Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Summary of solutions of the differential equation

$$y'' + a_1 y' + a_0 y = 0, \quad a_1, a_2 \in \mathbb{R},$$

and characteristic roots $r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_0}$.

1. Over damped systems: If $a_1^2 - 4a_0 > 0$, then,
   $$y_1(t) = e^{r_+ t}, \quad y_2(t) = e^{r_- t}.$$

2. Critically damped systems: If $a_1^2 - 4a_0 = 0$, then,
   $$y_1(t) = e^{-\frac{a_1}{2} t}, \quad y_2(t) = t e^{-\frac{a_1}{2} t}.$$

3. Under damped systems: If $a_1^2 - 4a_0 < 0$, then
   $$y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).$$
   with $\alpha = -\frac{a_1}{2}, \quad \beta = \frac{1}{2} \sqrt{4a_0 - a_1^2}$. Not damped: If $a_1 = 0$. 

Remark:
Different physical systems may have identical mathematical descriptions.
Mechanical and electrical oscillations (Sect. 2.7?)

- Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.
- **Application**: Mechanical Oscillations.
- Application: The RLC electrical circuit.

Application: Mechanical Oscillations.

Consider a spring attached to the ceiling, having rest length \( l \), with an attached mass \( m \).

- \((l + \Delta l)\) is called equilibrium position of the spring loaded with a mass \( m \).
- The coordinate \( y \) measures vertical deviations from the equilibrium position.

Forces acting on the system:

- Weight: \( F_g = mg \).
- Spring: \( F_s = -k(\Delta l + y) \). Hooke’s Law. (Small oscillations.)
- Damping: \( F_d(t) = -d y'(t) \). Fluid Resistance.

Newton’s Law: \( my''(t) = F_g + F_s(t) + F_d(t) \).
Application: Mechanical Oscillations.

Recall: \( F_g = mg \), \( F_s = -k(\Delta l + y) \), \( F_d(t) = -d y'(t) \).

\[
my''(t) = F_g + F_s(t) + F_d(t).
\]
That is, \( my''(t) = mg - k(\Delta l + y(t)) - d y'(t) \).

At equilibrium, \( y = 0 \), \( y' = 0 \), then \( k \Delta l = mg \). Hence

\[
my''(t) = -k y(t) - d y'(t)
\]

\[
my'' + dy' + ky = 0.
\]

To solve for the function \( y \), we need the characteristic equation

\[
Mr^2 + Dr + K = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2m} \left[ -d \pm \sqrt{d^2 - 4mk} \right].
\]

Application: Mechanical Oscillations.

Recall: \( my'' + dy' + ky = 0 \), and \( r_{\pm} = \frac{1}{2m} \left[ -d \pm \sqrt{d^2 - 4mk} \right] \).

Not damped oscillations: \( d = 0 \). No fluid friction.

\[
r_{\pm} = \pm \sqrt{-\frac{k}{m}}, \quad \omega_0 = \sqrt{\frac{k}{m}}, \quad r_{\pm} = \pm i\omega_0.
\]

\[
y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).
\]

Remarks:

- Fundamental Frequency: \( \omega_0 \); Period: \( T = \frac{2\pi}{\omega_0} \).
- Equivalent expression: \( y(t) = A \cos(\omega_0 t - \phi) \).
- Amplitude: \( A \); Phase shift: \( \phi \).
Application: Mechanical Oscillations.

Recall: Not damped oscillations:

\[ y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \iff y(t) = A \cos(\omega_0 t - \phi). \]

where \( \omega_0 = \sqrt{k/m} \) is the fundamental frequency, \( A \) is the amplitude, and \( \phi \) the initial phase shift of the oscillations.

(Recall that the oscillation period is \( T = \frac{2\pi}{\omega_0} \).)

**Proof:** Recall the trigonometric identity:

\[ A \cos(\omega_0 t - \phi) = A \cos(\omega_0 t) \cos(\phi) + A \sin(\omega_0 t) \sin(\phi). \]

Therefore, comparing the first and last expressions above,

\[
\begin{align*}
  c_1 &= A \cos(\phi) \\
  c_2 &= A \sin(\phi)
\end{align*}
\]
\[
\iff \begin{cases} 
  A &= \sqrt{c_1^2 + c_2^2} \\
  \phi &= \arctan\left(\frac{c_2}{c_1}\right).
\end{cases}
\]

Application: Mechanical Oscillations.

Damped Oscillations

Recall: \( m \ddot{y} + d \dot{y} + k y = 0 \), and \( r_{\pm} = \frac{1}{2m}[-d \pm \sqrt{d^2 - 4mk}] \).

Rewrite: \( r_{\pm} = -\frac{d}{2m} \pm \sqrt{\left(\frac{d}{2m}\right)^2 - \frac{k}{m}}. \)

Introduce: \( \omega_0 = \sqrt{\frac{k}{m}} \), and \( \omega_d = \frac{d}{2m} \). Hence

\[ r_{\pm} = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}. \]

**Remark:** We have three cases of damped oscillations:

(a) Over damped: \( \omega_d > \omega_0 \).
(b) Critically damped: \( \omega_d = \omega_0 \).
(c) Under damped: \( \omega_d < \omega_0 \).
Application: Mechanical Oscillations.

Recall: \( m y'' + d y' + k y = 0 \), and \( r_\pm = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2} \).

