Radioactive decay

Remarks:
(a) Radioactive substances randomly emit protons, electrons, radiation, and they are transformed in another substance.

(b) It can be seen that the time rate of change of the amount $N$ of a radioactive substances is proportional to the negative amount of radioactive substance.

$$N'(t) = -aN(t), \quad N(0) = N_0, \quad a > 0.$$  

(c) The integrating factor method implies $N(t) = N_0 e^{-at}$.

(d) The half-life is the time $\tau$ needed to get $N(\tau) = N_0/2$.

$$N_0 e^{-a\tau} = \frac{N_0}{2} \quad \Rightarrow \quad -a\tau = \ln \left( \frac{1}{2} \right) \quad \Rightarrow \quad \tau = \frac{\ln(2)}{a}.$$  

(e) Using the half-life, we get $N(t) = N_0 2^{-t/\tau}$.
Radioactive decay

Example
Remains containing 14% of the original amount of Carbon-14 are found. Knowing that Carbon-14 half-live is \( \tau = 5730 \) years, date the remains.

Solution: Set \( t = 0 \) when the organism dies. Since the amount \( N \) of Carbon-14 only decays after the organism dies,

\[
N(t) = N_0 2^{-t/\tau}, \quad \tau = 5730 \text{ years.}
\]

The remains contain 14% of the original amount at the time \( t \),

\[
\frac{N(t)}{N_0} = \frac{14}{100} \quad \Rightarrow \quad 2^{-t/\tau} = \frac{14}{100}
\]

\[
-t/\tau = \log_2(14/100) \quad \Rightarrow \quad t = \tau \log_2(100/14).
\]

The organism died 16,253 years ago.

Modeling with first order equations (Sect. 1.5).

- Radioactive decay.
  - Carbon-14 dating.
- **Salt in a water tank.**
  - The experimental device.
  - The main equations.
  - Analysis of the mathematical model.
  - Predictions for particular situations.
Problem: Describe the salt concentration in a tank with water if salty water comes in and goes out of the tank.

Main ideas of the test:

- Since the mass of salt and water is conserved, we construct a mathematical model for the salt concentration in water.
- The amount of salt in the tank depends on the salt concentration coming in and going out of the tank.
- The salt in the tank also depends on the water rates coming in and going out of the tank.
- To construct a model means to find the differential equation that takes into account the above properties of the system.
- Finding the solution to the differential equation with a particular initial condition means we can predict the evolution of the salt in the tank if we know the tank initial condition.

Modeling with first order equations (Sect. 1.5).

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The experimental device.

Definitions:

- \( r_i(t) \), \( r_o(t) \): Rates in and out of water entering and leaving the tank at the time \( t \).
- \( q_i(t) \), \( q_o(t) \): Salt concentration of the water entering and leaving the tank at the time \( t \).
- \( V(t) \): Water volume in the tank at the time \( t \).
- \( Q(t) \): Salt mass in the tank at the time \( t \).

Units:

- \([r_i(t)] = [r_o(t)] = \frac{\text{Volume}}{\text{Time}}\), \([q_i(t)] = [q_o(t)] = \frac{\text{Mass}}{\text{Volume}}\).
- \([V(t)] = \text{Volume}\), \([Q(t)] = \text{Mass}\).
Modeling with first order equations (Sect. 1.5).

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**The main equations.**

*Remark:* The mass conservation provides the main equations of the mathematical description for salt in water.

**Main equations:**

\[
\frac{d}{dt} V(t) = r_i(t) - r_o(t), \quad \text{Volume conservation,} \quad (1)
\]

\[
\frac{d}{dt} Q(t) = r_i(t) q_i(t) - r_o(t) q_o(t), \quad \text{Mass conservation,} \quad (2)
\]

\[
q_o(t) = \frac{Q(t)}{V(t)}, \quad \text{Instantaneously mixed,} \quad (3)
\]

\[
r_i, \ r_o : \text{Constants.} \quad (4)
\]
The main equations.

Remarks:
\[
\frac{dV}{dt} = \frac{\text{Volume}}{\text{Time}} = [r_i - r_o],
\]

\[
\frac{dQ}{dt} = \frac{\text{Mass}}{\text{Time}} = [r_i q_i - r_o q_o],
\]

\[
[r_i q_i - r_o q_o] = \frac{\text{Volume}}{\text{Time}} \frac{\text{Mass}}{\text{Volume}} = \frac{\text{Mass}}{\text{Time}}.
\]

Modeling with first order equations (Sect. 1.5).

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Analysis of the mathematical model.

Eqs. (4) and (1) imply

\[ V(t) = (r_i - r_o) t + V_0, \]

where \( V(0) = V_0 \) is the initial volume of water in the tank.

