Non-homogeneous equations (Sect. 2.6).

- We study: $y'' + p(t)\, y' + q(t)\, y = f(t)$.
- Method of variation of parameters.
- Using the method in an example.
- The proof of the variation of parameter method.
- Using the method in another example.

Method of variation of parameters.

Remarks:

- This is a general method to find solutions to equations having variable coefficients and non-homogeneous with a continuous but otherwise arbitrary source function,

$$y'' + p(t)\, y' + q(t)\, y = f(t).$$

- The variation of parameter method can be applied to more general equations than the undetermined coefficients method.
- The variation of parameter method usually takes more time to implement than the simpler method of undetermined coefficients.

Method of variation of parameters.

Theorem (Variation of parameters)

Let p, q, $f : (t_1, t_2) \to \mathbb{R}$ be continuous functions, then let functions y_1, $y_2 : (t_1, t_2) \to \mathbb{R}$ be linearly independent solutions to the homogeneous equation

$$y'' + p(t)\, y' + q(t)\, y = 0,$$

and let the function $W_{y_1 y_2}$ be the Wronskian of solutions y_1 and y_2. If the functions u_1 and u_2 are defined by

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{y_1 y_2}(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{y_1 y_2}(t)}\, dt,$$

then a particular solution y_p to the non-homogeneous differential equation $y'' + p(t)\, y' + q(t)\, y = f(t)$ is given by

$$y_p = u_1 y_1 + u_2 y_2.$$

Non-homogeneous equations (Sect. 2.6).

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Using the method in an example.

Example

Find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = 2e^t.$$

Solution:

First: Find fundamental solutions to the homogeneous equation. The characteristic equation is

$$r^2 - 5r + 6 = 0 \quad \Rightarrow \quad r = \frac{1}{2}\big(5 \pm \sqrt{25-24}\big) \quad \Rightarrow \quad \begin{cases} r_1 = 3, \\ r_2 = 2. \end{cases}$$

Hence, $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$. Compute their Wronskian,

$$W_{y_1 y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t}) \quad \Rightarrow \quad W_{y_1 y_2}(t) = -e^{5t}.$$

Second: We compute the functions u_1 and u_2. By definition,

$$u_1' = -\frac{y_2 f}{W_{y_1 y_2}}, \qquad u_2' = \frac{y_1 f}{W_{y_1 y_2}}.$$

Using the method in an example.

Example

Find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = 2e^t.$$

Solution: Recall: $y_1(t) = e^{3t}$, $y_2(t) = e^{2t}$, $W_{y_1 y_2}(t) = -e^{5t}$, and

$$u_1' = -\frac{y_2 f}{W_{y_1 y_2}}, \qquad u_2' = \frac{y_1 f}{W_{y_1 y_2}}.$$

$$u_1' = -e^{2t}(2e^t)(-e^{-5t}) \quad \Rightarrow \quad u_1' = 2e^{-2t} \quad \Rightarrow \quad u_1 = -e^{-2t},$$

$$u_2' = e^{3t}(2e^t)(-e^{-5t}) \quad \Rightarrow \quad u_2' = -2e^{-t} \quad \Rightarrow \quad u_2 = 2e^{-t}.$$

Third: The particular solution is

$$y_p = (-e^{-2t})(e^{3t}) + (2e^{-t})(e^{2t}) \quad \Rightarrow \quad y_p = e^t.$$

The general solution is $y(t) = c_1 e^{3t} + c_2 e^{2t} + e^t$, $c_1, c_2 \in \mathbb{R}$. $\triangleleft$

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The proof of the variation of parameter method.

Proof: Denote $L(y) = y'' + p(t)\, y' + q(t)\, y$.

We need to find y_p solution of $L(y_p) = f$.

We know y_1 and y_2 solutions of $L(y_1) = 0$ and $L(y_2) = 0$.

Idea: The reduction of order method: Find y_2 proposing $y_2 = uy_1$.

First idea: Propose that y_p is given by $y_p = u_1 y_1 + u_2 y_2$.

We hope that the equation for u_1 and u_2 will be simpler than the original equation for y_p, since y_1 and y_2 are solutions to the homogeneous equation. Compute:

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2',$$

$$y_p'' = u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''.$$

The proof of the variation of parameter method.

Proof: Then $L(y_p) = f$ is given by

$$\left[u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''\right]$$

$$p(t)\left[u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'\right] + q(t)\left[u_1 y_1 + u_2 y_2\right] = f(t).$$

$$u_1'' y_1 + u_2'' y_2 + 2(u_1' y_1' + u_2 y_2') + p\,(u_1' y_1 + u_2' y_2)$$
$$+u_1\,(y_1'' + p\,y_1' + q\,y_1) + u_2\,(y_2'' + p\,y_2' + q\,y_2) = f$$

Recall: $y_1'' + p\,y_1' + q\,y_1 = 0$ and $y_2'' + p\,y_2' + q\,y_2 = 0$. Hence,

$$u_1'' y_1 + u_2'' y_2 + 2(u_1' y_1' + u_2' y_2') + p\,(u_1' y_1 + u_2' y_2) = f$$

Second idea: Look for u_1 and u_2 that satisfy the extra equation

$$u_1' y_1 + u_2' y_2 = 0.$$

The proof of the variation of parameter method.