(a) Over damped: \( \omega_d > \omega_0 \). Two distinct real roots:

\[
y(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}.
\]

(b) Critically damped: \( \omega_d = \omega_0 \). Repeated real root \( r_+ = r_- = \hat{r} \):

\[
y(t) = (c_1 + c_2 t) e^{\hat{r} t}.
\]

(c) Under damped: \( \omega_d < \omega_0 \). Complex roots:

\[
y(t) = \left[ c_1 \cos(\beta t) + c_2 \sin(\beta t) \right] e^{-\omega_d t}
\]

\[
y(t) = A \cos(\beta t - \phi) e^{-\omega_d t}
\]

where \( r_\pm = -\omega_d \pm i\beta \), and \( \beta = \sqrt{\omega_0^2 - \omega_d^2} \).

Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant \( k = 5\text{Kg/Secs}^2 \) moving in a medium with damping constant \( d = 5\text{Kg/Secs} \), with initial conditions \( y(0) = \sqrt{3} \) and \( y'(0) = 0 \).

Solution: The equation is: \( my'' + dy' + ky = 0 \), with \( m = 5 \), \( k = 5 \), \( d = 5 \). The characteristic roots are

\[
r_\pm = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}, \quad \omega_d = \frac{d}{2m} = \frac{1}{2}, \quad \omega_0 = \sqrt{\frac{k}{m} = 1}.
\]

\[
r_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \quad \text{Under damped oscillations.}
\]

\[
y(t) = A \cos\left(\frac{\sqrt{3}}{2} t - \phi\right) e^{-t/2}.
\]
Application: Mechanical Oscillations.

Example

Find the movement of a 5Kg mass attached to a spring with constant \( k = 5\text{Kg/Secs}^2 \) moving in a medium with damping constant \( d = 5\text{Kg/Secs} \), with initial conditions \( y(0) = \sqrt{3} \) and \( y'(0) = 0 \).

Solution: Recall: \( y(t) = A \cos\left(\frac{\sqrt{3}}{2} t - \phi\right) e^{-t/2} \). Hence,

\[
y'(t) = -\frac{\sqrt{3}}{2} A \sin\left(\frac{\sqrt{3}}{2} t - \phi\right) e^{-t/2} - \frac{1}{2} A \cos\left(\frac{\sqrt{3}}{2} t - \phi\right) e^{-t/2}.
\]

The initial conditions:

\[
\sqrt{3} = y(0) = A \cos(\phi), \quad 0 = y'(0) = \frac{\sqrt{3}}{2} A \sin(\phi) - \frac{1}{2} A \cos(\phi).
\]

\[
\tan(\phi) = \frac{1}{\sqrt{3}} \implies \phi = \frac{\pi}{6}, \quad \Rightarrow \quad A = 2.
\]

We conclude: \( y(t) = 2 \cos\left(\frac{\sqrt{3}}{2} t - \frac{\pi}{6}\right) e^{-t/2} \). 

\[\triangleright\]

Mechanical and electrical oscillations (Sect. 2.7?)

- Review: On solutions of \( y'' + a_1 y' + a_0 y = 0 \).
- Application: Mechanical Oscillations.
- **Application:** The RLC electrical circuit.
The RLC electrical circuit.

Consider an electric circuit with resistance $R$, non-zero capacitor $C$, and non-zero inductance $L$, as in the figure.

Kirchhoff’s Law: The electric current flowing in the circuit satisfies:

$$L I''(t) + R I'(t) + \frac{1}{C} \int_{t_0}^{t} I(s) \, ds = 0.$$ 

Derivate both sides above: $L I''(t) + R I'(t) + \frac{1}{C} I(t) = 0$.

Divide by $L$: $I''(t) + 2\left(\frac{R}{2L}\right) I'(t) + \frac{1}{LC} I(t) = 0$.

Introduce $\alpha = \frac{R}{2L}$ and $\omega = \frac{1}{\sqrt{LC}}$, then $I'' + 2\alpha I' + \omega^2 I = 0$.

Example

Find real-valued fundamental solutions to $I'' + 2\alpha I' + \omega^2 I = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: The characteristic polynomial is $p(r) = r^2 + 2\alpha r + \omega^2$.

The roots are:

$$r_\pm = \frac{1}{2} \left[ -2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2} \right] \Rightarrow r_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.$$ 

Case (a) $R = 0$. This implies $\alpha = 0$, so $r_\pm = \pm i\omega$. Therefore,

$$I_1(t) = \cos(\omega t), \quad I_2(t) = \sin(\omega t).$$

Remark: When the circuit has no resistance, the current oscillates without dissipation.
Example
Find real-valued fundamental solutions to $l'' + 2\alpha l' + \omega^2 l = 0$, where $\alpha = R/(2L)$, $\omega^2 = 1/(LC)$, in the cases (a) (b) below.

Solution: Recall: $r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$.

Case (b) $R < \sqrt{4L/C}$. This implies

$$R^2 < \frac{4L}{C} \iff \frac{R^2}{4L^2} < \frac{1}{LC} \iff \alpha^2 < \omega^2.$$ 

Therefore, $r_{\pm} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$. The fundamental solutions are

$$l_1(t) = e^{-\alpha t} \cos(\sqrt{\omega^2 - \alpha^2} t), \quad l_2(t) = e^{-\alpha t} \sin(\sqrt{\omega^2 - \alpha^2} t).$$

The resistance $R$ damps the current oscillations.