Generalized sources (Sect. 4.4).

- The Dirac delta generalized function.
- Properties of Dirac’s delta.
- Relation between deltas and steps.
- Dirac’s delta in Physics.
- The Laplace Transform of Dirac’s delta.
- Differential equations with Dirac’s delta sources.
The Dirac delta generalized function.

Definition
Consider the sequence of functions for $n \geq 1$,

$$\delta_n(t) = \begin{cases} 
0, & t < 0 \\
 n, & 0 \leq t \leq \frac{1}{n} \\
0, & t > \frac{1}{n}.
\end{cases}$$

The Dirac delta generalized function is given by

$$\lim_{n \to \infty} \delta_n(t) = \delta(t), \quad t \in \mathbb{R}.$$

Remarks:
(a) There exist infinitely many sequences $\delta_n$ that define the same generalized function $\delta$.
(b) For example, compare with the sequences $\delta_n$ in the literature.

The Dirac delta generalized function.

Remarks:
(a) The Dirac $\delta$ is a function on the domain $\mathbb{R} - \{0\}$, and $\delta(t) = 0$ for $t \in \mathbb{R} - \{0\}$.
(b) $\delta$ at $t = 0$ is not defined, since $\delta(0) = \lim_{n \to \infty} n = +\infty$.
(c) $\delta$ is not a function on $\mathbb{R}$. 
Properties of Dirac's delta.

Remark: The Dirac $\delta$ is not a function on $\mathbb{R}$.

We define operations on Dirac's $\delta$ as limits $n \to \infty$ of the operation on the sequence elements $\delta_n$.

Definition

$$\delta(t - c) = \lim_{n \to \infty} \delta_n(t - c),$$

$$a \delta(t) + b \delta(t) = \lim_{n \to \infty} \left[a \delta_n(t) + b \delta_n(t)\right],$$

$$f(t) \delta(t) = \lim_{n \to \infty} \left[f(t) \delta_n(t)\right],$$

$$\int_a^b \delta(t) \, dt = \lim_{n \to \infty} \int_a^b \delta_n(t) \, dt,$$

$$\mathcal{L}[\delta] = \lim_{n \to \infty} \mathcal{L}[\delta_n].$$
Properties of Dirac's delta.

Theorem
\[ \int_{-a}^{a} \delta(t) \, dt = 1, \quad a > 0. \]

Proof:
\[ \int_{-a}^{a} \delta(t) \, dt = \lim_{n \to \infty} \int_{-a}^{a} \delta_n(t) \, dt = \lim_{n \to \infty} \int_{0}^{1/n} n \, dt \]
\[ \int_{-a}^{a} \delta(t) \, dt = \lim_{n \to \infty} \left[ n \left( t \bigg|_{0}^{1/n} \right) \right] = \lim_{n \to \infty} \left[ n \left( \frac{1}{n} \right) \right]. \]

We conclude: \[ \int_{-a}^{a} \delta(t) \, dt = 1. \]

Properties of Dirac's delta.

Theorem
If \( f : \mathbb{R} \to \mathbb{R} \) is continuous, \( t_0 \in \mathbb{R} \) and \( a > 0 \), then
\[ \int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) \, dt = f(t_0). \]

Proof: Introduce the change of variable \( \tau = t - t_0 \),
\[ I = \int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) \, dt = \int_{-a}^{a} \delta(\tau) f(\tau + t_0) \, d\tau, \]
\[ I = \lim_{n \to \infty} \int_{-a}^{a} \delta_n(\tau) f(\tau + t_0) \, d\tau = \lim_{n \to \infty} \int_{0}^{1/n} n f(\tau + t_0) \, d\tau \]
Therefore, \[ I = \lim_{n \to \infty} n \int_{0}^{1/n} F'(\tau + t_0) \, d\tau, \] where we introduced the primitive \( F(t) = \int f(t) \, dt \), that is, \( f(t) = F'(t) \).
Properties of Dirac’s delta.

Theorem
If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then
\[
\int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) \, dt = f(t_0).
\]

Proof: So, $I = \lim_{n \to \infty} n \int_0^{1/n} F'(\tau + t_0) \, d\tau$, with $f(t) = F'(t)$.
\[
I = \lim_{n \to \infty} n \left[ F(\tau + t_0) \bigg|_0^{1/n} \right] = \lim_{n \to \infty} n \left[ F\left( t_0 + \frac{1}{n} \right) - F(t_0) \right].
\]
\[
I = \lim_{n \to \infty} \frac{F\left( t_0 + \frac{1}{n} \right) - F(t_0)}{\frac{1}{n}} = F'(t_0) = f(t_0).
\]
We conclude: \[
\int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) \, dt = f(t_0).
\]

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Relation between deltas and steps.

Theorem

The sequence of functions for $n \geq 1$,

$$u_n(t) = \begin{cases} 
0, & t < 0 \\
nt, & 0 \leq t \leq \frac{1}{n} \\
1, & t > \frac{1}{n}.
\end{cases}$$

satisfies, for $t \in (-\infty, 0) \cup (0, 1/n) \cup (1/n, \infty)$, both equations,

$$u_n'(t) = \delta_n(t), \quad \lim_{n \to \infty} u_n(t) = u(t), \quad t \in \mathbb{R}.$$  

Remark:

- If we generalize the notion of derivative as
  $$u'(t) = \lim_{n \to \infty} u_n'(t),$$
  then holds $u'(t) = \delta(t)$.
- Dirac’s delta is a generalized derivative of the step function.

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Dirac’s delta in Physics.

