcal{L}[g] \), then
\[
\mathcal{L}[f * g] = \mathcal{L}[f] \mathcal{L}[g].
\]

Proof: The key step is to interchange two integrals. We start with the product of the Laplace transforms,
\[
\mathcal{L}[f] \mathcal{L}[g] = \left[ \int_0^\infty e^{-st} f(t) \, dt \right] \left[ \int_0^\infty e^{-s\tilde{t}} g(\tilde{t}) \, d\tilde{t} \right],
\]
\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-s\tilde{t}} g(\tilde{t}) \left( \int_0^\infty e^{-st} f(t) \, dt \right) d\tilde{t},
\]
\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_0^\infty e^{-s(t+\tilde{t})} f(t) \, dt \right) d\tilde{t}.
\]

Change variables: \( \tau = t + \tilde{t} \), hence \( d\tau = dt \);
\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left( \int_{\tilde{t}}^{\infty} e^{-s\tau} f(\tau - \tilde{t}) \, d\tau \right) d\tilde{t}.
\]

The key step: Switch the order of integration.
\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty \int_{\tilde{t}}^{\infty} e^{-s\tau} g(\tilde{t}) f(\tau - \tilde{t}) \, d\tau \, d\tilde{t}.
\]
Laplace Transform of a convolution.

Proof: Recall: \( \mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s \tau} g(\tilde{t}) f(\tau - \tilde{t}) d\tilde{t} d\tau. \)

Then, is straightforward to check that

\[
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-s \tau} \left( \int_0^\tau g(\tilde{t}) f(\tau - \tilde{t}) d\tilde{t} \right) d\tau, \\
\mathcal{L}[f] \mathcal{L}[g] = \int_0^\infty e^{-s \tau} (g * f)(\tau) d\tau \\
\mathcal{L}[f] \mathcal{L}[g] = \mathcal{L}[g * f]
\]

We conclude: \( \mathcal{L}[f * g] = \mathcal{L}[f] \mathcal{L}[g]. \)

\[\square\]

Laplace Transform of a convolution.

Example

Use convolutions to find the inverse Laplace Transform of 

\[ F(s) = \frac{3}{s^3(s^2 - 3)}. \]

Solution: We express \( F \) as a product of two Laplace Transforms,

\[ F(s) = 3 \frac{1}{s^3} \frac{1}{s^2 - 3} = \frac{3}{2} \frac{1}{\sqrt{3}} \left( \frac{2}{s^3} \right) \left( \frac{\sqrt{3}}{s^2 - 3} \right) \]

Recalling that \( \mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \) and \( \mathcal{L}[\sinh(at)] = \frac{a}{s^2 - a^2}, \)

\[ F(s) = \frac{\sqrt{3}}{2} \mathcal{L}[t^2] \mathcal{L}[\sinh(\sqrt{3} t)] = \frac{\sqrt{3}}{2} \mathcal{L}[t^2 * \sin(\sqrt{3} t)]. \]

We conclude that \( f(t) = \frac{\sqrt{3}}{2} \int_0^t \tau^2 \sinh(\sqrt{3}(t - \tau)) \) d\tau. \[\triangleq\]
Example
Compute $L[f(t)]$ where $f(t) = \int_0^t e^{-3(t-\tau)} \cos(2\tau) \, d\tau$.

Solution: The function $f$ is the convolution of two functions,

$$f(t) = (g * h)(t), \quad g(t) = \cos(2t), \quad h(t) = e^{-3t}.$$ 

Since $L[(g * h)(t)] = L[g(t)] L[h(t)]$, then,

$$F(s) = L \left[ \int_0^t e^{-3(t-\tau)} \cos(2\tau) \, d\tau \right] = L \left[ e^{-3t} \right] L \left[ \cos(2t) \right].$$

We conclude that $F(s) = \frac{s}{(s + 3)(s^2 + 4)}$. ◯

Example
Solve the IVP

$$y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0.$$  

Solution: Denote $G(s) = L[g(t)]$ and compute LT of the equation,

$$(s^2 - 5s + 6) L[y(t)] = L[g(t)] \Rightarrow L[y(t)] = \frac{1}{(s^2 - 5s + 6)} G(s).$$

Denoting $H(s) = \frac{1}{s^2 - 5s + 6}$, and $h(t) = L^{-1} [H(s)]$, then

$$L[y(t)] = H(s) \cdot G(s) \Rightarrow y(t) = (h * g)(t).$$

Function $h$ is simple to compute:

$$H(s) = \frac{1}{(s - 2)(s - 3)} = \frac{a}{(s - 2)} + \frac{b}{(s - 3)} = \frac{a(s - 3) + b(s - 2)}{(s - 2)(s - 3)}.$$
Laplace Transform of a convolution.

Example
Solve the IVP
\[ y'' - 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Then: \( 1 = a(s - 3) + b(s - 2) \). Evaluate at \( s = 2, \ 3 \).

\[ s = 2 \Rightarrow a = -1. \quad s = 3 \Rightarrow b = 1. \]

Therefore \( H(s) = -\frac{1}{(s - 2)} + \frac{1}{(s - 3)} \). Then
\[ h(t) = -e^{2t} + e^{3t}. \]

Recalling the formula \( y(t) = (h \ast g)(t) \), we get
\[ y(t) = \int_0^t (-e^{2\tau} + e^{3\tau}) g(t - \tau) \, d\tau. \]

Convolution solutions (Sect. 4.5).

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- **Impulse response solution.**
- Solution decomposition theorem.
Impulse response solution.

