On linear and non-linear equations. (Sect. 1.6).

- Review: Linear differential equations.
- Non-linear differential equations.
- Properties of solutions to non-linear ODE.
- Direction Fields.

Review: Linear differential equations.

**Theorem (Variable coefficients)**

Given continuous functions \( a, b : (t_1, t_2) \to \mathbb{R} \), with \( t_2 > t_1 \), and given constants \( t_0 \in (t_1, t_2) \), \( y_0 \in \mathbb{R} \), the IVP

\[
y' = -a(t) y + b(t), \quad y(t_0) = y_0,
\]

has the unique solution \( y : (t_1, t_2) \to \mathbb{R} \) given by

\[
y(t) = \frac{1}{\mu(t)} \left[ y_0 + \int_{t_0}^{t} \mu(s) b(s) \, ds \right], \tag{1}
\]

where the integrating factor function is given by

\[
\mu(t) = e^{A(t)}, \quad A(t) = \int_{t_0}^{t} a(s) \, ds.
\]

**Proof:** Based on the integration factor method.
Review: Linear differential equations.

Remarks:
- The Theorem above assumes that the coefficients $a, b$, are continuous in $(t_1, t_2) \subset \mathbb{R}$.
- The Theorem above implies:
  (a) There is an explicit expression for the solutions of a linear IVP, given in Eq. (1).
  (b) For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution to a linear IVP.
  (c) For every initial condition $y_0 \in \mathbb{R}$ the corresponding solution $y(t)$ of a linear IVP is defined for all $t \in (t_1, t_2)$.
- None of these properties holds for solutions to non-linear differential equations.

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Non-linear differential equations.

Definition
An ordinary differential equation \( y'(t) = f(t, y(t)) \) is called non-linear iff the function \( (t, u) \mapsto f(t, u) \) is non-linear in the second argument.

Example

(a) The differential equation \( y'(t) = \frac{t^2}{y^3(t)} \) is non-linear, since the function \( f(t, u) = \frac{t^2}{u^3} \) is non-linear in the second argument.

(b) The differential equation \( y'(t) = 2ty(t) + \ln(y(t)) \) is non-linear, since the function \( f(t, u) = 2tu + \ln(u) \) is non-linear in the second argument, due to the term \( \ln(u) \).

(c) The differential equation \( \frac{y'(t)}{y(t)} = 2t^2 \) is linear, since the function \( f(t, u) = 2t^2u \) is linear in the second argument.

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Properties of solutions to non-linear ODE.

Theorem (Non-linear ODE)
Fix a non-empty rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$ and fix a function $f : R \to \mathbb{R}$ denoted as $(t, u) \mapsto f(t, u)$. If the functions $f$ and $\partial_u f$ are continuous on $R$, and $(t_0, y_0) \in R$, then there exists a smaller open rectangle $\hat{R} \subset R$ with $(t_0, y_0) \in \hat{R}$ such that the IVP
\[ y'(t) = f(t, y(t)), \quad y(t_0) = y_0 \]
has a unique solution $y$ on the set $\hat{R} \subset \mathbb{R}^2$.

Remarks:
(i) There is no general explicit expression for the solution $y(t)$ to a non-linear ODE.
(ii) Non-uniqueness of solution to the IVP above may happen at points $(t, u) \in \mathbb{R}^2$ where $\partial_u f$ is not continuous.
(iii) Changing the initial data $y_0$ may change the domain on the variable $t$ where the solution $y(t)$ is defined.

Properties of solutions to non-linear ODE.

Example
Given non-zero constants $a_1, a_2, a_3, a_4$, find every solution $y$ of
\[ y' = \frac{t^2}{(y^4 + a_4 y^3 + a_3 y^2 + a_2 y + a_1)}. \]

Solution: The ODE is separable. So first, rewrite the equation as
\[ (y^4 + a_4 y^3 + a_3 y^2 + a_2 y + a_1) y' = t^2, \]
then we integrate in $t$ on both sides of the equation,
\[ \int (y^4 + a_4 y^3 + a_3 y^2 + a_2 y + a_1) y' \, dt = \int t^2 \, dt + c. \]
Introduce the substitution $u = y(t)$, so $du = y'(t) \, dt$,
\[ \int (u^4 + a_4 u^3 + a_3 u^2 + a_2 u + a_1) \, du = \int t^2 \, dt + c. \]
Properties of solutions to non-linear ODE.

Example
Given non-zero constants \( a_1, a_2, a_3, a_4 \), find every solution \( y \) of
\[
y' = \frac{t^2}{(y^4 + a_4 y^3 + a_3 y^2 + a_2 y + a_1)}.
\]

Solution:
Recall: \[
\int \left( u^4 + a_4 u^3 + a_3 u^2 + a_2 u + a_1 \right) du = \int t^2 dt + c.
\]
Integrate, and in the result substitute back the function \( y \):
\[
\frac{1}{5} y^5(t) + \frac{a_4}{4} y^4(t) + \frac{a_3}{3} y^3(t) + \frac{a_2}{2} y^2(t) + a_1 y(t) = \frac{t^3}{3} + c.
\]
The solution is in implicit form. It is the root of a polynomial degree five. There is no formula for the roots of a general polynomial degree five or bigger.

