The Euler equation (Sect. 3.2).

- We study the Euler Equation:
  \[(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.\]
- Solutions to the Euler equation near \(x_0\).
- The roots of the indicial polynomial.
  - Different real roots.
  - Repeated roots.
  - Different complex roots.

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The Euler equation

**Definition**

Given real constants \(p_0, q_0\), the *Euler differential equation* for the unknown \(y\) with singular point at \(x_0 \in \mathbb{R}\) is given by

\[(x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0.\]

**Remarks:**

- The Euler equation has variable coefficients.
- Functions \(y(x) = e^{rx}\) are **not** solutions of the Euler equation.
- The point \(x_0 \in \mathbb{R}\) is a singular point of the equation.
- The particular case \(x_0 = 0\) is given by
  \[x^2 y'' + p_0 x y' + q_0 y = 0.\]
We study the Euler Equation:

\[(x - x_0)^2 \, y'' + p_0 \, (x - x_0) \, y' + q_0 \, y = 0.\]

**Solutions to the Euler equation near** \(x_0**.

- The roots of the indicial polynomial.
  - Different real roots.
  - Repeated roots.
  - Different complex roots.

**Summary of the main idea:**

- The main idea to find solution to the constant coefficients equation \(y'' + a_1 \, y' + a_0 \, y = 0\) was to look for functions of the form \(y(x) = e^{rx}\). The exponential cancels out from the equation and we obtain an equation only for \(r\) without \(x\),

\[(r^2 + a_1 \, r + a_0) \, e^{rx} = 0 \iff (r^2 + a_1 \, r + a_0) = 0. \tag{1}\]

- In the case of the Euler equation \(x^2 \, y'' + p_0 \, x \, y' + q_0 \, y = 0\) the exponential functions \(e^{rx}\) do not have the property given in Eq. (1), since

\[(x^2 \, r^2 + p_0 \, x \, r + q_0) \, e^{rx} = 0 \iff x^2 \, r^2 + p_0 \, x \, r + q_0 = 0,

but the later equation still involves the variable \(x\).
Solutions to the Euler equation near $x_0$.

Summary of the main idea: Look for solutions like $y(x) = x^r$.

These functions have the following properties:

\[ y'(x) = r x^{r-1} \quad \Rightarrow \quad x y'(x) = r x^r; \]
\[ y''(x) = r(r-1) x^{r-2} \quad \Rightarrow \quad x^2 y''(x) = r(r-1) x^r. \]

Introduce $y = x^r$ into Euler's equation $x^2 y'' + p_0 x y' + q_0 y = 0$, for $x \neq 0$ we obtain

\[ \left[ r(r-1) + p_0 r + q_0 \right] x^r = 0 \quad \Leftrightarrow \quad r(r-1) + p_0 r + q_0 = 0. \]

The last equation involves only $r$, not $x$.

This equation is called the indicial equation, and is also called the Euler characteristic equation.

Solutions to the Euler equation near $x_0$.

Theorem (Euler equation, $x_0 = 0$)

Given $p_0, q_0, x_0 \in \mathbb{R}$, consider the Euler equation

\[ x^2 y'' + p_0 x y' + q_0 y = 0. \tag{2} \]

Let $r_+, r_-$ be solutions of $r(r-1) + p_0 r + q_0 = 0$.

(a) If $r_+ \neq r_-$, then a general solution of Eq. (2) is

\[ y(x) = c_0 |x|^{r_+} + c_1 |x|^{r_-}, \quad x \neq 0, \quad c_0, c_1 \in \mathbb{R} \text{ (or } \mathbb{C}). \]

(b) If $r_+ = r_- = \hat{r}$, then a real-valued general solution of Eq. (2) is

\[ y(x) = \left[ c_0 + c_1 \ln |x| \right] |x|^{\hat{r}}, \quad x \neq 0, \quad c_0, c_1 \in \mathbb{R}. \]

Given $x_1 \neq 0, y_0, y_1 \in \mathbb{R}$, there is a unique solution to the IVP

\[ x^2 y'' + p_0 x y' + q_0 y = 0, \quad y(x_1) = y_0, \quad y'(x_1) = y_1. \]
Solutions to the Euler equation near $x_0$.

