Review for Final Exam.

- Monday 12/09, 12:45-2:45pm in **CC-403**.
- Exam is cumulative, 12-14 problems.
- 5 grading attempts per problem.
- Problems similar to homeworks.
- Integration and LT tables provided.
- No notes, no books, no calculators.

- Heat Eq. and Fourier Series (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

Fourier Series

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We conclude: $f(x) = -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\sin[(2k-1)\pi x]$. $\triangleleft$

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We conclude: $f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{2}\right)$. $\triangleleft$

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We conclude: $f(x) = 1 + \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\left(\frac{(2k-1)\pi x}{2}\right).$ $\triangleleft$

Review for Final Exam.

- Heat Eq. and Fourier Series (Chptr.6).
- **Eigenvalue-Eigenfunction BVP (Chptr. 6).**
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

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$$0 = y(8) = c_2 \sin(\mu 8),$$

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Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

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Eigenvalue-Eigenfunction BVP.

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Find the non-negative eigenvalues and their eigenfunctions of

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Find the non-negative eigenvalues and their eigenfunctions of

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Solution: Case $\lambda > 0$.

Eigenvalue-Eigenfunction BVP.

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A Boundary Value Problem.

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Review for Final Exam.

- Heat Eq. and Fourier Series (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- **Systems of linear Equations (Chptr. 5).**
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

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Then, the general solution is

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Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

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Case $\lambda_+ = 3$,

$$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$

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$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2+1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

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Systems of linear Equations.

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We conclude: $\mathbf{x}(t) = \frac{5}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.$ $\triangleleft$

Review for Final Exam.

- Heat Eq. and Fourier Series (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- **Laplace transforms (Chptr. 4).**
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).

Laplace transforms.

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- Partial fraction decompositions, completing the squares.

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Laplace transforms.

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Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Compute $\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)]$

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$$H(s) = \frac{1}{s(s^2+9)}$$

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$$H(s) = \frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{(bs+c)}{(s^2+9)} = \frac{a(s^2+9) + (bs+c)s}{s(s^2+9)},$$

Laplace transforms.

Example

Use L.T. to find the solution to the IVP

$$y'' + 9y = u_5(t), \qquad y(0) = 3, \qquad y'(0) = 2.$$

Solution: Recall $\mathcal{L}[y] = 3\,\frac{s}{(s^2+9)} + \frac{2}{3}\,\frac{3}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}$.

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Therefore, we conclude that,

$$y(t) = 3\cos(3t) + \frac{2}{3}\,\sin(3t) + \frac{u_5(t)}{9}\Big[1 - \cos(3(t-5))\Big].$$

◁

Review for Final Exam.

- Heat Eq. and Fourier Series (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- **Second order linear equations (Chptr. 2).**
- First order differential equations (Chptr. 1).

Second order linear equations.

Summary: Solve $y'' + a_1\, y' + a_0\, y = g(t)$.

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First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

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First find fundamental solutions $y(t) = e^{rt}$ to the case $g = 0$, where r is a root of $p(r) = r^2 + a_1 r + a_0$.

(a) If $r_1 \neq r_2$, real,

Second order linear equations.

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Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*.

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Second order linear equations.

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Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients:

Second order linear equations.

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Summary:

Non-homogeneous equations: $g \neq 0$.

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Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution y_p using the guessing table, $g \to y_p$.

Second order linear equations.

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Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method.

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

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Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

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Look for a solution $y_2(x) = v(x)\,y_1(x)$,

Second order linear equations.

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Look for a solution $y_2(x) = v(x)\, y_1(x)$, and find an equation for v.

Second order linear equations.

Example

Knowing that $y_1(x) = x^2$ solves $x^2\, y'' - 4x\, y' + 6y = 0$, with $x > 0$, find a second solution y_2 not proportional to y_1.

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$$v'' = 0$$

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$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x$$

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Choose $c_1 = 0$, $c_2 = 1$.

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$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2 x\, y_1.$$

Choose $c_1 = 0$, $c_2 = 1$. Hence $y_2(x) = x^3$, and $y_1(x) = x^2$. $\triangleleft$

Second order linear equations.