Remarks:

(a) Dirac’s delta generalized function is useful to describe impulsive forces in mechanical systems.

(b) An impulsive force transmits a finite momentum in an infinitely short time.

(c) For example: The momentum transmitted to a pendulum when hit by a hammer. Newton’s law of motion says,

\[ m v'(t) = F(t), \quad \text{with} \quad F(t) = F_0 \delta(t - t_0). \]

The momentum transfer is:

\[ \Delta I = \lim_{\Delta t \to 0} mv(t)\bigg|_{t_0+\Delta t}^{t_0+\Delta t} = \lim_{\Delta t \to 0} \int_{t_0-\Delta t}^{t_0+\Delta t} F(t) \, dt = F_0. \]

That is, \( \Delta I = F_0 \).

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The Laplace Transform of Dirac’s delta.

Recall: The Laplace Transform can be generalized from functions to $\delta$, as follows, $L[\delta(t - c)] = \lim_{n \to \infty} L[\delta_n(t - c)]$.

Theorem

$$L[\delta(t - c)] = e^{-cs}.$$

Proof:

$$L[\delta(t - c)] = \lim_{n \to \infty} L[\delta_n(t - c)], \quad \delta_n(t) = n \left[ u(t) - u(t - \frac{1}{n}) \right].$$

$$L[\delta(t - c)] = \lim_{n \to \infty} n \left( L[u(t - c)] - L[u(t - c - \frac{1}{n})] \right)$$

$$L[\delta(t - c)] = \lim_{n \to \infty} n \left( \frac{e^{-cs}}{s} - \frac{e^{-(c + \frac{1}{n})s}}{s} \right) = e^{-cs} \lim_{n \to \infty} \frac{1 - e^{-\frac{s}{n}}}{\left( \frac{s}{n} \right)}. $$

This is a singular limit, $\frac{0}{0}$. Use l’Hôpital rule.

The Laplace Transform of Dirac’s delta.

Proof: Recall: $L[\delta(t - c)] = e^{-cs} \lim_{n \to \infty} \frac{1 - e^{-\frac{s}{n}}}{\left( \frac{s}{n} \right)}$.

$$\lim_{n \to \infty} \frac{1 - e^{-\frac{s}{n}}}{\left( \frac{s}{n} \right)} = \lim_{n \to \infty} \frac{-\frac{s}{n^2} e^{-\frac{s}{n}}}{\left( -\frac{s}{n^2} \right)} = \lim_{n \to \infty} e^{-\frac{s}{n}} = 1. $$

We therefore conclude that $L[\delta(t - c)] = e^{-cs}$. □

Remarks:

(a) This result is consistent with a previous result:

$$\int_{t_0 - a}^{t_0 + a} \delta(t - t_0) f(t) \, dt = f(t_0).$$

(b) $L[\delta(t - c)] = \int_0^\infty \delta(t - c) e^{-st} \, dt = e^{-cs}.$

(c) $L[\delta(t - c) f(t)] = \int_0^\infty \delta(t - c) e^{-st} f(t) \, dt = e^{-cs} f(c).$
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Differential equations with Dirac’s delta sources.

**Example**
Find the solution $y$ to the initial value problem

$$y'' - y = -20 \delta(t - 3), \quad y(0) = 1, \quad y'(0) = 0.$$ 

**Solution:** Compute: $\mathcal{L}[y''] - \mathcal{L}[y] = -20 \mathcal{L}[\delta(t - 3)].$

$$\mathcal{L}[y''] = s^2 \mathcal{L}[y] - s y(0) - y'(0) \quad \Rightarrow \quad (s^2 - 1) \mathcal{L}[y] - s = -20 e^{-3s},$$

We arrive to the equation $\mathcal{L}[y] = \frac{s}{(s^2 - 1)} - 20 e^{-3s} \frac{1}{(s^2 - 1)}$, 

$$\mathcal{L}[y] = \mathcal{L}[\cosh(t)] - 20 \mathcal{L}[u(t - 3) \sinh(t - 3)],$$

We conclude: $y(t) = \cosh(t) - 20 u(t - 3) \sinh(t - 3).$  ◀
Example
Find the solution to the initial value problem
\[ y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Compute:
\[ L[y''] + 4L[y] = L[\delta(t - \pi)] - L[\delta(t - 2\pi)], \]
\[ (s^2 + 4)L[y] = e^{-\pi s} - e^{-2\pi s} \Rightarrow L[y] = \frac{e^{-\pi s}}{(s^2 + 4)} - \frac{e^{-2\pi s}}{(s^2 + 4)}, \]
that is,
\[ L[y] = \frac{1}{2} L\left[ u(t - \pi) \sin(2(t - \pi)) \right] - \frac{1}{2} L\left[ u(t - 2\pi) \sin(2(t - 2\pi)) \right]. \]

Recall:
\[ e^{-cs} L[f(t)] = L[u(t - c) f(t - c)]. \]
Therefore,
\[ L[y] = \frac{1}{2} L\left[ u(t - \pi) \sin(2(t - \pi)) \right] - \frac{1}{2} L\left[ u(t - 2\pi) \sin(2(t - 2\pi)) \right]. \]

This implies that,
\[ y(t) = \frac{1}{2} u(t - \pi) \sin(2(t - \pi)) - \frac{1}{2} u(t - 2\pi) \sin(2(t - 2\pi)), \]
We conclude:
\[ y(t) = \frac{1}{2} \left[ u(t - \pi) - u(t - 2\pi) \right] \sin(2t). \]