Theorem (Euler equation, $x_0 \neq 0$)

Given $p_0$, $q_0$, $x_0 \in \mathbb{R}$, consider the Euler equation

$$(x-x_0)^2 y'' + p_0(x-x_0) y' + q_0 y = 0. \quad (3)$$

Let $r_+$, $r_-$ be solutions of $r(r-1) + p_0r + q_0 = 0$.

(a) If $r_+ \neq r_-$, then a general solution of Eq. (3) is

$$y(x) = c_0|x-x_0|^{r_+} + c_1|x-x_0|^{r_-}, \quad x \neq x_0, \quad c_0, c_1 \in \mathbb{R} \text{ (or } \mathbb{C}).$$

(b) If $r_+ = r_- = \hat{r}$, then a real-valued general solution of Eq. (3) is

$$y(x) = \left[c_0 + c_1 \ln |x-x_0|\right]|x-x_0|^\hat{r}, \quad x \neq x_0, \quad c_0, c_1 \in \mathbb{R}.$$

Given $x_1 \neq x_0$, $y_0$, $y_1 \in \mathbb{R}$, there is a unique solution to the IVP

$$(x-x_0)^2 y'' + p_0(x-x_0) y' + q_0 y = 0, \quad y(x_1) = y_0, \quad y'(x_1) = y_1.$$
Different real roots.

Example
Find the general solution of the Euler equation
\[ x^2 y'' + 4x y' + 2y = 0. \]

Solution: We look for solutions of the form \( y(x) = x^r \),
\[ x y'(x) = r x^r, \quad x^2 y''(x) = r(r - 1) x^r. \]

Introduce \( y(x) = x^r \) into Euler equation,
\[ \left[ r(r - 1) + 4r + 2 \right] x^r = 0 \Leftrightarrow r(r - 1) + 4r + 2 = 0. \]

The solutions of \( r^2 + 3r + 2 = 0 \) are given by
\[ r_{\pm} = \frac{1}{2} \left[ -3 \pm \sqrt{9 - 8} \right] \Rightarrow r_+ = -1 \quad r_- = -2. \]

The general solution is \( y(x) = c_1 |x|^{-1} + c_2 |x|^{-2} \).  

The Euler equation (Sect. 3.2).

- We study the Euler Equation:
  \[ (x - x_0)^2 y'' + p_0 (x - x_0) y' + q_0 y = 0. \]
- Solutions to the Euler equation near \( x_0 \).
- The roots of the indicial polynomial.
  - Different real roots.
  - Repeated roots.
  - Different complex roots.
Repeated roots.

Example
Find the general solution of \( x^2 y'' - 3x y' + 4y = 0 \).

Solution: We look for solutions of the form \( y(x) = x^r \),

\[
x y'(x) = rx^r, \quad x^2 y''(x) = r(r-1)x^r.
\]

Introduce \( y(x) = x^r \) into Euler equation,

\[
[r(r-1) - 3r + 4] x^r = 0 \iff r(r-1) - 3r + 4 = 0.
\]

The solutions of \( r^2 - 4r + 4 = 0 \) are given by

\[
r_\pm = \frac{1}{2} [4 \pm \sqrt{16 - 16}] \Rightarrow r_+ = r_- = 2.
\]

Two linearly independent solutions are

\[
y_1(x) = x^2, \quad y_2 = x^2 \ln(|x|).
\]

The general solution is \( y(x) = c_1 x^2 + c_2 x^2 \ln(|x|) \).  

The Euler equation (Sect. 3.2).

- We study the Euler Equation:
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- Solutions to the Euler equation near \( x_0 \).
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Different complex roots.