**Definition**

The *impulse response solution* is the solution $y_\delta$ to the IVP

$$y''_\delta + a_1 y'_\delta + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0.$$ 

**Computing Laplace Transforms,**

$$L[y_\delta] = \frac{1}{s^2 + a_1 s + a_0}.$$ 

Denoting the characteristic polynomial by $p(s) = s^2 + a_1 s + a_0$,

$$y_\delta = L^{-1}\left[\frac{1}{p(s)}\right].$$

**Summary:** The impulse response solution is the inverse Laplace Transform of the reciprocal of the equation characteristic polynomial.

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Impulse response solution.

**Recall:** The impulse response solution is $y_\delta$ solution of the IVP

$$y''_\delta + a_1 y'_\delta + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0.$$ 

**Example**

Find the solution (impulse response at $t = c$) of the IVP

$$y''_{\delta_c} + 2 y'_{\delta_c} + 2 y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.$$ 

**Solution:**

$$L[y''_{\delta_c}] + 2 L[y'_{\delta_c}] + 2 L[y_{\delta_c}] = L[\delta(t - c)].$$

$$(s^2 + 2s + 2) L[y_{\delta_c}] = e^{-cs} \quad \Rightarrow \quad L[y_{\delta_c}] = \frac{e^{-cs}}{s^2 + 2s + 2}.$$
Impulse response solution.

Example
Find the solution (impulse response at \( t = c \)) of the IVP
\[
y''_{\delta_c} + 2y'_{\delta_c} + 2y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.
\]

Solution: Recall: \( \mathcal{L}[y_{\delta_c}] = \frac{e^{-cs}}{(s^2 + 2s + 2)}. \)

Find the roots of the denominator,
\[
s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s = \frac{1}{2} [-2 \pm \sqrt{4 - 8}]
\]
Complex roots. We complete the square:
\[
s^2 + 2s + 2 = \left[ s^2 + 2\left(\frac{2}{2}\right)s + 1 \right] - 1 + 2 = (s + 1)^2 + 1.
\]
Therefore, \( \mathcal{L}[y_{\delta_c}] = \frac{e^{-cs}}{(s + 1)^2 + 1}. \)

Impulse response solution.

Example
Find the solution (impulse response at \( t = c \)) of the IVP
\[
y''_{\delta_c} + 2y'_{\delta_c} + 2y_{\delta_c} = \delta(t - c), \quad y_{\delta_c}(0) = 0, \quad y'_{\delta_c}(0) = 0, \quad c \in \mathbb{R}.
\]

Solution: Recall: \( \mathcal{L}[y_{\delta_c}] = \frac{e^{-cs}}{(s + 1)^2 + 1}. \)

Recall: \( \mathcal{L}[\sin(t)] = \frac{1}{s^2 + 1}, \) and \( \mathcal{L}[f](s - c) = \mathcal{L}[e^{ct} f(t)]. \)

\[
\frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t} \sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_{\delta_c}] = e^{-cs} \mathcal{L}[e^{-t} \sin(t)].
\]

Since \( e^{-cs} \mathcal{L}[f](s) = \mathcal{L}[u(t - c) f(t - c)], \)
we conclude \( y_{\delta_c}(t) = u(t - c) e^{-(t-c)} \sin(t - c). \)
Convolution solutions (Sect. 4.5).

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- **Solution decomposition theorem.**

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Solution decomposition theorem.

**Theorem (Solution decomposition)**

The solution $y$ to the IVP

$$y'' + a_1 y' + a_0 y = g(t), \quad y(0) = y_0, \quad y'(0) = y'_1,$$

can be decomposed as

$$y(t) = y_h(t) + (y_\delta \ast g)(t),$$

where $y_h$ is the solution of the homogeneous IVP

$$y_h'' + a_1 y_h' + a_0 y_h = 0, \quad y_h(0) = y_0, \quad y_h'(0) = y'_1,$$

and $y_\delta$ is the impulse response solution, that is,

$$y_\delta'' + a_1 y_\delta' + a_0 y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$
Example
Use the Solution Decomposition Theorem to express the solution of

\[ y'' + 2y' + 2y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1. \]

Solution: \( \mathcal{L}[y''] + 2 \mathcal{L}[y'] + 2 \mathcal{L}[y] = \mathcal{L}[\sin(at)], \) and recall,

\[ \mathcal{L}[y''] = s^2 \mathcal{L}[y] - s - 1, \quad \mathcal{L}[y'] = s \mathcal{L}[y] - 1. \]

\[ (s^2 + 2s + 2) \mathcal{L}[y] - s + 1 - 2 = \mathcal{L}[\sin(at)]. \]

\[ \mathcal{L}[y] = \frac{(s + 1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)} \mathcal{L}[\sin(at)]. \]
Solution decomposition theorem.

Proof: Compute: \( L[y''] + a_1 L[y'] + a_0 L[y] = L[g(t)] \), and recall,

\[
L[y''] = s^2 L[y] - sy_0 - y_1, \quad L[y'] = s L[y] - y_0.
\]

\[
(s^2 + a_1 s + a_0) L[y] - sy_0 - y_1 - a_1 y_0 = L[g(t)].
\]

\[
L[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)} L[g(t)].
\]

Recall: \( L[y_h] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} \), and \( L[y_0] = \frac{1}{(s^2 + a_1 s + a_0)} \).

Since, \( L[y] = L[y_h] + L[y_0] L[g(t)] \), so \( y(t) = y_h(t) + (y_0 \ast g)(t) \).

Equivalently: \( y(t) = y_h(t) + \int_0^t y_0(\tau) g(t - \tau) d\tau \). \( \square \)