Example

Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

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Second order linear equations.

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Second order linear equations.

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But this $y_p = k\,e^{-t}$ is solution of the homogeneous equation.

Then propose $y_p(t) = kt\,e^{-t}$.

Second order linear equations.

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Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$.

Second order linear equations.

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$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt\,e^{-t}$. This is correct, since te^{-t} is not solution of the homogeneous equation.

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Second order linear equations.

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$$y_p' = k\,e^{-t} - kt\,e^{-t},$$

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$$(-2 + t - 2 + 2t - 3t)\,k\,e^{-t} = 3\,e^{-t} \;\Rightarrow\; -4k = 3 \;\Rightarrow\; k = -\frac{3}{4}.$$

We obtain: $y_p(t) = -\frac{3}{4}t\,e^{-t}$.

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Second order linear equations.

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(4) Find the general solution: $y(t) = c_1\,e^{3t} + c_2\,e^{-t} - \frac{3}{4}\,t\,e^{-t}$.

Second order linear equations.

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Second order linear equations.

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Second order linear equations.

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Find the solution y to the initial value problem

$$y'' - 2y' - 3y = 3\,e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4}\,t\,e^{-t}$.

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Since $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$, we obtain,

$$y(t) = \frac{1}{2}\left(e^{3t} + e^{-t}\right) - \frac{3}{4}t\,e^{-t}.$$

◁

Review for Final Exam.

- Heat Eq. and Fourier Series (Chptr.6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- **First order differential equations (Chptr. 1).**

First order differential equations.

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- Linear, first order equations: $y' + p(t)\,y = q(t)$.

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- No modeling problems from Sect. 2.3.

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Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

First order differential equations.

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   (Just by looking at it: $y' + a(t)\, y = b(t)$.)

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Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
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   (Few manipulations: $h(y)\,y' = g(t)$.)

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   (Few manipulations: $h(y)\,y' = g(t)$.)
4. Homogeneous equations.
   (Several manipulations: $y' = F(y/t)$.)

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5. Exact equations.
   (Check one equation: $N\, y' + M = 0$, and $\partial_t N = \partial_y M$.)

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6. Exact equation with integrating factor.
   (Very complicated to check.)

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Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

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$$y = x\,v, \quad y' = x\,v' + v \quad x\,v' + v = \frac{1 + v + v^2}{v}.$$

$$x\,v' = \frac{1 + v + v^2}{v} - v$$

First order differential equations.

Example

Find all solutions of $y' = \dfrac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in x and y on every term is the same number, two in this example. The equation is homogeneous.

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First order differential equations.

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Find the solution y to the initial value problem

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First order differential equations.

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Solution: This is a Bernoulli equation,

First order differential equations.

Example

Find the solution y to the initial value problem

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First order differential equations.

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Find the solution y to the initial value problem

$$y' + y + e^{2x}\, y^3 = 0, \qquad y(0) = \frac{1}{3}.$$

Solution: This is a Bernoulli equation, $y' + y = -e^{2x}\, y^n, \quad n = 3.$

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Divide by y^3.

First order differential equations.

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First order differential equations.

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$$\frac{1}{3} = y_+(0) = \frac{1}{\sqrt{c}} \quad \Rightarrow \quad c = 9$$

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$$\frac{1}{3} = y_+(0) = \frac{1}{\sqrt{c}} \quad \Rightarrow \quad c = 9 \quad \Rightarrow \quad y(x) = \frac{e^{-x}}{\sqrt{2x + 9}}. \quad \triangleleft$$

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Find all solutions of $2xy^2 + 2y + 2x^2 y\, y' + 2x\, y' = 0$.

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Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0.$

Solution: Re-write the equation is a more organized way,

First order differential equations.

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Find all solutions of $2xy^2 + 2y + 2x^2y\,y' + 2x\,y' = 0.$

Solution: Re-write the equation is a more organized way,

$$[2x^2y + 2x]\,y' + [2xy^2 + 2y] = 0.$$

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