Review for Exam 2.

- 6 or 7 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Variation of parameters (2.6).
  - Undetermined coefficients (2.5).
  - Constant coefficients, homogeneous, (2.2)-(2.4).
  - Reduction order method, (2.4.2).
  - Second order variable coefficients, (2.1).
  - First order homogeneous (1.3.2).

Review for Exam 2.

- 5 problems.
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- Exam covers:
  - **Variation of parameters (2.6).**
  - Undetermined coefficients (2.5).
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Variation of parameters (2.6).

Theorem (Variation of parameters)

Let p, q, $f : (t_1, t_2) \to \mathbb{R}$ be continuous functions, then let functions y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ be linearly independent solutions to the homogeneous equation

$$y'' + p(t)\,y' + q(t)\,y = 0,$$

and let the function $W_{y_1 y_2}$ be the Wronskian of solutions y_1 and y_2. If the functions u_1 and u_2 are defined by

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{y_1 y_2}(t)}\,dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{y_1 y_2}(t)}\,dt,$$

then a particular solution y_p to the non-homogeneous differential equation $y'' + p(t)\,y' + q(t)\,y = f(t)$ is given by

$$y_p = u_1 y_1 + u_2 y_2.$$

Variation of parameters (2.6).

Proof: Summary: If u_1 and u_2 satisfy $\left\{ \begin{matrix} u_1' y_1 + u_2' y_2 = 0, \\ u_1' y_1' + u_2' y_2' = f, \end{matrix} \right\}$ then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$.

The equations above are simple to solve for u_1 and u_2,

$$u_2' = -\frac{y_1}{y_2}\,u_1' \quad \Rightarrow \quad u_1' y_1' - \frac{y_1 y_2'}{y_2}\,u_1' = f \quad \Rightarrow \quad u_1' \Big(\frac{y_1' y_2 - y_1 y_2'}{y_2} \Big) = f.$$

Since $W_{y_1 y_2} = y_1 y_2' - y_1' y_2$, then $\displaystyle u_1' = -\frac{y_2 f}{W_{y_1 y_2}} \quad \Rightarrow \quad u_2' = \frac{y_1 f}{W_{y_1 y_2}}$.

Integrating in the variable t we obtain

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{y_1 y_2}(t)}\,dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{y_1 y_2}(t)}\,dt,$$

This establishes the Theorem. $\square$

Variation of parameters (2.6).

Example

Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: First, write the equation in the form of the Theorem. That is, divide the whole equation by t^2,

$$y'' - \frac{2}{t^2}\, y = 3 - \frac{1}{t^2} \quad \Rightarrow \quad f(t) = 3 - \frac{1}{t^2}.$$

We know that $y_1 = t^2$ and $y_2 = 1/t$. Their Wronskian is

$$W_{y_1 y_2}(t) = (t^2)\Big(\frac{-1}{t^2}\Big) - (2t)\Big(\frac{1}{t}\Big) \quad \Rightarrow \quad W_{y_1 y_2}(t) = -3.$$

Variation of parameters (2.6).

Example

Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: $y_1 = t^2$, $y_2 = 1/t$, $f(t) = 3 - \dfrac{1}{t^2}$, $W_{y_1 y_2}(t) = -3$.

We now compute y_1 and u_2,

$$u_1' = -\frac{1}{t}\left(3 - \frac{1}{t^2}\right)\frac{1}{-3} = \frac{1}{t} - \frac{1}{3}\, t^{-3} \quad \Rightarrow \quad u_1 = \ln(t) + \frac{1}{6} t^{-2},$$

$$u_2' = (t^2)\left(3 - \frac{1}{t^2}\right)\frac{1}{-3} = -t^2 + \frac{1}{3} \quad \Rightarrow \quad u_2 = -\frac{1}{3}\, t^3 + \frac{1}{3}\, t.$$

Variation of parameters (2.6).

Example

Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: The particular solution $\tilde{y}_p = u_1 y_1 + u_2 y_2$ is

$$\tilde{y}_p = \Big[\ln(t) + \frac{1}{6} t^{-2}\Big](t^2) + \frac{1}{3}(-t^3 + t)(t^{-1})$$

$$\tilde{y}_p = t^2 \ln(t) + \frac{1}{6} - \frac{1}{3}\, t^2 + \frac{1}{3} = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\, t^2$$

$$\tilde{y}_p = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\, y_1(t).$$

A simpler expression is $y_p = t^2 \ln(t) + \dfrac{1}{2}$. ◁

Variation of parameters (2.6).

