We study: \( y'' + p(t) y' + q(t) y = f(t) \).

Method of variation of parameters.

Using the method in an example.

The proof of the variation of parameter method.

Using the method in another example.

Method of variation of parameters.

Remarks:

This is a general method to find solutions to equations having variable coefficients and non-homogeneous with a continuous but otherwise arbitrary source function, \( y'' + p(t) y' + q(t) y = f(t) \).

The variation of parameter method can be applied to more general equations than the undetermined coefficients method.

The variation of parameter method usually takes more time to implement than the simpler method of undetermined coefficients.
Method of variation of parameters.

**Theorem (Variation of parameters)**

Let $p, q, f : (t_1, t_2) \to \mathbb{R}$ be continuous functions, then let functions $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ be linearly independent solutions to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0,$$

and let the function $W_{y_1y_2}$ be the Wronskian of solutions $y_1$ and $y_2$. If the functions $u_1$ and $u_2$ are defined by

$$u_1(t) = \int -\frac{y_2(t)f(t)}{W_{y_1y_2}(t)} \, dt, \quad u_2(t) = \int \frac{y_1(t)f(t)}{W_{y_1y_2}(t)} \, dt,$$

then a particular solution $y_p$ to the non-homogeneous differential equation $y'' + p(t)y' + q(t)y = f(t)$ is given by

$$y_p = u_1y_1 + u_2y_2.$$

Non-homogeneous equations (Sect. 2.6).

- We study: $y'' + p(t)y' + q(t)y = f(t)$.
- Method of variation of parameters.
- **Using the method in an example.**
- The proof of the variation of parameter method.
- Using the method in another example.
Using the method in an example.

Example
Find the general solution of the inhomogeneous equation
\[ y'' - 5y' + 6y = 2e^t. \]

Solution:
First: Find fundamental solutions to the homogeneous equation. The characteristic equation is
\[ r^2 - 5r + 6 = 0 \quad \Rightarrow \quad r = \frac{1}{2}(5 \pm \sqrt{25 - 24}) \quad \Rightarrow \quad \begin{cases} r_1 = 3, \\ r_2 = 2. \end{cases} \]
Hence, \( y_1(t) = e^{3t} \) and \( y_2(t) = e^{2t} \). Compute their Wronskian,
\[ W_{y_1y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t}) \quad \Rightarrow \quad W_{y_1y_2}(t) = -e^{5t}. \]

Second: We compute the functions \( u_1 \) and \( u_2 \). By definition,
\[ u_1' = -\frac{y_2f}{W_{y_1y_2}}, \quad u_2' = \frac{y_1f}{W_{y_1y_2}}. \]

Third: The particular solution is
\[ y_p = (-e^{2t})(e^{3t}) + (2e^{t})(e^{2t}) \quad \Rightarrow \quad y_p = e^t. \]
The general solution is
\[ y(t) = c_1e^{3t} + c_2e^{2t} + e^t, \quad c_1, c_2 \in \mathbb{R}. \]
Non-homogeneous equations (Sect. 2.6).

- We study: \( y'' + p(t) y' + q(t) y = f(t) \).
- Method of variation of parameters.
- Using the method in an example.
- **The proof of the variation of parameter method.**
- Using the method in another example.

The proof of the variation of parameter method.

**Proof:** Denote \( L(y) = y'' + p(t) y' + q(t) y \).

We need to find \( y_p \) solution of \( L(y_p) = f \).

We know \( y_1 \) and \( y_2 \) solutions of \( L(y_1) = 0 \) and \( L(y_2) = 0 \).

Idea: The reduction of order method: Find \( y_2 \) proposing \( y_2 = u y_1 \).

**First idea:** Propose that \( y_p \) is given by \( y_p = u_1 y_1 + u_2 y_2 \).

We hope that the equation for \( u_1 \) and \( u_2 \) will be simpler than the original equation for \( y_p \), since \( y_1 \) and \( y_2 \) are solutions to the homogeneous equation.

Compute:

\[
y'_p = u'_1 y_1 + u_1 y'_1 + u'_2 y_2 + u_2 y'_2,
\]

\[
y''_p = u''_1 y_1 + 2u'_1 y'_1 + u_1 y''_1 + u''_2 y_2 + 2u'_2 y'_2 + u_2 y''_2.
\]
The proof of the variation of parameter method.

Proof: Then $L(y_p) = f$ is given by

$$
[u''_1 y_1 + 2u'_1 y'_1 + u''_1 y'_1 + u''_2 y_2 + 2u'_2 y'_2 + u_2 y''_2]
$$

$$
p(t)[u'_1 y_1 + u_1 y'_1 + u'_2 y_2 + u_2 y'_2] + q(t)[u_1 y_1 + u_2 y_2] = f(t).
$$

Then

$$
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) + p(u'_1 y_1 + u'_2 y_2)
$$

$$
+ u_1 (y''_1 + p y'_1 + q y_1) + u_2 (y''_2 + p y'_2 + q y_2) = f
$$

Recall: $y''_1 + p y'_1 + q y_1 = 0$ and $y''_2 + p y'_2 + q y_2 = 0$. Hence,

$$
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) + p(u'_1 y_1 + u'_2 y_2) = f
$$

Second idea: Look for $u_1$ and $u_2$ that satisfy the extra equation

$$
u'_1 y_1 + u'_2 y_2 = 0.
$$

The proof of the variation of parameter method.

Proof: Recall: $u'_1 y_1 + u'_2 y_2 = 0$ and

$$
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) + p(u'_1 y_1 + u'_2 y_2) = f.
$$

These two equations imply that $L(y_p) = f$ is

$$
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) = f.
$$

From $u'_1 y_1 + u'_2 y_2 = 0$ we get $[u'_1 y_1 + u'_2 y_2]' = 0$, that is

$$
u''_1 y_1 + u''_2 y_2 + (u'_1 y'_1 + u'_2 y'_2) = 0.
$$

This information in $L(y_p) = f$ implies

$$
u'_1 y'_1 + u'_2 y'_2 = f.
$$

Summary: If $u_1$ and $u_2$ satisfy $u'_1 y_1 + u'_2 y_2 = 0$ and $u'_1 y'_1 + u'_2 y'_2 = f$, then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$. 
The proof of the variation of parameter method.

Proof: Summary: If \( u_1 \) and \( u_2 \) satisfy \[
\begin{align*}
    u_1'y_1 + u_2'y_2 &= 0, \\
    u_1'y_1' + u_2'y_2' &= f,
\end{align*}
\] then \( y_p = u_1y_1 + u_2y_2 \) satisfies \( L(y_p) = f \).

The equations above are simple to solve for \( u_1 \) and \( u_2 \),
\[
    u_2' = -\frac{y_1}{y_2} u_1' \implies u_1'y_1' - \frac{y_1'y_2'}{y_2} u_1' = f \implies u_1' \left( \frac{y_1'y_2 - y_1'y_2'}{y_2} \right) = f.
\]

Since \( W_{y_1y_2} = y_1y'_2 - y'_1y_2 \), then \( u_1' = -\frac{y_2f}{W_{y_1y_2}} \implies u_2' = \frac{y_1f}{W_{y_1y_2}} \).

Integrating in the variable \( t \) we obtain
\[
    u_1(t) = \int -\frac{y_2(t)f(t)}{W_{y_1y_2}(t)} \, dt, \quad u_2(t) = \int \frac{y_1(t)f(t)}{W_{y_1y_2}(t)} \, dt,
\]
This establishes the Theorem. \( \square \)

Non-homogeneous equations (Sect. 2.6).

- We study: \( y'' + p(t)y' + q(t)y = f(t) \).
- Method of variation of parameters.
- Using the method in an example.
- The proof of the variation of parameter method.
- **Using the method in another example.**
Using the method in another example.

Example
Find a particular solution to the differential equation
\[ t^2 y'' - 2y = 3t^2 - 1, \]
knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: First, write the equation in the form of the Theorem. That is, divide the whole equation by \( t^2 \),

\[ y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2} \Rightarrow f(t) = 3 - \frac{1}{t^2}. \]

We know that \( y_1 = t^2 \) and \( y_2 = 1/t \). Their Wronskian is

\[ W_{y_1y_2}(t) = (t^2)\left(\frac{-1}{t^2}\right) - (2t)\left(\frac{1}{t}\right) \Rightarrow W_{y_1y_2}(t) = -3. \]

Using the method in another example.

Example
Find a particular solution to the differential equation
\[ t^2 y'' - 2y = 3t^2 - 1, \]
knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: \( y_1 = t^2, \ y_2 = 1/t, \ f(t) = 3 - \frac{1}{t^2}, \ W_{y_1y_2}(t) = -3. \)

We now compute \( y_1 \) and \( u_2 \),

\[ u_1' = -\frac{1}{t} \left( 3 - \frac{1}{t^2} \right) \frac{1}{-3} = \frac{1}{t} - \frac{1}{3} t^{-3} \Rightarrow u_1 = \ln(t) + \frac{1}{6} t^{-2}, \]

\[ u_2' = (t^2) \left( 3 - \frac{1}{t^2} \right) \frac{1}{-3} = -t^2 + \frac{1}{3} \Rightarrow u_2 = -\frac{1}{3} t^3 + \frac{1}{3} t. \]
Using the method in another example.

Example
Find a particular solution to the differential equation
\[ t^2 y'' - 2y = 3t^2 - 1, \]
knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: The particular solution \( \tilde{y}_p = u_1 y_1 + u_2 y_2 \) is
\[
\tilde{y}_p = \left[ \ln(t) + \frac{1}{6} t^{-2} \right] (t^2) + \frac{1}{3} (-t^3 + t)(t^{-1})
\]
\[
\tilde{y}_p = t^2 \ln(t) + \frac{1}{6} - \frac{1}{3} t^2 + \frac{1}{3} = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3} t^2
\]
\[
\tilde{y}_p = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3} y_1(t).
\]
A simpler expression is \( y_p = t^2 \ln(t) + \frac{1}{2} \).

Using the method in another example.

Example
Find a particular solution to the differential equation
\[ t^2 y'' - 2y = 3t^2 - 1, \]
knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: If we do not remember the formulas for \( u_1, u_2 \), we can always solve the system
\[
u_1' y_1 + u_2' y_2 = 0
\]
\[
u_1' y_1' + u_2' y_2' = f.
\]
\[
t^2 u_1' + u_2' \frac{1}{t} = 0, \quad 2t u_1' + u_2' \frac{(-1)}{t^2} = 3 - \frac{1}{t^2}.
\]
\[
u_2' = -t^3 u_1' \Rightarrow 2t u_1' + t u_1' = 3 - \frac{1}{t^2} \Rightarrow \begin{cases} u_1' = \frac{1}{t} - \frac{1}{3t^3} \\ u_2' = -t^2 + \frac{1}{3} \end{cases}.
\]