The Laplace Transform of step functions (Sect. 4.3).

Last Lecture

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.

This Lecture

- Differential equations with discontinuous sources.

Equations with discontinuous sources (Sect. 4.3).

- Differential equations with discontinuous sources.
- We solve the IVPs:

(a) Example 1:

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

(b) Example 2:

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

(c) Example 3:

$$y'' + y' + \frac{5}{4}\,y = g(t), \quad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t), & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

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Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Compute the Laplace transform of the whole equation,

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[u(t-4)] = \frac{e^{-4s}}{s}.$$

From the previous Section we know that

$$\big[s\,\mathcal{L}[y] - y(0)\big] + 2\,\mathcal{L}[y] = \frac{e^{-4s}}{s} \quad \Rightarrow \quad (s+2)\,\mathcal{L}[y] = y(0) + \frac{e^{-4s}}{s}.$$

Introduce the initial condition, $\mathcal{L}[y] = \dfrac{3}{(s+2)} + e^{-4s}\,\dfrac{1}{s(s+2)}$,

Use the table: $\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\dfrac{1}{s(s+2)}$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\dfrac{1}{s(s+2)}$.

We need to invert the Laplace transform on the last term.
Partial fractions:

$$\frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{(s+2)} = \frac{a(s+2)+bs}{s(s+2)} = \frac{(a+b)\,s + (2a)}{s(s+2)}$$

We get, $a+b=0$, $2a=1$. We obtain: $a = \dfrac{1}{2}$, $b = -\dfrac{1}{2}$. Hence,

$$\frac{1}{s(s+2)} = \frac{1}{2}\left[\frac{1}{s} - \frac{1}{(s+2)}\right].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\dfrac{1}{s(s+2)} = \dfrac{1}{2}\left[\dfrac{1}{s} - \dfrac{1}{(s+2)}\right]$.

The algebraic equation for $\mathcal{L}[y]$ has the form,

$$\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\left[e^{-4s}\,\frac{1}{s} - e^{-4s}\,\frac{1}{(s+2)}\right].$$

$$\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\Big(\mathcal{L}[u(t-4)] - \mathcal{L}\big[u(t-4)\,e^{-2(t-4)}\big]\Big).$$

We conclude that

$$y(t) = 3e^{-2t} + \frac{1}{2}\,u(t-4)\Big[1 - e^{-2(t-4)}\Big]. \qquad \triangleleft$$

Equations with discontinuous sources (Sect. 4.3).

- Differential equations with discontinuous sources.
- We solve the IVPs:
  (a) Example 1:

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

  (b) **Example 2:**

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

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$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t), & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Differential equations with discontinuous sources.

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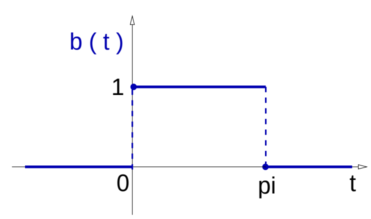
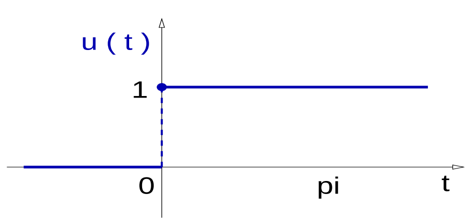
Use the Laplace transform to find the solution of the IVP

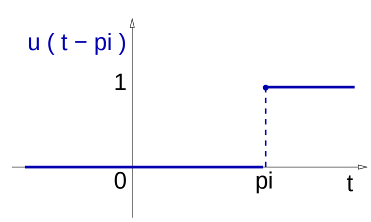
$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution:

Rewrite the source function using step functions.







Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\, y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $b(t) = u(t) - u(t - \pi)$

Now is simple to find $\mathcal{L}[b]$, since

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t - \pi)] = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

So, the source is $\mathcal{L}[b(t)] = \left(1 - e^{-\pi s}\right) \frac{1}{s}$, and the equation is

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\, \mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \frac{1}{s}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\, y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\, \mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \frac{1}{s}$.

The initial conditions imply: $\mathcal{L}[y''] = s^2\, \mathcal{L}[y]$ and $\mathcal{L}[y'] = s\, \mathcal{L}[y]$.

Therefore, $\left(s^2 + s + \frac{5}{4}\right) \mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \frac{1}{s}$.

We arrive at the expression: $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$.

Denoting: $H(s) = \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$,

we obtain, $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) H(s)$.

In other words: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\, H(s)\big]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\, H(s)\big]$.

Denoting: $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, the $\mathcal{L}[\;]$ properties imply

$$\mathcal{L}^{-1}\big[e^{-\pi s} H(s)\big] = u(t - \pi)\, h(t - \pi).$$

Therefore, the solution has the form

$$y(t) = h(t) - u(t - \pi)\, h(t - \pi).$$

We only need to find $h(t) = \mathcal{L}^{-1}\Big[\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\frac{1}{s\,(s^2 + s + \frac{5}{4})}\Big]$.

Partial fractions: Find the zeros of the denominator,

$$s_\pm = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big] \quad \Rightarrow \quad \text{Complex roots.}$$

The partial fraction decomposition is:

$$H(s) = \frac{1}{(s^2 + s + \frac{5}{4})\,s} = \frac{a}{s} + \frac{(bs + c)}{(s^2 + s + \frac{5}{4})}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $H(s) = \frac{1}{(s^2 + s + \frac{5}{4})\,s} = \frac{a}{s} + \frac{(bs + c)}{(s^2 + s + \frac{5}{4})}$.

The partial fraction decomposition is:

$$1 = a\left(s^2 + s + \frac{5}{4}\right) + s\,(bs + c) = (a + b)\,s^2 + (a + c)\,s + \frac{5}{4}a.$$

This equation implies that a, b, and c, are solutions of

$$a + b = 0, \quad a + c = 0, \quad \frac{5}{4}\,a = 1.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $a = \frac{4}{5}$, $b = -\frac{4}{5}$, $c = -\frac{4}{5}$.

Hence, we have found that,

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)s} = \frac{4}{5}\Big[\frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)}\Big]$$

We have to compute the inverse Laplace Transform

$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)}\Big]$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)}\Big]$.

In this case we complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)\,s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4} = \Big(s + \frac{1}{2}\Big)^2 + 1.$$

So: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s} - \frac{(s+1)}{\left[\left(s + \frac{1}{2}\right)^2 + 1\right]}\Big]$.

That is, $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\left(s + \frac{1}{2}\right) + \frac{1}{2}}{\left[\left(s + \frac{1}{2}\right)^2 + 1\right]}\Big]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0,\\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0,\pi) \\ 0, & t \in [\pi,\infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{(s+\frac{1}{2})+\frac{1}{2}}{\big[(s+\frac{1}{2})^2+1\big]}\Big].$

$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{(s+\frac{1}{2})}{\big[(s+\frac{1}{2})^2+1\big]}\Big] - \frac{2}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{\big[(s+\frac{1}{2})^2+1\big]}\Big].$$

Recall: $\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t)$. Hence,

$$h(t) = \frac{4}{5}\Big[1 - e^{-t/2}\cos(t) - \frac{1}{2}\,e^{-t/2}\sin(t)\Big].$$

We conclude: $y(t) = h(t) + u(t-\pi)h(t-\pi)$. $\triangleleft$

Equations with discontinuous sources (Sect. 4.3).

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    $$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0,\\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0,\pi) \\ 0, & t \in [\pi,\infty). \end{cases}$$

  - (c) **Example 3:**

    $$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0,\\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t), & t \in [0,\pi) \\ 0, & t \in [\pi,\infty). \end{cases}$$

Differential equations with discontinuous sources.

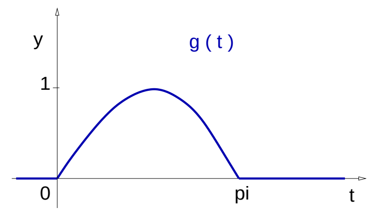
Example

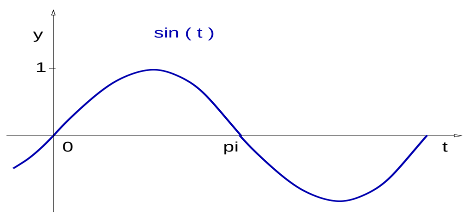
Use the Laplace transform to find the solution of the IVP

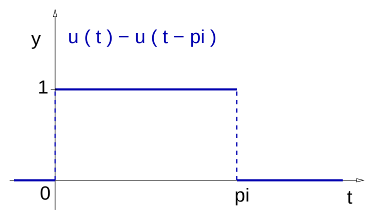
$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution:

Rewrite the source function using step functions.







Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $g(t) = \big[u(t) - u(t-\pi)\big]\,\sin(t).$

Recall the identity: $\sin(t) = -\sin(t-\pi)$. Then,

$$g(t) = u(t)\,\sin(t) - u(t-\pi)\,\sin(t),$$

$$g(t) = u(t)\,\sin(t) + u(t-\pi)\,\sin(t-\pi).$$

Now is simple to find $\mathcal{L}[g]$, since

$$\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

Recall the Laplace transform of the differential equation

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$ and $\mathcal{L}[y'] = s\,\mathcal{L}[y]$.