Example

Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: If we do not remember the formulas for u_1, u_2, we can always solve the system

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f.$$

$$t^2\, u_1' + u_2' \frac{1}{t} = 0, \quad 2t\, u_1' + u_2' \frac{(-1)}{t^2} = 3 - \frac{1}{t^2}.$$

$$u_2' = -t^3\, u_1' \;\Rightarrow\; 2t\, u_1' + t\, u_1' = 3 - \frac{1}{t^2} \;\Rightarrow\; \begin{cases} u_1' = \dfrac{1}{t} - \dfrac{1}{3t^3} \\[2ex] u_2' = -t^2 + \dfrac{1}{3}. \end{cases}$$

Variation of parameters (2.6).

Example

Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2}\, e^{-2x}.$$

Solution: We find the solutions of the homogeneous equation,

$$r^2 + 4r + 4 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[-4 \pm \sqrt{16 - 16}\right] \quad \Rightarrow \quad r_\pm = -2.$$

Fundamental solutions of the homogeneous equations are

$$y_1 = e^{-2x}, \quad y_2 = x\, e^{-2x}.$$

We now compute their Wronskian,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & x\, e^{-2x} \\ -2e^{-2x} & (1-2x)\, e^{-2x} \end{vmatrix} = (1-2x)\, e^{-4x} + 2x\, e^{-4x}.$$

Hence $W = e^{-4x}$.

Variation of parameters (2.6).

Example

Use the variation of parameters to find the general solution of

$$y'' + 4y' + 4y = x^{-2}\, e^{-2x}.$$

Solution: $y_1 = e^{-2x}$, $y_2 = x\, e^{-2x}$, $g = x^{-2}\, e^{-2x}$, $W = e^{-4x}$.

Now we find the functions u_1 and u_2,

$$u_1' = -\frac{y_2 g}{W} = -\frac{x\, e^{-2x}\, x^{-2}\, e{-2x}}{e^{-4x}} = -\frac{1}{x} \quad \Rightarrow \quad u_1 = -\ln|x|.$$

$$u_2' = \frac{y_1 g}{W} = \frac{e^{-2x}\, x^{-2}\, e{-2x}}{e^{-4x}} = x^{-2} \quad \Rightarrow \quad u_2 = -\frac{1}{x}.$$

$$y_p = u_1 y_1 + u_2 y_2 = -\ln|x|\, e^{-2x} - \frac{1}{x}\, xe^{-2x} = -(1 + \ln|x|)\, e^{-2x}.$$

Since $\tilde{y}_p = -\ln|x|\, e^{-2x}$ is solution, $y = (c_1 + c_2 x - \ln|x|)\, e^{-2x}$. $\lhd$

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Undetermined coefficients (2.5).

Guessing Solution Table.

$f_i(t)$ (K, m, a, b, given.)	$y_{p_i}(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
Kt^m	$k_m t^m + k_{m-1} t^{m-1} + \cdots + k_0$
$K \cos(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$K \sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$Kt^m e^{at}$	$e^{at}(k_m t^m + \cdots + k_0)$
$Ke^{at} \cos(bt)$	$e^{at}\left[k_1 \cos(bt) + k_2 \sin(bt)\right]$
$KKe^{at} \sin(bt)$	$e^{at}\left[k_1 \cos(bt) + k_2 \sin(bt)\right]$
$Kt^m \cos(bt)$	$\left(k_m t^m + \cdots + k_0\right)\left[a_1 \cos(bt) + a_2 \sin(bt)\right]$
$Kt^m \sin(bt)$	$\left(k_m t^m + \cdots + k_0\right)\left[a_1 \cos(bt) + a_2 \sin(bt)\right]$

Undetermined coefficients (2.5).

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: We know that the general solution to homogeneous equation is $y(t) = c_1 e^{4t} + c_2 e^{-t}$.

Following the table: Since $f = 2\sin(t)$, then we guess

$$y_p = k_1 \sin(t) + k_2 \cos(t).$$

This guess satisfies $L(y_p) \neq 0$.

Compute: $y_p' = k_1 \cos(t) - k_2 \sin(t)$, $y_p'' = -k_1 \sin(t) - k_2 \cos(t)$.

$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$
$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

Undetermined coefficients (2.5).