Example
Find the general solution of the Euler equation
\[ x^2 y'' - 3x y' + 13y = 0. \]

Solution: We look for solutions of the form \( y(x) = x^r \),
\[ xy'(x) = rx^r, \quad x^2 y''(x) = r(r-1)x^r. \]
Introduce \( y(x) = x^r \) into Euler equation
\[ [r(r-1) - 3r + 13] x^r = 0 \iff r(r-1) - 3r + 13 = 0. \]
The solutions of the indicial equation \( r^2 - 4r + 13 = 0 \) are
\[ r_\pm = \frac{1}{2} [4 \pm \sqrt{16 - 52}] \Rightarrow r_\pm = \frac{1}{2} [4 \pm \sqrt{-36}] \Rightarrow \begin{cases} r_+ = 2 + 3i, \\ r_- = 2 - 3i. \end{cases} \]
The general solution is \( y(x) = c_1 |x|^{(2+3i)} + c_2 |x|^{(2-3i)}. \)

Different complex roots.

Theorem (Real-valued fundamental solutions)
If \( p_0, q_0 \in \mathbb{R} \) satisfy that \( [(p_0 - 1)^2 - 4q_0] < 0 \), then the indicial polynomial \( p(r) = r(r-1) + p_0r + q_0 \) of the Euler equation
\[ x^2 y'' + p_0 x y' + q_0 y = 0 \quad (4) \]
has complex roots \( r_* = \alpha + i\beta \) and \( r_- = \alpha - i\beta \), where
\[ \alpha = -\frac{(p_0 - 1)}{2}, \quad \beta = \frac{1}{2} \sqrt{4q_0 - (p_0 - 1)^2}. \]
A complex-valued fundamental set of solution to Eq. (4) is
\[ \tilde{y}_1(x) = |x|^{(\alpha + i\beta)}, \quad \tilde{y}_2(x) = |x|^{(\alpha - i\beta)}. \]
A real-valued fundamental set of solutions to Eq. (4) is
\[ y_1(x) = |x|^{\alpha} \cos(\beta \ln |x|), \quad y_2(x) = |x|^{\alpha} \sin(\beta \ln |x|). \]
Different complex roots.

**Proof:** Given $\tilde{y}_1 = |x|^{(\alpha + i\beta)}$ and $\tilde{y}_2 = |x|^{(\alpha - i\beta)}$, introduce

$$y_1 = \frac{1}{2}(\tilde{y}_1 + \tilde{y}_2), \quad y_2 = \frac{1}{2i}(\tilde{y}_1 - \tilde{y}_2).$$

Use another Euler equation to rewrite $\tilde{y}_1$ and $\tilde{y}_2$,

$$\tilde{y}_1 = |x|^{(\alpha + i\beta)} = |x|^\alpha |x|^{i\beta} = |x|^\alpha e^{i\ln(|x|)} = |x|^\alpha e^{i\beta \ln(|x|)}.$$

$$\tilde{y}_1 = |x|^\alpha [\cos(\beta \ln |x|) + i \sin(\beta \ln |x|)],$$

$$\tilde{y}_2 = |x|^\alpha [\cos(\beta \ln |x|) - i \sin(\beta \ln |x|)].$$

We conclude that

$$y_1(x) = |x|^\alpha \cos(\beta \ln |x|), \quad y_2(x) = |x|^\alpha \sin(\beta \ln |x|).$$

\[\square\]

Different complex roots.

**Example**

Find a real-valued general solution of the Euler equation

$$x^2 y'' - 3x y' + 13y = 0.$$

**Solution:** The indicial equation is $r(r-1) - 3r + 13 = 0$.

The solutions of the indicial equations are

$$r^2 - 4r + 13 = 0 \Rightarrow r_+ = 2 + 3i, \quad r_- = 2 - 3i.$$

A complex-valued general solution is

$$y(x) = \tilde{c}_1 |x|^{(2+3i)} + \tilde{c}_2 |x|^{(2-3i)}, \quad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C}.$$

A real-valued general solution is

$$y(x) = c_1 |x|^2 \cos(3 \ln |x|) + c_2 |x|^2 \sin(3 \ln |x|), \quad c_1, c_2 \in \mathbb{R}. \quad \triangleq$$