Exact equations (Sect. 1.4).

- Exact differential equations.
- The Poincaré Lemma.
- Implicit solutions and the potential function.
- Generalization: The integrating factor method.

**Exact differential equations.**

**Definition**

Given an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$ and continuously differentiable functions $M, N : R \to \mathbb{R}$, denoted as $(t, u) \mapsto M(t, u)$ and $(t, u) \mapsto N(t, u)$, the differential equation in the unknown function $y : (t_1, t_2) \to \mathbb{R}$ given by

$$N(t, y(t)) y'(t) + M(t, y(t)) = 0$$

is called **exact** iff for every point $(t, u) \in R$ holds

$$\partial_t N(t, u) = \partial_u M(t, u)$$

**Recall:** we use the notation: $\partial_t N = \frac{\partial N}{\partial t}$, and $\partial_u M = \frac{\partial M}{\partial u}$. 
Exact differential equations.

Example
Show whether the differential equation below is exact,

\[ 2ty'(t) + 2t + y^2(t) = 0. \]

Solution: We first identify the functions \( N \) and \( M \),

\[ [2ty(t)] y'(t) + [2t + y^2(t)] = 0 \Rightarrow \begin{cases} N(t, u) = 2tu, \\ M(t, u) = 2t + u^2. \end{cases} \]

The equation is exact iff \( \partial_t N = \partial_u M \). Since

\[ N(t, u) = 2tu \Rightarrow \partial_t N(t, u) = 2u, \]
\[ M(t, u) = 2t + u^2 \Rightarrow \partial_u M(t, u) = 2u. \]

We conclude: \( \partial_t N(t, u) = \partial_u M(t, u) \).

Remark: The ODE above is not separable and non-linear.

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Exact differential equations.

Example
Show whether the differential equation below is exact,

\[ \sin(t)y'(t) + t^2e^{y(t)}y'(t) - y'(t) = -y(t)\cos(t) - 2te^{y(t)}. \]

Solution: We first identify the functions \( N \) and \( M \), if we write

\[ [\sin(t) + t^2e^{y(t)} - 1] y'(t) + [y(t)\cos(t) + 2te^{y(t)}] = 0, \]

we can see that

\[ N(t, u) = \sin(t) + t^2e^u - 1 \Rightarrow \partial_t N(t, u) = \cos(t) + 2te^u, \]
\[ M(t, u) = u\cos(t) + 2te^u \Rightarrow \partial_u M(t, u) = \cos(t) + 2te^u. \]

The equation is exact, since \( \partial_t N(t, u) = \partial_u M(t, u) \).
Example
Show whether the linear differential equation below is exact,

\[ y'(t) = -a(t)y(t) + b(t), \quad a(t) \neq 0. \]

Solution: We first find the functions \( N \) and \( M \),

\[ y' + a(t)y - b(t) = 0 \quad \Rightarrow \quad \begin{cases} N(t, u) = 1, \\ M(t, u) = a(t)u - b(t). \end{cases} \]

The differential equation is not exact, since

\[ N(t, u) = 1 \quad \Rightarrow \quad \partial_t N(t, u) = 0, \]

\[ M(t, u) = a(t)u - b(t) \quad \Rightarrow \quad \partial_u M(t, u) = a(t). \]

This implies that \( \partial_t N(t, u) \neq \partial_u M(t, u). \)
The Poincaré Lemma.

Remark: The coefficients $N$ and $M$ of an exact equations are the derivatives of a potential function $\psi$.

Lemma (Poincaré)

Given an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, the continuously differentiable functions $M, N : R \to \mathbb{R}$ satisfy the equation

$$\partial_t N(t, u) = \partial_u M(t, u)$$

iff there exists a twice continuously differentiable function $\psi : R \to \mathbb{R}$, called potential function, such that for all $(t, u) \in R$ holds

$$\partial_u \psi(t, u) = N(t, u), \quad \partial_t \psi(t, u) = M(t, u).$$

Proof: ($\Leftarrow$) Simple:

$$\begin{align*}
\partial_t N &= \partial_t \partial_u \psi, \\
\partial_u M &= \partial_u \partial_t \psi,
\end{align*}$$

$\Rightarrow \partial_t N = \partial_u M.$

($\Rightarrow$) Difficult: Poincaré, 1880.

Example

Show that the function $\psi(t, u) = t^2 + tu^2$ is the potential function for the exact differential equation

$$2ty(t)y'(t) + 2t + y^2(t) = 0.$$

Solution: We already saw that the differential equation above is exact, since the functions $M$ and $N$,

$$\begin{align*}
N(t, u) &= 2tu, \\
M(t, u) &= 2t + u^2
\end{align*}$$

$\Rightarrow \partial_t N = 2u = \partial_u M.$

The potential function is $\psi(t, u) = t^2 + tu^2$, since

$$\begin{align*}
\partial_t \psi &= 2t + u^2 = M, \\
\partial_u \psi &= 2tu = N.
\end{align*}$$

Remark: The Poincaré Lemma only states necessary and sufficient conditions on $N$ and $M$ for the existence of $\psi$. 
Exact equations (Sect. 1.4).

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Implicit solutions and the potential function.

**Theorem (Exact differential equations)**

Let $M, N : R \to \mathbb{R}$ be continuously differentiable functions on an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$. If the differential equation

$$N(t, y(t)) y'(t) + M(t, y(t)) = 0 \quad (1)$$

is exact, then every solution $y : (t_1, t_2) \to \mathbb{R}$ must satisfy the algebraic equation

$$\psi(t, y(t)) = c,$$

where $c \in \mathbb{R}$ and $\psi : R \to \mathbb{R}$ is a potential function for Eq. (1).

**Proof:**

$$0 = N(t, y) y' + M(t, y) = \partial_y \psi(t, y) \frac{dy}{dt} + \partial_t \psi(t, y).$$

$$0 = \frac{d}{dt} \psi(t, y(t)) \iff \psi(t, y(t)) = c. \quad \square$$
Implicit solutions and the potential function.

Example
Find all solutions $y$ to the equation

$$\left[ \sin(t) + t^2 e^{y(t)} - 1 \right] y'(t) + y(t) \cos(t) + 2te^{y(t)} = 0.$$ 

Solution: Recall: The equation is exact,

$$N(t, u) = \sin(t) + t^2 e^u - 1 \quad \Rightarrow \quad \partial_t N(t, u) = \cos(t) + 2te^u,$$

$$M(t, u) = u \cos(t) + 2te^u \quad \Rightarrow \quad \partial_u M(t, u) = \cos(t) + 2te^u,$$

hence, $\partial_t N = \partial_u M$. Poincaré Lemma says the exists $\psi$,

$$\partial_u \psi(t, u) = N(t, u), \quad \partial_t \psi(t, u) = M(t, u).$$

These are actually equations for $\psi$. From the first one,

$$\psi(t, u) = \int \left[ \sin(t) + t^2 e^u - 1 \right] du + g(t).$$
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**Remark:**
Sometimes a non-exact equation can we transformed into an exact equation multiplying the equation by an integrating factor. Just like in the case of linear differential equations.

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**Generalization: The integrating factor method.**

**Theorem (Integrating factor)**

Let $M, N : R \rightarrow \mathbb{R}$ be continuously differentiable functions on $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, with $N \neq 0$. If the equation

$$N(t, y(t)) y'(t) + M(t, y(t)) = 0$$

is not exact, that is, $\partial_t N(t, u) \neq \partial_u M(t, u)$, and if the function

$$\frac{1}{N(t, u)} \left[ \partial_u M(t, u) - \partial_t N(t, u) \right]$$

does not depend on the variable $u$, then the equation

$$\mu(t)[N(t, y(t)) y'(t) + M(t, y(t))] = 0$$

is exact, where

$$\frac{\mu'(t)}{\mu(t)} = \frac{1}{N(t, u)} \left[ \partial_u M(t, u) - \partial_t N(t, u) \right].$$
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[
[t^2 + ty(t)] y'(t) + [3t y(t) + y^2(t)] = 0.
\]

Solution: The equation is not exact:
\[
N(t, u) = t^2 + tu \quad \Rightarrow \quad \partial_t N(t, u) = 2t + u,
\]
\[
M(t, u) = 3tu + u^2 \quad \Rightarrow \quad \partial_u M(t, u) = 3t + 2u,
\]
hence $\partial_t N \neq \partial_u M$. We now verify whether the extra condition in Theorem above holds:
\[
\frac{[\partial_u M(t, u) - \partial_t N(t, u)]}{N(t, u)} = \frac{1}{(t^2 + tu)} [(3t + 2u) - (2t + u)]
\]
\[
\frac{[\partial_u M(t, u) - \partial_t N(t, u)]}{N(t, u)} = \frac{1}{t(t+u)} (t+u) = \frac{1}{t}.
\]

Therefore, the equation below is exact:
\[
[t^3 + t^2 y(t)] y'(t) + [3t^2 y(t) + ty^2(t)] = 0.
\]
Generalization: The integrating factor method.

Example
Find all solutions $y$ to the differential equation
\[ [t^2 + t y(t)] y'(t) + [3t y(t) + y^2(t)] = 0. \]

Solution: $[t^3 + t^2 y(t)] y'(t) + [3t^2 y(t) + t y^2(t)] = 0.$

This equation is exact:
\[ \tilde{N}(t, u) = t^3 + t^2 u \quad \Rightarrow \quad \partial_t \tilde{N}(t, u) = 3t^2 + 2tu, \]
\[ \tilde{M}(t, u) = 3t^2 u + tu^2 \quad \Rightarrow \quad \partial_u \tilde{M}(t, u) = 3t^2 + 2tu, \]
that is, $\partial_t \tilde{N} = \partial_u \tilde{M}$. Therefore, there exists $\psi$ such that
\[ \partial_u \psi(t, u) = \tilde{N}(t, u), \quad \partial_t \psi(t, u) = \tilde{M}(t, u). \]

From the first equation above we obtain
\[ \partial_u \psi = t^3 + t^2 u \quad \Rightarrow \quad \psi(t, u) = \int (t^3 + t^2 u) \, du + g(t). \]

Integrating, $\psi(t, u) = t^3 u + \frac{1}{2} t^2 u^2 + g(t)$.

Introduce $\psi$ in $\partial_t \psi = \tilde{M}$, where $\tilde{M} = 3t^2 u + tu^2$. So,
\[ \partial_t \psi(t, u) = 3t^2 u + tu^2 + g'(t) = \tilde{M}(t, u) = 3t^2 u + tu^2, \]

So $g'(t) = 0$ and we choose $g(t) = 0$. We conclude that a
potential function is $\psi(t, u) = t^3 u + \frac{1}{2} t^2 u^2$.

And every solution $y$ satisfies $t^3 y(t) + \frac{1}{2} t^2 [y(t)]^2 = c.$  \[ \triangle \]