Eqs. (3) and (2) imply

\[ \frac{d}{dt} Q(t) = r_i q_i(t) - r_o \frac{Q(t)}{V(t)}. \]

Eqs. (5) and (6) imply

\[ \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t). \]

Analysis of the mathematical model.

Recall: \( \frac{d}{dt} Q(t) = r_i q_i(t) - \frac{r_o}{(r_i - r_o) t + V_0} Q(t) \).

Notation: \( a(t) = -\frac{r_o}{(r_i - r_o) t + V_0} \), and \( b(t) = r_i q_i(t) \).

The main equation of the description is given by

\[ Q'(t) = a(t) Q(t) + b(t). \]

Linear ODE for \( Q \). Solution: Integrating factor method.

\[ Q(t) = e^{A(t)} \left[ Q_0 + \int_0^t e^{-A(s)} b(s) \, ds \right] \]

with \( Q(0) = Q_0 \), and \( A(t) = \int_0^t a(s) \, ds \).
Radioactive decay.
  ▶ Carbon-14 dating.

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Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Always holds \( Q'(t) = a(t) Q(t) + b(t) \).
In this case:
\[
a(t) = -\frac{r_o}{(r_i - r_o) t + V_0} \Rightarrow a(t) = -\frac{r}{V_0} = -a_0.
\]
\[
b(t) = r_i q_i(t) \Rightarrow b(t) = rq_i = b_0.
\]
We need to solve the IVP:
\[
Q'(t) = -a_0 Q(t) + b_0, \quad Q(0) = Q_0.
\]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r, q_i, Q_0 \) and \( V_0 \) are given, find \( Q(t) \).

Solution: Recall the IVP: \( Q'(t) + a_0 Q(t) = b_0, \quad Q(0) = Q_0 \).

Integrating factor method:

\[
A(t) = a_0 t, \quad \mu(t) = e^{a_0 t}, \quad e^{a_0 t} Q(t) = Q_0 + \int_0^t e^{a_0 s} b_0 \, ds.
\]

\[
Q(t) = e^{-a_0 t} \left[ Q_0 + \frac{b_0}{a_0} (e^{a_0 t} - 1) \right] = \left( Q_0 - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0}.
\]

But \( \frac{b_0}{a_0} = r q_i \frac{V_0}{r} = q_i V_0 \), and \( a_0 = \frac{r}{V_0} \). We conclude:

\[
Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0.
\]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) and \( q_i \) are constants.
If \( r = 2 \) liters/min, \( q_i = 0 \), \( V_0 = 200 \) liters, \( Q_0/V_0 = 1 \) grams/liter,
find \( t_1 \) such that \( q(t_1) = Q(t_1)/V(t_1) \) is 1\% the initial value.

Solution: This problem is a particular case \( q_i = 0 \) of the previous Example. Since
\[ Q(t) = (Q_0 - q_i V_0) e^{-rt/V_0} + q_i V_0, \]
we get
\[ Q(t) = Q_0 e^{-rt/V_0}. \]
Since \( V(t) = (r_i - r_o) t + V_0 \) and \( r_i = r_o \), we obtain \( V(t) = V_0 \).
So \( q(t) = Q(t)/V(t) \) is given by \( q(t) = \frac{Q_0}{V_0} e^{-rt/V_0} \). Therefore,
\[ \frac{1}{100} \frac{Q_0}{V_0} = q(t_1) = \frac{Q_0}{V_0} e^{-rt_1/V_0} \Rightarrow e^{-rt_1/V_0} = \frac{1}{100}. \]
Predictions for particular situations.

Example
Assume that \( r_i = r_o = r \) are constants. If \( r = 5 \times 10^6 \) gal/year, \( q_i(t) = 2 + \sin(2t) \) grams/gal, \( V_0 = 10^6 \) gal, \( Q_0 = 0 \), find \( Q(t) \).

Solution: Recall: \( Q'(t) = a(t) Q(t) + b(t) \). In this case:

\[
a(t) = -\frac{r_o}{(r_i - r_o)t + V_0} \implies a(t) = -\frac{r}{V_0} = -a_0,
\]

\[
b(t) = r_i q_i(t) \implies b(t) = r \left[ 2 + \sin(2t) \right].
\]

We need to solve the IVP: \( Q'(t) = -a_0 Q(t) + b(t), \ Q(0) = 0 \).

\[
e^{a_0 t} Q(t) = \int_0^t e^{a_0 s} b(s) \, ds.
\]

We conclude: \( Q(t) = re^{-rt/V_0} \int_0^t e^{rs/V_0} \left[ 2 + \sin(2s) \right] \, ds \).