The integrating factor method (Sect. 1.1)

- Overview of differential equations.
- Linear Ordinary Differential Equations.
- The integrating factor method.
  - Constant coefficients.
  - The Initial Value Problem.

Overview of differential equations.

Definition

A *differential equation* is an equation, where the unknown is a function, and both the function and its derivative appear in the equation.

Remark: There are two main types of differential equations:

- Ordinary Differential Equations (ODE): Derivatives with respect to only one variable appear in the equation.

  Example:
  Newton's second law of motion: $m\,\mathbf{a} = \mathbf{F}$.

- Partial differential Equations (PDE): Partial derivatives of two or more variables appear in the equation.

  Example:
  The wave equation for sound propagation in air.

Overview of differential equations.

Example

Newton's second law of motion is an ODE: The unknown is $\mathbf{x}(t)$, the particle position as function of time t and the equation is

$$\frac{d^2}{dt^2}\mathbf{x}(t) = \frac{1}{m}\,\mathbf{F}(t, \mathbf{x}(t)),$$

with m the particle mass and $\mathbf{F}$ the force acting on the particle.

Example

The wave equation is a PDE: The unknown is $u(t, x)$, a function that depends on two variables, and the equation is

$$\frac{\partial^2}{\partial t^2}u(t, x) = v^2 \frac{\partial^2}{\partial x^2}u(t, x),$$

with v the wave speed. Sound propagation in air is described by a wave equation, where u represents the air pressure.

Overview of differential equations.

Remark: Differential equations are a central part in a physical description of nature:

- Classical Mechanics:
  - Newton's second law of motion. (ODE)
  - Lagrange's equations. (ODE)
- Electromagnetism:
  - Maxwell's equations. (PDE)
- Quantum Mechanics:
  - Schrödinger's equation. (PDE)
- General Relativity:
  - Einstein equation. (PDE)
- Quantum Electrodynamics:
  - The equations of QED. (PDE).

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Linear Ordinary Differential Equations

Remark: Given a function $y : \mathbb{R} \to \mathbb{R}$, we use the notation

$$y'(t) = \frac{dy}{dt}(t).$$

Definition

Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, a *first order ODE* in the unknown function $y : \mathbb{R} \to \mathbb{R}$ is the equation

$$y'(t) = f(t, y(t)).$$

The first order ODE above is called *linear* iff there exist functions $a, b : \mathbb{R} \to \mathbb{R}$ such that $f(t, y) = a(t)\, y + b(t)$. That is, f is linear on its argument y, hence a first order linear ODE is given by

$$y'(t) = a(t)\, y(t) + b(t).$$

Linear Ordinary Differential Equations

Example

A first order linear ODE is given by

$$y'(t) = -2\,y(t) + 3.$$

In this case function $a(t) = -2$ and $b(t) = 3$. Since these function do not depend on t, the equation above is called of constant coefficients.

Example

A first order linear ODE is given by

$$y'(t) = -\frac{2}{t}\,y(t) + 4t.$$

In this case function $a(t) = -2/t$ and $b(t) = 4t$. Since these functions depend on t, the equation above is called of variable coefficients.

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The integrating factor method.

Remark: Solutions to first order linear ODE can be obtained using the integrating factor method.

Theorem (Constant coefficients)

Given constants $a, b \in \mathbb{R}$ with $a \neq 0$, the linear differential equation

$$y'(t) = a\, y(t) + b$$

has infinitely many solutions, one for each value of $c \in \mathbb{R}$, given by

$$y(t) = c\, e^{at} - \frac{b}{a}.$$

Remark: A proof is given in the Lecture Notes. Here we present the main idea of the proof, showing and exponential integrating factor. In the Lecture Notes it is shown that this is essentially the only integrating factor.

The integrating factor method.

Main ideas of the Proof: Write down the differential equation as

$$y'(t) - a\, y(t) = b.$$

Key idea: The left-hand side above is a total derivative if we multiply it by the exponential e^{-at}. Indeed,

$$e^{-at} y' - a\, e^{-at}\, y = b\, e^{-at} \;\Leftrightarrow\; e^{-at} y' + \big(e^{-at}\big)'\, y = b\, e^{-at}.$$

This is the key idea, because the derivative of a product implies

$$\big[e^{-at}\, y(t)\big]' = b\, e^{-at}.$$

The exponential e^{-at} is called an integrating factor. Indeed, we can now integrate the equation!

$$e^{-at}\, y = -\frac{b}{a} e^{-at} + c \quad \Leftrightarrow \quad y(t) = c\, e^{at} - \frac{b}{a}. \qquad \square$$

The integrating factor method.

Example

Find all functions y solution of the ODE $y' = 2y + 3$.

Solution: Write down the differential equation as $y' - 2\,y = 3$.
Key idea: The left-hand side above is a total derivative if we multiply it by the exponential e^{-2t}. Indeed,

$$e^{-2t}y' - 2\,e^{-2t}\,y = 3\,e^{-2t} \;\Leftrightarrow\; e^{-2t}y' + \left(e^{-2t}\right)'\,y = 3\,e^{-2t}.$$

This is the key idea, because the derivative of a product implies

$$\left[e^{-2t}\,y\right]' = 3\,e^{-2t}.$$

The exponential e^{-2t} is called an integrating factor. Integrating,

$$e^{-2t}\,y = -\frac{3}{2}e^{-2t} + c \quad\Leftrightarrow\quad y(t) = c\,e^{2t} - \frac{3}{2}.$$

□

The integrating factor method.

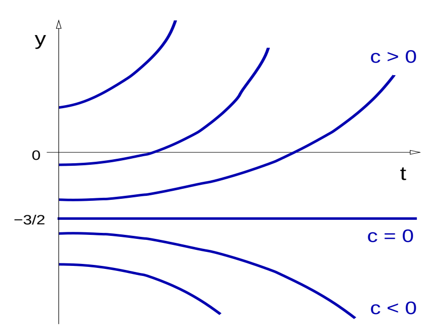
Example

Find all functions y solution of the ODE $y' = 2y + 3$.

Solution:

We concluded that the ODE has infinitely many solutions, given by

$$y(t) = c\,e^{2t} - \frac{3}{2}, \qquad c \in \mathbb{R}.$$



Since we did one integration, it is reasonable that the solution contains a constant of integration, $c \in \mathbb{R}$.

Verification: $y' = 2c\,e^{2t}$, but we know that $2c\,e^{2t} = 2y + 3$, therefore we conclude that y satisfies the ODE $y' = 2y + 3$. ◁

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The Initial Value Problem.

Definition

The *Initial Value Problem* (IVP) for a linear ODE is the following: Given functions $a, b : \mathbb{R} \to \mathbb{R}$ and constants t_0, $y_0 \in R$, find a solution $y : \mathbb{R} \to \mathbb{R}$ of the problem

$$y' = a(t)\, y + b(t), \qquad y(t_0) = y_0.$$

Remark: The initial condition selects one solution of the ODE.

Theorem (Constant coefficients)

Given constants $a, b, t_0, y_0 \in \mathbb{R}$, with $a \neq 0$, the initial value problem

$$y' = a\, y + b, \qquad y(t_0) = y_0$$

has the unique solution

$$y(t) = \Big(y_0 + \frac{b}{a}\Big) e^{a(t-t_0)} - \frac{b}{a}.$$

The Initial Value Problem.

Example

Find the solution to the initial value problem

$$y' = 2y + 3, \qquad y(0) = 1.$$

Solution: Every solution of the ODE above is given by

$$y(t) = c\, e^{2t} - \frac{3}{2}, \qquad c \in \mathbb{R}.$$

The initial condition $y(0) = 1$ selects only one solution:

$$1 = y(0) = c - \frac{3}{2} \quad \Rightarrow \quad c = \frac{5}{2}.$$

We conclude that $y(t) = \frac{5}{2}\, e^{2t} - \frac{3}{2}$. $\triangleleft$

The integrating factor method.

Example

Find the solution y to the IVP $y' = -3y + 1$, $y(0) = 1$.

Solution: Write down the differential equation as $y' + 3\,y = 1$.
Key idea: The left-hand side above is a total derivative if we multiply it by the exponential e^{3t}. Indeed,

$$e^{3t} y' + 3\, e^{3t}\, y = e^{3t} \;\Leftrightarrow\; e^{3t} y' + \big(e^{3t}\big)'\, y = e^{3t}.$$

This is the key idea, because the derivative of a product implies

$$\big[e^{3t}\, y\big]' = e^{3t}.$$

The exponential e^{3t} is called an integrating factor. Integrating,

$$e^{3t}\, y = \frac{1}{3} e^{3t} + c \quad \Leftrightarrow \quad y(t) = c\, e^{-3t} + \frac{1}{3}.$$

$\square$

The integrating factor method.

Example

Find the solution y to the IVP $y' = -3y + 1$, $y(0) = 1$.

Solution: Every solution of the ODE above is given by

$$y(t) = c\, e^{-3t} + \frac{1}{3}, \qquad c \in \mathbb{R}.$$

The initial condition $y(0) = 2$ selects only one solution:

$$1 = y(0) = c + \frac{1}{3} \quad \Rightarrow \quad c = \frac{2}{3}.$$

We conclude that $y(t) = \frac{2}{3}\, e^{-3t} + \frac{1}{3}$. $\triangleleft$