Review 1 for Exam 1.

- 6 or 7 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks, webwork.
- Exam covers:
  - Linear equations (1.1), (1.2).
  - Bernoulli equation (1.2).
  - Separable equations (1.3).
  - Homogeneous equations (1.3).
  - Exact equations (1.4).
  - Exact equations with integrating factors (1.4).
  - Modeling (1.5).

Exam overview

Remark:
- Exam problems will be: Solve this equation. We don’t tell you if the equation is linear, separable, etc. You must find that out.
- If you know what type of equation is, then the equation is simple to solve.
- The difficult part in Exam 1 is to know what type of equation is the one you have to solve.
Exam overview

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t) y = b(t)$.)

2. Bernoulli equations.
   (Just by looking at it: $y' + a(t) y = b(t) y^n$.)

   (Few manipulations: $h(y) y' = g(t)$.)

   (Several manipulations: $y' = F(y/t)$.)

5. Exact equations.
   (Check one equation: $Ny' + M = 0$, and $\partial_t N = \partial_y M$.)

6. Exact equation with integrating factor.
   (Very complicated to check.)

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Example
Find every solution $y$ to the equation $(t^2 + y^2)(t + y y') + 2 = 0$.

Solution: Rewrite the equation in a more standard way:

$$(t^2 + y^2)y y' + (t^2 + y^2)t + 2 = 0 \iff y' = -\frac{(t^2 + y^2)t + 2}{(t^2 + y^2)y}.$$  

So the equation must be exact or exact with integrating factor.

$$N = t^2 y + y^3 \quad \Rightarrow \quad \partial_t N = 2ty.$$  
$$M = t^3 + ty^2 + 2 \quad \Rightarrow \quad \partial_y M = 2ty.$$  

The equation is exact: $\partial_t N = \partial_y M$. 
Example
Find the explicit solution \( y \) to the IVP

\[
y' = \frac{t(t^2 + e^t)}{4y^3}, \quad y(0) = -\sqrt{2}.
\]

Solution: Not linear. Bernoulli with \( n = -3 \). Numerator depends only on \( t \), denominator depends only on \( y \): Separable.

\[
4y^3y' = t^3 + te^t \quad \Rightarrow \quad \int 4y^3 y' \, dt = \int (t^3 + te^t) \, dt + c
\]

The usual substitution: \( u = y(t) \) implies \( du = y'(t) \, dt \),

\[
\int 4u^3 \, du = \int (t^3 + te^t) \, dt + c \quad \Rightarrow \quad u^4 = \frac{t^4}{4} + \int te^t \, dt + c.
\]
Example
Find the explicit solution $y$ to the IVP
$$y' = \frac{t(t^2 + e^t)}{4y^3}, \quad y(0) = -\sqrt{2}.$$ 

Solution: Recall: $u^4 = \frac{t^4}{4} + \int te^t \, dt + c$. Integration by parts:
$$f = t, \quad g' = e^t; \quad f' = 1, \quad g = e^t, \quad \Rightarrow \quad \int te^t \, dt = te^t - \int e^t \, dt = (t-1)e^t.$$

We obtain: $y^4(t) = \frac{t^4}{4} + (t-1)e^t + c$. The initial condition:
$$(-\sqrt{2})^4 = 0 + (0-1) + c \Rightarrow 4 = -1 + c \Rightarrow c = 5.$$

We conclude: $y^4(t) = \frac{t^4}{4} + (t-1)e^t + 5$. Implicit form.
Example

Find every solution $y$ of the equation $y' = \frac{3y^2 - t^2}{2ty}$.

Solution: Not linear. Bernoulli $n = -1$: $y' = \frac{3y}{2t} - \frac{t}{2y}$.

Not separable. Every term on the right hand side is of the form $t^ny^m$ with $n + m = 2$. Homogeneous.

$$y' = \frac{3y^2 - t^2}{2ty} \cdot \left(\frac{1}{t^2}\right) \Rightarrow y' = \frac{3\left(\frac{y}{t}\right)^2 - 1}{2\left(\frac{y}{t}\right)}.$$

We introduce the change of unknown:

$$v = \frac{y}{t} \Rightarrow y = t\,v \Rightarrow y' = v + t\,v'.$$

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Example

Find every solution $y$ of the equation $y' = \frac{3y^2 - t^2}{2ty}$.

Solution: $y' = \frac{3\left(\frac{y}{t}\right)^2 - 1}{2\left(\frac{y}{t}\right)}$, $v = \frac{y}{t}$, $y' = v + t\,v'$.

$$v + t\,v' = \frac{3v^2 - 1}{2v} \Rightarrow t\,v' = \frac{3v^2 - 1}{2v} - v = \frac{3v^2 - 1 - 2v^2}{2v}$$

$$t\,v' = \frac{v^2 - 1}{2v} \Rightarrow \frac{2v}{v^2 - 1} \, v' = \frac{1}{t}.$$

This is a separable equation for $v$: $\int \frac{2v}{v^2 - 1} \, v' \, dt = \int \frac{1}{t} \, dt + c$. 
Example
Find every solution $y$ of the equation $y' = \frac{3y^2 - t^2}{2ty}$.

Solution: $\int \frac{2v}{v^2 - 1} \, v' \, dt = \int \frac{1}{t} \, dt + c$.