Proof: Recall: $u_1' y_1 + u_2' y_2 = 0$ and

$$u_1'' y_1 + u_2'' y_2 + 2(u_1' y_1' + u_2' y_2') + p\,(u_1' y_1 + u_2' y_2) = f.$$

These two equations imply that $L(y_p) = f$ is

$$u_1'' y_1 + u_2'' y_2 + 2(u_1' y_1' + u_2' y_2') = f.$$

From $u_1' y_1 + u_2' y_2 = 0$ we get $\left[u_1' y_1 + u_2' y_2\right]' = 0$, that is

$$u_1'' y_1 + u_2'' y_2 + (u_1' y_1' + u_2' y_2') = 0.$$

This information in $L(y_p) = f$ implies

$$u_1' y_1' + u_2' y_2' = f.$$

Summary: If u_1 and u_2 satisfy $u_1' y_1 + u_2' y_2 = 0$ and $u_1' y_1' + u_2' y_2' = f$, then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$.

The proof of the variation of parameter method.

Proof: Summary: If u_1 and u_2 satisfy $\left\{\begin{array}{l} u_1' y_1 + u_2' y_2 = 0, \\ u_1' y_1' + u_2' y_2' = f, \end{array}\right\}$ then $y_p = u_1 y_1 + u_2 y_2$ satisfies $L(y_p) = f$.

The equations above are simple to solve for u_1 and u_2,

$$u_2' = -\frac{y_1}{y_2}\, u_1' \quad \Rightarrow \quad u_1' y_1' - \frac{y_1 y_2'}{y_2}\, u_1' = f \quad \Rightarrow \quad u_1' \Big(\frac{y_1' y_2 - y_1 y_2'}{y_2} \Big) = f.$$

Since $W_{y_1 y_2} = y_1 y_2' - y_1' y_2$, then $u_1' = -\dfrac{y_2 f}{W_{y_1 y_2}} \quad \Rightarrow \quad u_2' = \dfrac{y_1 f}{W_{y_1 y_2}}$.

Integrating in the variable t we obtain

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{y_1 y_2}(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{y_1 y_2}(t)}\, dt,$$

This establishes the Theorem. $\square$

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Using the method in another example.

Example

Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: First, write the equation in the form of the Theorem. That is, divide the whole equation by t^2,

$$y'' - \frac{2}{t^2}\, y = 3 - \frac{1}{t^2} \quad \Rightarrow \quad f(t) = 3 - \frac{1}{t^2}.$$

We know that $y_1 = t^2$ and $y_2 = 1/t$. Their Wronskian is

$$W_{y_1 y_2}(t) = (t^2)\Big(\frac{-1}{t^2}\Big) - (2t)\Big(\frac{1}{t}\Big) \quad \Rightarrow \quad W_{y_1 y_2}(t) = -3.$$

Using the method in another example.

Example

Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: $y_1 = t^2$, $y_2 = 1/t$, $f(t) = 3 - \dfrac{1}{t^2}$, $W_{y_1 y_2}(t) = -3$.

We now compute y_1 and u_2,

$$u_1' = -\frac{1}{t}\left(3 - \frac{1}{t^2}\right)\frac{1}{-3} = \frac{1}{t} - \frac{1}{3}\, t^{-3} \quad \Rightarrow \quad u_1 = \ln(t) + \frac{1}{6} t^{-2},$$

$$u_2' = (t^2)\left(3 - \frac{1}{t^2}\right)\frac{1}{-3} = -t^2 + \frac{1}{3} \quad \Rightarrow \quad u_2 = -\frac{1}{3}\, t^3 + \frac{1}{3} t.$$

Using the method in another example.

Example

Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: The particular solution $\tilde{y}_p = u_1 y_1 + u_2 y_2$ is

$$\tilde{y}_p = \Big[\ln(t) + \frac{1}{6} t^{-2}\Big](t^2) + \frac{1}{3}(-t^3 + t)(t^{-1})$$

$$\tilde{y}_p = t^2 \ln(t) + \frac{1}{6} - \frac{1}{3}\, t^2 + \frac{1}{3} = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\, t^2$$

$$\tilde{y}_p = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\, y_1(t).$$

A simpler expression is $y_p = t^2 \ln(t) + \frac{1}{2}$. $\triangleleft$

Using the method in another example.

Example

Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that the functions $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: If we do not remember the formulas for u_1, u_2, we can always solve the system

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f. \end{aligned}$$

$$t^2\, u_1' + u_2' \frac{1}{t} = 0, \quad 2t\, u_1' + u_2' \frac{(-1)}{t^2} = 3 - \frac{1}{t^2}.$$

$$u_2' = -t^3\, u_1' \;\Rightarrow\; 2t\, u_1' + t\, u_1' = 3 - \frac{1}{t^2} \;\Rightarrow\; \begin{cases} u_1' = \dfrac{1}{t} - \dfrac{1}{3t^3} \\ u_2' = -t^2 + \dfrac{1}{3}. \end{cases}$$