Therefore, $\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \left(1 + e^{-\pi s}\right)\frac{1}{(s^2+1)}$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \left(1 + e^{-\pi s}\right)\frac{1}{(s^2+1)}$.

$$\mathcal{L}[y] = \left(1 + e^{-\pi s}\right)\frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2+1)}.$$

Introduce the function $H(s) = \dfrac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2+1)}$.

Then, $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)]$, and

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)\left(s^2 + 1\right)}.$$

Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}\left[-1 \pm \sqrt{1 - 5}\right] \quad \Rightarrow \quad \text{Complex roots.}$$

The partial fraction decomposition is:

$$\frac{1}{\left(s^2 + s + \frac{5}{4}\right)\left(s^2 + 1\right)} = \frac{(as + b)}{\left(s^2 + s + \frac{5}{4}\right)} + \frac{(cs + d)}{(s^2 + 1)}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\displaystyle \frac{1}{\left(s^2 + s + \frac{5}{4}\right)\left(s^2 + 1\right)} = \frac{(as + b)}{\left(s^2 + s + \frac{5}{4}\right)} + \frac{(cs + d)}{(s^2 + 1)}.$

Therefore, we get

$$1 = (as + b)(s^2 + 1) + (cs + d)\Big(s^2 + s + \frac{5}{4}\Big),$$

$$1 = (a + c)\,s^3 + (b + c + d)\,s^2 + \Big(a + \frac{5}{4}\,c + d\Big)\,s + \Big(b + \frac{5}{4}\,d\Big).$$

This equation implies that a, b, c, and d, are solutions of

$$a + c = 0, \quad b + c + d = 0, \quad a + \frac{5}{4}\,c + d = 0, \quad b + \frac{5}{4}\,d = 1.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $a = \frac{16}{17}$, $b = \frac{12}{17}$, $c = -\frac{16}{17}$, $d = \frac{4}{17}$.

We have found: $H(s) = \frac{4}{17}\Big[\frac{(4s+3)}{(s^2+s+\frac{5}{4})} + \frac{(-4s+1)}{(s^2+1)}\Big]$.

Complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)\,s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4} = \Big(s + \frac{1}{2}\Big)^2 + 1.$$

$$H(s) = \frac{4}{17}\Big[\frac{(4s+3)}{\big[\big(s+\frac{1}{2}\big)^2 + 1\big]} + \frac{(-4s+1)}{(s^2+1)}\Big].$$

Differential equations with discontinuous sources.

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Use the Laplace transform to find the solution of the IVP

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Solution: So: $H(s) = \frac{4}{17}\Big[\frac{(4s+3)}{\big[\big(s+\frac{1}{2}\big)^2 + 1\big]} + \frac{(-4s+1)}{(s^2+1)}\Big]$.

Rewrite the polynomial in the numerator,

$$(4s+3) = 4\Big(s + \frac{1}{2} - \frac{1}{2}\Big) + 3 = 4\Big(s + \frac{1}{2}\Big) + 1,$$

$$H(s) = \frac{4}{17}\Big[4\,\frac{\big(s+\frac{1}{2}\big)}{\big[\big(s+\frac{1}{2}\big)^2+1\big]} + \frac{1}{\big[\big(s+\frac{1}{2}\big)^2+1\big]} - 4\,\frac{s}{(s^2+1)} + \frac{1}{(s^2+1)}\Big],$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution:

$$H(s) = \frac{4}{17}\Big[4\,\frac{\left(s+\frac{1}{2}\right)}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} + \frac{1}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} - 4\,\frac{s}{(s^2+1)} + \frac{1}{(s^2+1)}\Big],$$

Use the Laplace Transform table to get $H(s)$ equal to

$$H(s) = \frac{4}{17}\Big[4\,\mathcal{L}\big[e^{-t/2}\cos(t)\big] + \mathcal{L}\big[e^{-t/2}\sin(t)\big] - 4\,\mathcal{L}[\cos(t)] + \mathcal{L}[\sin(t)]\Big].$$

$$H(s) = \mathcal{L}\Big[\frac{4}{17}\Big(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big)\Big].$$

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Example

Use the Laplace transform to find the solution of the IVP

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Solution: Recall:

$$H(s) = \mathcal{L}\Big[\frac{4}{17}\Big(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big)\Big].$$

Denote:

$$h(t) = \frac{4}{17}\Big[4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big].$$

Then, $H(s) = \mathcal{L}[h(t)]$. Recalling: $\mathcal{L}[y(t)] = H(s) + e^{-\pi s}\,H(s)$,

$$\mathcal{L}[y(t)] = \mathcal{L}[h(t)] + e^{-\pi s}\,\mathcal{L}[h(t)].$$

We conclude: $y(t) = h(t) + u(t-\pi)h(t-\pi)$. $\lhd$