The substitution $u = v^2 - 1$ implies $du = 2v \, v' \, dt$. So,

$$\int \frac{du}{u} = \int \frac{1}{t} \, dt + c \implies \ln(|u|) = \ln(|t|) + c \implies |u| = c_1 |t|.$$ 

where $c_1 = e^c$. Substitute back: $|v^2 - 1| = c_1 |t|$. Finally, $v = y/t$,

$$\left| \frac{y^2}{t^2} - 1 \right| = c_1 |t| \implies \left| y^2 - t^2 \right| = c_1 |t|^3.$$ 


Example
A water tank initially has $V_0 = 100$ liters of water with $Q_0$ grams of salt. At $t_0 = 0$ fresh water is poured into the tank. The salt in the tank is always well mixed. Find the rates $r_i$ and $r_o$ such that:

(a) The tank water volume is constant.

(b) The time to reduce the salt in the tank to one percent of the initial value is $t_1 = 25$ min.

Solution:
Part (a): Water volume constant implies $r_i = r_o = r$. Then $V'(t) = 0$, so $V(t) = V_0$.

Part (b): First find the salt in the tank $Q(t)$: $\frac{dQ}{dt} = r_i q_i - r_o q_o(t)$.

Incoming fresh water: $q_i = 0$. Mixing: $q_o(t) = Q(t)/V(t)$.

$$\frac{dQ}{dt} = - \frac{r}{V_0} Q(t) \implies Q(t) = Q_0 e^{-rt/V_0}.$$
Example
A water tank initially has $V_0 = 100$ liters of water with $Q_0$ grams of salt. At $t_0 = 0$ fresh water is poured into the tank. The salt in the tank is always well mixed. Find the rates $r_i$ and $r_o$ such that:
(a) The tank water volume is constant.
(b) The time to reduce the salt in the tank to one percent of the initial value is $t_1 = 25$ min.

Solution: Recall: $Q(t) = Q_0 e^{-rt/V_0}$. Condition for $r$:
\[ Q(t_1) = \frac{Q_0}{100} \Rightarrow Q_0 e^{-rt_1/V_0} = \frac{Q_0}{100} \Rightarrow -\frac{rt_1}{V_0} = \ln\left(\frac{1}{100}\right). \]
\[ \frac{rt_1}{V_0} = \ln(100) \Rightarrow r = \frac{V_0}{t_1} \ln(100) \Rightarrow r = 4 \ln(100). \]
\[ \triangledown \]

Example
Find the solution $y$ to the IVP
\[ y' = \frac{2}{t} y - \frac{\sin(t)}{t} y^2, \quad y(2\pi) = 2\pi, \quad t > 0. \]

\[ \frac{y'}{y^2} - \frac{2}{t} \frac{1}{y} = -\frac{\sin(t)}{t}, \quad v = \frac{1}{y} \Rightarrow v' = -\frac{y'}{y^2}. \]
\[ -v' - \frac{2}{t} v = -\frac{\sin(t)}{t} \Rightarrow v' + \frac{2}{t} v = \frac{\sin(t)}{t}. \]
We solve the linear equation with the integrating factor method.
\[ A(t) = \int \frac{2}{t} \, dt = 2\ln(t) = \ln(t^2) \Rightarrow \mu(t) = t^2. \]
Example
Find the solution $y$ to the IVP

$$y' = \frac{2}{t}y - \frac{\sin(t)}{t}y^2, \quad y(2\pi) = 2\pi, \quad t > 0.$$ 

Solution: Recall: $\mu(t) = t^2$. Then,

$$t^2\left(v' + \frac{2}{t}v\right) = t^2\frac{\sin(t)}{t} \quad \Rightarrow \quad (t^2 v)' = t \sin(t).$$

Integrating: $t^2 v = \int t \sin(t)\,dt + c$. The right hand side can be computed integrating by parts,

$$\int t \sin(t)\,dt = -t \cos(t) + \int \cos(t)\,dt, \quad \left\{ \begin{array}{l} f = t, \quad g' = \sin(t), \\ f' = 1, \quad g = -\cos(t). \end{array} \right.$$
Example
Find the integrating factor that converts the equation below into an exact equation, where
\[
\left(x^3 e^y + \frac{x}{y}\right) y' + (2x^2 e^y + 1) = 0.
\]

Solution: We first verify if the equation is not exact.
\[
N = \left(x^3 e^y + \frac{x}{y}\right) \Rightarrow \partial_x N = 3x^2 e^y + \frac{1}{y}.
\]
\[
M = (2x^2 e^y + 1) = 0 \Rightarrow \partial_y M = 2x^2 e^y.
\]
So the equation is not exact. We now compute
\[
\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \left(3x^2 e^y + \frac{1}{y}\right)}{\left(x^3 e^y + \frac{x}{y}\right)} = \frac{-x^2 e^y - \frac{1}{y}}{x\left(x^2 e^y + \frac{1}{y}\right)} = -\frac{1}{x}.
\]

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Example
Find the integrating factor that converts the equation below into an exact equation, where
\[
\left(x^3 e^y + \frac{x}{y}\right) y' + (2x^2 e^y + 1) = 0.
\]

Solution: Recall: \(\frac{\partial_y M - \partial_x N}{N} = -\frac{1}{x}\). Therefore,
\[
\frac{\mu'(x)}{\mu(x)} = -\frac{1}{x} \Rightarrow \ln(\mu) = -\ln(x) = \ln\left(\frac{1}{x}\right) \Rightarrow \mu(x) = \frac{1}{x}.
\]
So the equation \(\left(x^2 e^y + \frac{1}{y}\right) y' + \left(2xe^y + \frac{1}{x}\right) = 0\) is exact. Indeed,
\[
\partial_x \tilde{N} = 2xe^y, \quad \partial_y \tilde{M} = 2xe^y,
\]
\[
\Rightarrow \partial_x \tilde{N} = \partial_y \tilde{M}.
\]