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: Recall:

$$L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)]$$
$$-4[k_1 \sin(t) + k_2 \cos(t)] = 2\sin(t),$$

$$(-5k_1 + 3k_2)\sin(t) + (-3k_1 - 5k_2)\cos(t) = 2\sin(t).$$

This equation holds for all $t \in \mathbb{R}$. In particular, at $t = \frac{\pi}{2}$, $t = 0$.

$$\left.\begin{aligned} -5k_1 + 3k_2 &= 2, \\ -3k_1 - 5k_2 &= 0, \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} k_1 &= -\frac{5}{17}, \\ k_2 &= \frac{3}{17}. \end{aligned}\right.$$

Undetermined coefficients (2.5).

Example

Find all the solutions to the inhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: Recall: $k_1 = -\dfrac{5}{17}$ and $k_2 = \dfrac{3}{17}$.

So the particular solution to the inhomogeneous equation is

$$y_p(t) = \frac{1}{17}\big[-5\sin(t) + 3\cos(t)\big].$$

The general solution is

$$y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{1}{17}\big[-5\sin(t) + 3\cos(t)\big]. \qquad \triangleleft$$

Undetermined coefficients (2.5)

Example

Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}$$

Solution: Find the solutions of the homogeneous problem,

$$r^2 + 4 = 0 \quad \Rightarrow \quad r_{\pm} = \pm 2i.$$

$$y_1 = \cos(2x), \quad y_2 = \sin(2x).$$

Start with the first source, $f_1(x) = 3\sin(2x)$.
The function $\tilde{y}_{p_1} = k_1 \sin(2x) + k_2 \cos(2x)$ is the wrong guess, since it is solution of the homogeneous equation. We guess:

$$y_p = x\big[k_1 \sin(2x) + k_2 \cos(2x)\big].$$

$$y_p' = \big[k_1 \sin(2x) + k_2 \cos(2x)\big] + 2x\big[k_1 \cos(2x) - k_2 \sin(2x)\big].$$

$$y_p'' = 4\big[k_1 \cos(2x) - k_2 \sin(2x)\big] + 4x\big[-k_1 \sin(2x) - k_2 \cos(2x)\big].$$

Undetermined coefficients (2.5)

Example

Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}.$$

Solution: Recall: $y_1 = \sin(2x)$, and $y_2 = \cos(2x)$.

$$4\big[k_1\cos(2x) - k_2\sin(2x)\big] + 4x\big[-k_1\sin(2x) - k_2\cos(2x)\big] + 4x\big[k_1\sin(2x) + k_2\cos(2x)\big] = 3\sin(2x),$$

Therefore, $4\big[k_1\cos(2x) - k_2\sin(2x)\big] = 3\sin(2x)$.

Evaluating at $x = 0$ and $x = \pi/4$ we get

$$4k_1 = 0, \quad -4k_2 = 3 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = -\frac{3}{4}.$$

Therefore, $y_{p_1} = -\frac{3}{4}\,x\cos(2x)$.

Undetermined coefficients (2.5)

Example

Use the undetermined coefficients to find the general solution of

$$y'' + 4y = 3\sin(2x) + e^{3x}.$$

Solution: Recall: $y_{p_1} = -\frac{3}{4}\,x\cos(2x)$.

We now compute y_{p_2} for $f_2(x) = e^{3x}$.

We guess: $y_{p_2} = k\,e^{3x}$. Then, $y''_{p_2} = 9\,e^{3x}$,

$$(9+4)ke^{3x} = e^{3x} \quad \Rightarrow \quad k = \frac{1}{13} \quad \Rightarrow \quad y_{p_2} = \frac{1}{13}\,e^{3x}.$$

Therefore, the general solution is

$$y(x) = c_1\sin(2x) + \Big(c_2 - \frac{3}{4}\,x\Big)\cos(2x) + \frac{1}{13}\,e^{3x}. \qquad \triangleleft$$

Undetermined coefficients (2.5).

Example

- For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess

$$y_p(t) = \left[k_1 \sin(t) + k_2 \cos(t)\right] e^{2t}.$$

- For $y'' - 3y' - 4y = 2t^2 e^{3t}$, guess

$$y_p(t) = \left(k_0 + k_1 t + k_2 t^2\right) e^{3t}.$$

- For $y'' - 3y' - 4y = 3t\,\sin(t)$, guess

$$y_p(t) = (1 + k_1 t)\left[k_2 \sin(t) + k_3 \cos(t)\right].$$