Review: Second order linear ODE.

Definition
Given functions \( a_1, a_0, b : \mathbb{R} \rightarrow \mathbb{R} \), the differential equation in the unknown function \( y : \mathbb{R} \rightarrow \mathbb{R} \) given by

\[
y'' + a_1(t) y' + a_0(t) y = b(t)
\]

is called a second order linear differential equation. If \( b = 0 \), the equation is called homogeneous. If the coefficients \( a_1, a_2 \in \mathbb{R} \) are constants, the equation is called of constant coefficients.

Theorem (Superposition property)
If the functions \( y_1 \) and \( y_2 \) are solutions to the homogeneous linear equation

\[
y'' + a_1(t) y' + a_0(t) y = 0,
\]

then the linear combination \( c_1 y_1(t) + c_2 y_2(t) \) is also a solution for any constants \( c_1, c_2 \in \mathbb{R} \).
Second order linear ODE (Sect. 2.2).

- **Idea**: Solving constant coefficients equations.
- The characteristic equation.
- Solution formulas for constant coefficients equations.

Idea: Solving constant coefficients equations.

**Remark:** Just by trial and error one can find solutions to second order, constant coefficients, homogeneous, linear differential equations. We present the main ideas with an example.

**Example**
Find solutions to the equation $y'' + 5y' + 6y = 0$.

**Solution:** We look for solutions proportional to exponentials $e^{rt}$, for an appropriate constant $r \in \mathbb{R}$, since the exponential can be canceled out from the equation.

If $y(t) = e^{rt}$, then $y'(t) = re^{rt}$, and $y''(t) = r^2 e^{rt}$. Hence

$$(r^2 + 5r + 6)e^{rt} = 0 \iff r^2 + 5r + 6 = 0.$$ 

That is, $r$ must be a root of the polynomial $p(r) = r^2 + 5r + 6$.

This polynomial is called the **characteristic polynomial** of the differential equation.
Idea: Solving constant coefficients equations.

Example
Find solutions to the equation $y'' + 5y' + 6y = 0$.

Solution: Recall: $p(r) = r^2 + 5r + 6$.

The roots of the characteristic polynomial are

$$r = \frac{1}{2} \left(-5 \pm \sqrt{25 - 24}\right) = \frac{1}{2} (-5 \pm 1) \Rightarrow \{ r_1 = -2, r_2 = -3 \}.$$ 

Therefore, we have found two solutions to the ODE,

$$y_1(t) = e^{-2t}, \quad y_2(t) = e^{-3t}.$$ 

Their superposition provides infinitely many solutions,

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \quad c_1, c_2 \in \mathbb{R}.$$ 

Summary: The differential equation $y'' + 5y' + 6y = 0$ has infinitely many solutions,

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \quad c_1, c_2 \in \mathbb{R}.$$ 

Remarks:

- There are two free constants in the solution found above.
- The ODE above is second order, so two integrations must be done to find the solution. This explains the origin of the two free constants in the solution.
- An IVP for a second order differential equation will have a unique solution if the IVP contains two initial conditions.
Second order linear ODE (Sect. 2.2).

- Idea: Solving constant coefficients equations.
- **The characteristic equation.**
- Solution formulas for constant coefficients equations.

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**The characteristic equation.**

**Definition**

Given a second order linear homogeneous differential equation with constant coefficients

\[ y'' + a_1 y' + a_0 = 0, \tag{1} \]

the *characteristic polynomial* and the *characteristic equation* associated with the differential equation in (1) are, respectively,

\[ p(r) = r^2 + a_1 r + a_0, \quad p(r) = 0. \]

**Remark:** If \( r_1, r_2 \) are the solutions of the characteristic equation and \( c_1, c_2 \) are constants, then we will show that the general solution of Eq. (1) is given by

\[ y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \]
Example
Find the solution \( y \) of the initial value problem
\[
y'' + 5y' + 6 = 0, \quad y(0) = 1, \quad y'(0) = -1.
\]

Solution: A solution of the differential equation above is
\[
y(t) = c_1 e^{-2t} + c_2 e^{-3t}.
\]
We now find the constants \( c_1 \) and \( c_2 \) that satisfy the initial conditions above:
\[
1 = y(0) = c_1 + c_2, \quad -1 = y'(0) = -2c_1 - 3c_2.
\]
\[
c_1 = 1 - c_2 \Rightarrow 1 = 2(1 - c_2) + 3c_2 \Rightarrow c_2 = -1 \Rightarrow c_1 = 2.
\]
Therefore, the unique solution to the initial value problem is
\[
y(t) = 2e^{-2t} - e^{-3t}.
\]

Example
Find the general solution \( y \) of the differential equation
\[
2y'' - 3y' + y = 0.
\]

Solution: We look for every solution of the form \( y(t) = e^{rt} \), where \( r \) is a solution of the characteristic equation
\[
2r^2 - 3r + 1 = 0 \Rightarrow r = \frac{1}{4} (3 \pm \sqrt{9 - 8}) \Rightarrow \begin{cases} r_1 &= 1, \\ r_2 &= \frac{1}{2}. \end{cases}
\]
Therefore, the general solution of the equation above is
\[
y(t) = c_1 e^t + c_2 e^{t/2},
\]
where \( c_1, c_2 \) are arbitrary constants.
Second order linear ODE (Sect. 2.2).

- Idea: Solving constant coefficients equations.
- The characteristic equation.
- **Solution formulas for constant coefficients equations.**

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Solution formulas for constant coefficients equations.

**Theorem (Constant coefficients)**

Given real constants $a_1$, $a_0$, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$ y'' + a_1 y' + a_0 y = 0. $$

Let $r_+, r_- \in \mathbb{R}$ be the roots of the characteristic polynomial

$$ p(r) = r^2 + a_1 r + a_0, $$

and let $c_0$, $c_1$ be arbitrary constants. Then, the general solution of the differential equation is given by:

(a) If $r_+ \neq r_-$, real or complex, then

$$ y(t) = c_0 e^{r_+ t} + c_1 e^{r_- t}. $$

(b) If $r_+ = r_- = \hat{r} \in \mathbb{R}$, then

$$ y(t) = c_0 e^{\hat{r} t} + c_1 t e^{\hat{r} t}. $$

Furthermore, given real constants $t_0$, $y_0$ and $y_1$, there is a unique solution to the initial value problem

$$ y'' + a_1 y' + a_0 y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1. $$