There is no explicit expression for solutions \( y \) of the ODE.  

Properties of solutions to non-linear ODE.

Example
Find every solution \( y \) of the initial value problem
\[
y'(t) = y^{1/3}(t), \quad y(0) = 0.
\]

Remark: The equation above is non-linear, separable, and the function \( f(t, u) = u^{1/3} \) has derivative
\[
\partial_u f = \frac{1}{3} \frac{1}{u^{2/3}},
\]
so \( \partial_u f \) is not continuous at \( u = 0 \).

The initial condition above is precisely where \( f \) is not continuous.

Solution: There are two solutions to the IVP above:
The first solution is
\[
y_1(t) = 0.
\]
Properties of solutions to non-linear ODE.

Example
Find every solution $y$ of the initial value problem
\[ y'(t) = y^{1/3}(t), \quad y(0) = 0. \]

Solution: The second solution is obtained as follows:
\[
\int [y(t)]^{-1/3} y'(t) \, dt = \int dt + c.
\]
Then, the substitution $u = y(t)$, with $du = y'(t) \, dt$, implies that
\[
\int u^{-1/3} \, du = \int dt + c \quad \Rightarrow \quad \frac{3}{2} [y(t)]^{2/3} = t + c,
\]
\[ y(t) = \left[ \frac{2}{3} (t + c) \right]^{3/2} \Rightarrow 0 = y(0) = \left( \frac{2}{3} c \right)^{3/2} \Rightarrow c = 0. \]
So, the second solution is: $y_2(t) = \left( \frac{2}{3} t \right)^{3/2}$. Recall $y_1(t) = 0$. ◀

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Properties of solutions to non-linear ODE.

Example
Find the solution $y$ to the initial value problem
\[ y'(t) = y^2(t), \quad y(0) = y_0. \]

Solution: This is a separable equation. So,
\[
\int \frac{y'}{y^2} \, dt = \int dt + c \quad \Rightarrow \quad -\frac{1}{y} = t + c \quad \Rightarrow \quad y(t) = -\frac{1}{t + c}.
\]
Using the initial condition in the expression above,
\[ y_0 = y(0) = -\frac{1}{c} \quad \Rightarrow \quad c = -\frac{1}{y_0} \quad \Rightarrow \quad y(t) = \frac{1}{\left( \frac{1}{y_0} - t \right)}.
\]
This solution diverges at $t = 1/y_0$, so its domain is $\mathbb{R} - \{y_0\}$.
The solution domain depends on the values of the initial data $y_0$. ◀
Properties of solutions to non-linear ODE.

Summary:

▶ Linear ODE:
  (a) There is an explicit expression for the solution of a linear IVP.
  (b) For every initial condition \( y_0 \in \mathbb{R} \) there exists a unique solution to a linear IVP.
  (c) The domain of the solution of a linear IVP is defined for every initial condition \( y_0 \in \mathbb{R} \).

▶ Non-linear ODE:
  (i) There is no general explicit expression for the solution \( y(t) \) to a non-linear ODE.
  (ii) Non-uniqueness of solution to a non-linear IVP may happen at points \((t, u) \in \mathbb{R}^2\)
       where \( \partial_u f \) is not continuous.
  (iii) Changing the initial data \( y_0 \) may change the domain on the variable \( t \) where the solution \( y(t) \) is defined.

On linear and non-linear equations. (Sect. 1.6).

▶ Review: Linear differential equations.
▶ Non-linear differential equations.
▶ Properties of solutions to non-linear ODE.
▶ Direction Fields.
Direction Fields.

Remarks:
- One does not need to solve a differential equation \( y'(t) = f(t, y(t)) \) to have a qualitative idea of the solution.
- Recall that \( y'(t) \) represents the slope of the tangent line to the graph of function \( y \) at the point \((t, y(t))\).
- A differential equation provides these slopes, \( f(t, y(t)) \), for every point \((t, y(t))\).
- Key idea: Graph the function \( f(t, y) \) on the \( yt \)-plane, not as points, but as slopes of small segments.

Definition
A Direction Field for the differential equation \( y'(t) = f(t, y(t)) \) is the graph on the \( yt \)-plane of the values \( f(t, y) \) as slopes of small segments.

Example
We know that the solution of \( y' = y \) are the exponentials \( y(t) = y_0 e^t \). The graph of these solution is simple.
So is the direction field:
Direction Fields.

Example
The solution of \( y' = \sin(y) \) is simple to compute. The equation is separable. After some calculations the implicit solution are

\[
\ln \left| \frac{\csc(y_0) + \cot(y)}{\csc(y) + \cot(y)} \right| = t.
\]

for \( y_0 \in \mathbb{R} \). The graph of these solution is not simple to do. But the direction field is simple to plot:

Direction Fields.

Example
The solution of \( y' = \frac{(1 + y^3)}{(1 + t^2)} \) could be hard to compute. But the direction field is simple to plot: