Review for Final Exam.

- Monday 12/09, 12:45-2:45pm in **CC-403**.
- Exam is cumulative, 12-14 problems.
- 5 grading attempts per problem.
- Problems similar to homeworks.
- Integration and LT tables provided.
- No notes, no books, no calculators.

- Heat Eq. and Fourier Series (Chptr. 6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).
Fourier Series

Example

Find the Fourier series of the odd-periodic extension of the function $f(x) = 1$ for $x \in (-1, 0)$.
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Solution: The Fourier series is

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  f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right].
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Since $f$ is odd and periodic, then the Fourier Series is a Sine Series,
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b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx
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\]

\[
b_n = 2 \int_{0}^{1} (-1)^n \sin(n\pi x) \, dx = (-2) \frac{(-1)^n}{n\pi} \cos(n\pi x) \bigg|_{0}^{1},
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\[
b_n = \frac{2}{n\pi} \left[ \cos(n\pi) - 1 \right]
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\[
b_n = \frac{2}{n\pi} \left[ \cos(n\pi) - 1 \right] \quad \Rightarrow \quad b_n = \frac{2}{n\pi} \left[ (-1)^n - 1 \right].
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If $n = 2k$,
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Solution: Recall: \( b_n = \frac{2}{n\pi} \left( (-1)^n - 1 \right) \).

If \( n = 2k \), then \( b_{2k} = \frac{2}{2k\pi} \left( (-1)^{2k} - 1 \right) \).
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If \( n = 2k \), then \( b_{2k} = \frac{2}{2k\pi} [(-1)^{2k} - 1] = 0 \).
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If \(n = 2k - 1\),
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b_{(2k-1)} = \frac{2}{(2k - 1)\pi} [(-1)^{2k-1} - 1]\]
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$$b_{2k-1} = \frac{2}{(2k-1)\pi} [(-1)^{2k-1} - 1] = -\frac{4}{(2k-1)\pi}.$$
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b_{(2k-1)} = \frac{2}{(2k-1)\pi} \left[ (-1)^{2k-1} - 1 \right] = -\frac{4}{(2k-1)\pi}.
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We conclude: \( f(x) = -\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin[(2k-1)\pi x] \).
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Find the Fourier series of the odd-periodic extension of the function $f(x) = 2 - x$ for $x \in (0, 2)$. 

Solution:
The Fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right].$$

Since $f$ is odd and periodic, then the Fourier Series is a Sine Series, that is, $a_n = 0$.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx,$$

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$$b_n = \int_{0}^{2} (2 - x) \sin \left( \frac{n\pi x}{2} \right) \, dx.$$
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$$ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx, $$
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Find the Fourier series of the odd-periodic extension of the function \( f(x) = 2 - x \) for \( x \in (0, 2) \).

Solution: \( b_n = 2 \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \, dx - \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) \, dx \).
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$$\int \sin\left(\frac{n\pi x}{2}\right) dx = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right),$$
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\[
\int \sin \left( \frac{n\pi x}{2} \right) \, dx = \frac{-2}{n\pi} \cos \left( \frac{n\pi x}{2} \right),
\]

The other integral is done by parts,
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$$\int \sin \left( \frac{n\pi x}{2} \right) \, dx = \frac{-2}{n\pi} \cos \left( \frac{n\pi x}{2} \right),$$

The other integral is done by parts,

$$I = \int x \sin \left( \frac{n\pi x}{2} \right) \, dx,$$

where

$$u = x, \quad v' = \sin \left( \frac{n\pi x}{2} \right).$$
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\]

The other integral is done by parts,

\[
l = \int x \sin\left(\frac{n\pi x}{2}\right) \, dx,
\]

\[
\begin{aligned}
   u &= x, & v' &= \sin\left(\frac{n\pi x}{2}\right) \\
   u' &= 1, & v &= -\frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)
\end{aligned}
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Find the Fourier series of the odd-periodic extension of the function \( f(x) = 2 - x \) for \( x \in (0, 2) \).

Solution: \( b_n = 2 \int_0^2 \sin \left( \frac{n \pi x}{2} \right) \, dx - \int_0^2 x \sin \left( \frac{n \pi x}{2} \right) \, dx \).

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&u = x, \\
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\end{aligned}
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\[
\begin{aligned}
u' = 1, \\
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\[
I = \frac{-2x}{n \pi} \cos \left( \frac{n \pi x}{2} \right) - \int \left( \frac{-2}{n \pi} \right) \cos \left( \frac{n \pi x}{2} \right) \, dx.
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Solution: 

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I = -\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) - \int \left(\frac{-2}{n\pi}\right) \cos\left(\frac{n\pi x}{2}\right) \, dx.
\]

\[
I = -\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right).
\]

\[\text{\(\Rightarrow\)} \quad b_n = 4n\pi.\]
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Solution: \( I = \frac{-2x}{n\pi} \cos \left( \frac{n\pi x}{2} \right) - \int \left( \frac{-2}{n\pi} \right) \cos \left( \frac{n\pi x}{2} \right) \, dx \).

\[ I = -\frac{2x}{n\pi} \cos \left( \frac{n\pi x}{2} \right) + \left( \frac{2}{n\pi} \right)^2 \sin \left( \frac{n\pi x}{2} \right). \]

So, we get

\[ b_n = 2 \left[ \frac{-2}{n\pi} \cos \left( \frac{n\pi x}{2} \right) \right]_0^2 + \left[ \frac{2x}{n\pi} \cos \left( \frac{n\pi x}{2} \right) \right]_0^2 - \left( \frac{2}{n\pi} \right)^2 \sin \left( \frac{n\pi x}{2} \right)_0^2 \]
Fourier Series

Example

Find the Fourier series of the odd-periodic extension of the function \( f(x) = 2 - x \) for \( x \in (0, 2) \).

Solution: \( l = \frac{-2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) - \int \left(\frac{-2}{n\pi}\right) \cos\left(\frac{n\pi x}{2}\right) \, dx \).

\[
l = -\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right). \quad \text{So, we get}
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b_n = 2 \left[\frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]_0^2 + \left[\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]_0^2 - \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)\bigg|_0^2
\]

\[
b_n = -\frac{4}{n\pi} \cos(n\pi) - 1 + \left[\frac{4}{n\pi} \cos(n\pi) - 0\right]
\]
Fourier Series

Example
Find the Fourier series of the odd-periodic extension of the function \( f(x) = 2 - x \) for \( x \in (0, 2) \).

Solution: \( I = \frac{-2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) - \int \left(\frac{-2}{n\pi}\right) \cos\left(\frac{n\pi x}{2}\right) \, dx \).

\[ I = -\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right). \]

So, we get

\[ b_n = 2\left[\frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]_0^2 + \left[\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]_0^2 - \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)\right|_0^2 \]

\[ b_n = \frac{-4}{n\pi} \left[\cos(n\pi) - 1\right] + \left[\frac{4}{n\pi} \cos(n\pi) - 0\right] \Rightarrow b_n = \frac{4}{n\pi}. \]
Fourier Series

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$\int_{0}^{2} \left[\frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]^2 + \left[\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]^2 - \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right) \, dx$.

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We conclude: $f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{2}\right)$. \(\triangleq\)
Fourier Series

Example
Find the Fourier series of the even-periodic extension of the function \( f(x) = 2 - x \) for \( x \in (0, 2) \).
Fourier Series

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Solution: The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)].$$
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Solution: $a_n = 2 \int_0^2 \cos\left(\frac{n\pi x}{2}\right) \, dx - \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) \, dx$. 
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\int \cos \left( \frac{n\pi x}{2} \right) \, dx = \frac{2}{n\pi} \sin \left( \frac{n\pi x}{2} \right),
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The other integral is done by parts,

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I = \int x \cos\left(\frac{n\pi x}{2}\right) \, dx,
\]

\[
\begin{aligned}
&\quad \left\{ \begin{array}{c}
u = x,
\quad v' = \cos\left(\frac{n\pi x}{2}\right) \\
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Find the Fourier series of the even-periodic extension of the function $f(x) = 2 - x$ for $x \in (0, 2)$.

Solution: Recall: $I = \frac{2x}{n\pi} \sin \left( \frac{n\pi x}{2} \right) - \int \frac{2}{n\pi} \sin \left( \frac{n\pi x}{2} \right) dx$. 
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$I = \frac{2x}{n\pi} \sin \left( \frac{n\pi x}{2} \right) + \left( \frac{2}{n\pi} \right)^2 \cos \left( \frac{n\pi x}{2} \right)$. So, we get

$$a_n = 2 \left[ \frac{2}{n\pi} \sin \left( \frac{n\pi x}{2} \right) \right]_0^2 - \left[ \frac{2x}{n\pi} \sin \left( \frac{n\pi x}{2} \right) \right]_0^2 - \left( \frac{2}{n\pi} \right)^2 \cos \left( \frac{n\pi x}{2} \right)_0^2$$
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Example
Find the Fourier series of the even-periodic extension of the function \( f(x) = 2 - x \) for \( x \in (0, 2) \).

Solution: Recall: \( b_n = 0, \ a_0 = 2, \ a_n = \frac{4}{n^2 \pi^2} [1 - (-1)^n] \).
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If \( n = 2k \),
Fourier Series

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If \( n = 2k \), then \( a_{2k} = \frac{4}{(2k)^2\pi^2} \left[ 1 - (-1)^{2k} \right] \).
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Find the Fourier series of the even-periodic extension of the function $f(x) = 2 - x$ for $x \in (0, 2)$.

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If $n = 2k$, then $a_{2k} = \frac{4}{(2k)^2 \pi^2} \left[ 1 - (-1)^{2k} \right] = 0$. 

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Solution: Recall: \( b_n = 0, \quad a_0 = 2, \quad a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n] \).

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a_{(2k-1)} = \frac{4}{(2k - 1)^2\pi^2} \left[ 1 - (-1)^{2k-1} \right] \]
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\[
a_{2k-1} = \frac{4}{(2k - 1)^2 \pi^2} \left[ 1 - (-1)^{2k-1} \right] = \frac{8}{(2k - 1)^2 \pi^2}.
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a_{2k-1} = \frac{4}{(2k-1)^2 \pi^2} [1 - (-1)^{2k-1}] = \frac{8}{(2k-1)^2 \pi^2}.
\]

We conclude: \( f(x) = 1 + \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos\left(\frac{(2k-1)\pi x}{2}\right). \) \( \triangle \)
Review for Final Exam.

- Heat Eq. and Fourier Series (Chptr. 6).
- **Eigenvalue-Eigenfunction BVP (Chptr. 6).**
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).
Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(8) = 0.$$
Example

Find the positive eigenvalues and their eigenfunctions of

\[ y'' + \lambda y = 0, \quad y(0) = 0, \quad y(8) = 0. \]

Solution: Since \( \lambda > 0 \),

\[ \text{Since } \lambda > 0, \]

\[ \text{The general solution is } y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x). \]

\[ \text{The boundary conditions imply: } y(0) = 0 = c_1 \Rightarrow y(x) = c_2 \sin(\mu x). \]

\[ 0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0 \Rightarrow \sin(\mu 8) = 0. \]

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Find the positive eigenvalues and their eigenfunctions of

\[ y'' + \lambda y = 0, \quad y(0) = 0, \quad y(8) = 0. \]

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Find the non-negative eigenvalues and their eigenfunctions of
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\[ 0 = y'(0) \]
Example
Find the non-negative eigenvalues and their eigenfunctions of
\[ y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(8) = 0. \]

Solution: The case \( \lambda = 0 \). The general solution is
\[ y(x) = c_1 + c_2 x. \]
The B.C. imply:
\[ 0 = y'(0) = c_2 \]
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Find the non-negative eigenvalues and their eigenfunctions of
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Solution: The case \( \lambda = 0 \). The general solution is
\[ y(x) = c_1 + c_2 x. \]

The B.C. imply:
\[ 0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \]
Eigenvalue-Eigenfunction BVP.

Example
Find the non-negative eigenvalues and their eigenfunctions of
\[ y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(8) = 0. \]

Solution: The case \( \lambda = 0 \). The general solution is
\[ y(x) = c_1 + c_2 x. \]
The B.C. imply:
\[ 0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0. \]
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Find the non-negative eigenvalues and their eigenfunctions of
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\[ y(x) = c_1 + c_2 x. \]

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\[ 0 = y'(0) = c_2 \quad \Rightarrow \quad y(x) = c_1, \quad y'(x) = 0. \]

\[ 0 = y'(8) = 0. \]

Then, choosing \( c_1 = 1 \), holds,

\[ \lambda = 0, \quad y_0(x) = 1. \]
A Boundary Value Problem.

Example
Find the solution of the BVP

\[ y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0. \]
A Boundary Value Problem.

Example
Find the solution of the BVP

\[ y'' + y = 0, \quad y'(0) = 1, \quad y(\pi/3) = 0. \]

Solution: \( y(x) = e^{rx} \) implies that \( r \) is solution of

\[ p(r) = r^2 + 1 = 0 \implies r = \pm i. \]

The general solution is

\[ y(x) = c_1 \cos(x) + c_2 \sin(x). \]

Then,

\[ y'(x) = -c_1 \sin(x) + c_2 \cos(x). \]

The B.C. imply:

\[ 1 = y'(0) = c_2 \implies y(x) = c_1 \cos(x) + \sin(x). \]

\[ 0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \implies c_1 = -\sin(\pi/3) \cos(\pi/3). \]

\[ c_1 = -\frac{\sqrt{3}}{2} \cdot \frac{1}{2} = -\frac{\sqrt{3}}{4}. \]

\[ y(x) = -\frac{\sqrt{3}}{4} \cos(x) + \sin(x). \]
A Boundary Value Problem.

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\[ 0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3) \quad \Rightarrow \quad c_1 = -\frac{\sin(\pi/3)}{\cos(\pi/3)}. \]

\[ c_1 = -\frac{\sqrt{3}/2}{1/2} \]
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\[ c_1 = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3} \quad \Rightarrow \quad y(x) = -\sqrt{3} \cos(x) + \sin(x). \]
Review for Final Exam.

- Heat Eq. and Fourier Series (Chptr. 6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- **Systems of linear Equations (Chptr. 5).**
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).
Systems of linear Equations.

**Summary:** Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.
Systems of linear Equations.

**Summary:** Find solutions of \( x' = A x \), with \( A \) a 2 \( \times \) 2 matrix.

First find the eigenvalues \( \lambda_i \); and the eigenvectors \( \mathbf{v}^{(i)} \) of \( A \).
Systems of linear Equations.

**Summary**: Find solutions of $\mathbf{x}' = A \mathbf{x}$, with $A$ a $2 \times 2$ matrix. First find the eigenvalues $\lambda_i$ and the eigenvectors $\mathbf{v}^{(i)}$ of $A$.

(a) If $\lambda_1 \neq \lambda_2$, real, 

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$, the complex-valued fundamental solutions $\mathbf{x}^{(\pm)} = Ae^{\alpha t}(a \pm bi)$.

Real-valued fundamental solutions are $\mathbf{x}^{(1)} = e^{\alpha t}(a \cos(\beta t) - b \sin(\beta t))$ and $\mathbf{x}^{(2)} = e^{\alpha t}(a \sin(\beta t) + b \cos(\beta t))$. 


Systems of linear Equations.

Summary: Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix. First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{v^{(1)}, v^{(2)}\}$ are linearly independent,
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First find the eigenvalues $\lambda_i$ and the eigenvectors $\mathbf{v}^{(i)}$ of $A$.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{v}^{(2)} e^{\lambda_2 t}$. 

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**Summary:** Find solutions of $\mathbf{x}' = A \mathbf{x}$, with $A$ a $2 \times 2$ matrix.

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(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(t) = c_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{v}^{(2)} e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex,
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(a) If $\lambda_1 \neq \lambda_2$, real, then $\{v^{(1)}, v^{(2)}\}$ are linearly independent, and the general solution is $x(x) = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $v^{(\pm)} = a \pm bi$, 

$$x^{(\pm)}(t) = e^{\alpha t} \left[ a \cos(\beta t) - b \sin(\beta t) \right] \pm ie^{\alpha t} \left[ a \sin(\beta t) + b \cos(\beta t) \right].$$
Systems of linear Equations.

Summary: Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{v^{(1)}, v^{(2)}\}$ are linearly independent, and the general solution is $x(x) = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $v^{(\pm)} = a \pm bi$, the complex-valued fundamental solutions $x^{(\pm)} = (a \pm bi) e^{(\alpha \pm \beta i)t}$.
Systems of linear Equations.

**Summary:** Find solutions of $\mathbf{x}' = A \mathbf{x}$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $\mathbf{v}^{(i)}$ of $A$.

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(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = a \pm b i$, the complex-valued fundamental solutions are $\mathbf{x}^{(\pm)} = (a \pm b i) e^{(\alpha \pm \beta i)t}$.

$$\mathbf{x}^{(\pm)} = e^{\alpha t} (a \pm bi) \left[ \cos(\beta t) + i \sin(\beta t) \right].$$
Summary: Find solutions of \( \mathbf{x}' = A \mathbf{x} \), with \( A \) a \( 2 \times 2 \) matrix.

First find the eigenvalues \( \lambda_i \) and the eigenvectors \( \mathbf{v}^{(i)} \) of \( A \).

(a) If \( \lambda_1 \neq \lambda_2 \), real, then \( \{ \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \} \) are linearly independent, and the general solution is

\[
\mathbf{x}(t) = c_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{v}^{(2)} e^{\lambda_2 t}.
\]

(b) If \( \lambda_1 \neq \lambda_2 \), complex, then denoting \( \lambda_{\pm} = \alpha \pm \beta i \) and \( \mathbf{v}^{(\pm)} = a \pm b i \), the complex-valued fundamental solutions

\[
\mathbf{x}^{(\pm)} = (a \pm b i) e^{(\alpha\pm\beta i)t}
\]

\[
\mathbf{x}^{(\pm)} = e^{\alpha t} (a \pm b i) \left[ \cos(\beta t) + i \sin(\beta t) \right].
\]

\[
\mathbf{x}^{(\pm)} = e^{\alpha t} \left[ a \cos(\beta t) - b \sin(\beta t) \right] \pm i e^{\alpha t} \left[ a \sin(\beta t) + b \cos(\beta t) \right].
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Systems of linear Equations.

Summary: Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{v^{(1)}, v^{(2)}\}$ are linearly independent, and the general solution is

$$x(t) = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}.$$  

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_{\pm} = \alpha \pm \beta i$ and $v^{(\pm)} = a \pm b i$, the complex-valued fundamental solutions

$$x^{(\pm)} = (a \pm bi) e^{(\alpha \pm \beta i)t},$$

$$x^{(\pm)} = e^{\alpha t} (a \pm bi) \left[ \cos(\beta t) + i \sin(\beta t) \right].$$

Real-valued fundamental solutions are
Systems of linear Equations.

**Summary:** Find solutions of $\mathbf{x}' = A \mathbf{x}$, with $A$ a $2 \times 2$ matrix.

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(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_{\pm} = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = a \pm b i$, the complex-valued fundamental solutions

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\mathbf{x}^{(\pm)} = (a \pm b i) e^{(\alpha \pm \beta i) t}
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\mathbf{x}^{(1)} = e^{\alpha t} \left[ a \cos(\beta t) - b \sin(\beta t) \right],
$$
System of linear Equations.

**Summary:** Find solutions of \( \mathbf{x}' = A \mathbf{x} \), with \( A \) a \( 2 \times 2 \) matrix.

First find the eigenvalues \( \lambda_i \) and the eigenvectors \( \mathbf{v}^{(i)} \) of \( A \).

(a) If \( \lambda_1 \neq \lambda_2 \), real, then \( \{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\} \) are linearly independent, and the general solution is \( \mathbf{x}(x) = c_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{v}^{(2)} e^{\lambda_2 t} \).

(b) If \( \lambda_1 \neq \lambda_2 \), complex, then denoting \( \lambda_{\pm} = \alpha \pm \beta i \) and \( \mathbf{v}^{(\pm)} = a \pm b i \), the complex-valued fundamental solutions

\[
\mathbf{x}^{(\pm)} = (a \pm b i) e^{(\alpha \pm \beta i)t}
\]

\[
\mathbf{x}^{(\pm)} = e^{\alpha t} (a \pm b i) [\cos(\beta t) + i \sin(\beta t)].
\]  

Real-valued fundamental solutions are

\[
\mathbf{x}^{(1)} = e^{\alpha t} [a \cos(\beta t) - b \sin(\beta t)],
\]

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\]
Systems of linear Equations.

**Summary:** Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$. 
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**Summary:** Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real,
Systems of linear Equations.

**Summary:** Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{v^{(1)}, v^{(2)}\}$ are linearly independent,
Systems of linear Equations.

**Summary:** Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{v^{(1)}, v^{(2)}\}$ are linearly independent, then the general solution is

$$x(x) = c_1 v^{(1)} e^{\lambda t} + c_2 v^{(2)} e^{\lambda t}.$$
Systems of linear Equations.

**Summary:** Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

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$$x(x) = c_1 v^{(1)} e^{\lambda t} + c_2 v^{(2)} e^{\lambda t}.$$ 

(d) If $\lambda_1 = \lambda_2 = \lambda$, real,
Summary: Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix. First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{v^{(1)}, v^{(2)}\}$ are linearly independent, then the general solution is

$$x(x) = c_1 v^{(1)} e^{\lambda t} + c_2 v^{(2)} e^{\lambda t}.$$ 

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $v$, then
**Systems of linear Equations.**

**Summary:** Find solutions of \( x' = Ax \), with \( A \) a \( 2 \times 2 \) matrix.

First find the eigenvalues \( \lambda_i \) and the eigenvectors \( \mathbf{v}^{(i)} \) of \( A \).

(c) If \( \lambda_1 = \lambda_2 = \lambda \), real, and their eigenvectors \( \{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\} \) are linearly independent, then the general solution is

\[
x(x) = c_1 \mathbf{v}^{(1)} e^{\lambda t} + c_2 \mathbf{v}^{(2)} e^{\lambda t}.
\]

(d) If \( \lambda_1 = \lambda_2 = \lambda \), real, and there is only one eigendirection \( \mathbf{v} \), then find \( \mathbf{w} \) solution of \( (A - \lambda I)\mathbf{w} = \mathbf{v} \).
Systems of linear Equations.

Summary: Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix.

First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{v^{(1)}, v^{(2)}\}$ are linearly independent, then the general solution is

$$x(x) = c_1 v^{(1)} e^{\lambda t} + c_2 v^{(2)} e^{\lambda t}.$$ 

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $v$, then find $w$ solution of $(A - \lambda I)w = v$. Then fundamental solutions to the differential equation are given by

$$x^{(1)} = v e^{\lambda t},$$
Systems of linear Equations.

**Summary:** Find solutions of $x' = Ax$, with $A$ a $2 \times 2$ matrix. First find the eigenvalues $\lambda_i$ and the eigenvectors $v^{(i)}$ of $A$.

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Systems of linear Equations.

Summary: Find solutions of \( \mathbf{x}' = A \mathbf{x} \), with \( A \) a \( 2 \times 2 \) matrix.

First find the eigenvalues \( \lambda_i \) and the eigenvectors \( \mathbf{v}^{(i)} \) of \( A \).

(c) If \( \lambda_1 = \lambda_2 = \lambda \), real, and their eigenvectors \( \{ \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \} \) are linearly independent, then the general solution is

\[
\mathbf{x}(x) = c_1 \mathbf{v}^{(1)} e^{\lambda t} + c_2 \mathbf{v}^{(2)} e^{\lambda t}.
\]

(d) If \( \lambda_1 = \lambda_2 = \lambda \), real, and there is only one eigendirection \( \mathbf{v} \), then find \( \mathbf{w} \) solution of \( (A - \lambda I)\mathbf{w} = \mathbf{v} \). Then fundamental solutions to the differential equation are given by

\[
\mathbf{x}^{(1)} = \mathbf{v} e^{\lambda t}, \quad \mathbf{x}^{(2)} = (\mathbf{v} t + \mathbf{w}) e^{\lambda t}.
\]

Then, the general solution is

\[
\mathbf{x} = c_1 \mathbf{v} e^{\lambda t} + c_2 (\mathbf{v} t + \mathbf{w}) e^{\lambda t}.
\]
Systems of linear Equations.

Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)
Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix}
\]

\[
= (1 - \lambda)(-1 - \lambda) - 2(4)
\]

\[
= \lambda^2 - 1 - 8 = 0
\]

\[
\Rightarrow \lambda = \pm 3
\]

Case \( \lambda = 3 \),

\[
A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix}
\rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}
\Rightarrow v_1 = 2v_2
\]

\[
v_1(+) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

Case \( \lambda = -3 \),

\[
A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}
\rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\Rightarrow v_1 = -v_2
\]

\[
v_1(-) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]
Systems of linear Equations.

Example

Find the solution to: $x' = Ax$, $x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$.

Solution:

$p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8$
Systems of linear Equations.

Example

Find the solution to: \( \mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

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\[\mathbf{v}(+) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}(-) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.\]
Example

Find the solution to: \( \mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

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p(\lambda) = \lambda^2 - 9 = 0
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Systems of linear Equations.

Example
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Case \( \lambda_+ = 3, \)
Systems of linear Equations.

Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

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Case \( \lambda_+ = 3, \)

\[A - 3I\]
Systems of linear Equations.

Example

Find the solution to: $x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$.

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Case $\lambda_+ = 3,$

$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix}$
Example

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Systems of linear Equations.

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$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \implies \nu_1 = 2\nu_2$
Systems of linear Equations.

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Find the solution to: \( \mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

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A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
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Systems of linear Equations.

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Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

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Case \( \lambda_- = -3, \)
Systems of linear Equations.

Example

Find the solution to: \( \mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution:

\[
p(\lambda) = \begin{vmatrix} 1 - \lambda & 4 \\ 2 & -1 - \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8,
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p(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 3.
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Case \( \lambda_+ = 3, \)

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A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \mathbf{v}(+) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

Case \( \lambda_- = -3, \)

\[
A + 3I
\]
Systems of linear Equations.

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Systems of linear Equations.

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Case \( \lambda_+ = 3 \),
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A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = 2\mathbf{v}_2 \quad \Rightarrow \quad \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

Case \( \lambda_- = -3 \),
\[
A + 3I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\]
Systems of linear Equations.

Example

Find the solution to: $\mathbf{x}' = A \mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$.

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$p(\lambda) = \begin{vmatrix} (1 - \lambda) & 4 \\ 2 & (-1 - \lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 1) - 8 = \lambda^2 - 1 - 8$,

$p(\lambda) = \lambda^2 - 9 = 0 \implies \lambda_{\pm} = \pm 3$.

Case $\lambda_{+} = 3$,

$A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \implies v_1 = 2v_2 \implies \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

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Systems of linear Equations.

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A - 3I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = 2 \mathbf{v}_2 \quad \Rightarrow \quad \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
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\[
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\]
Systems of linear Equations.

Example

Find the solution to: $x' = Ax$, $x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$.

Solution: Recall: $\lambda_{\pm} = \pm 3$, $v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $v^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
Example

Find the solution to: \( \mathbf{x}' = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution: Recall: \( \lambda_{\pm} = \pm 3, \quad \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \)

The general solution is \( \mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}. \)
Systems of linear Equations.

Example

Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

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The initial condition implies,

\[
\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0)
\]
Systems of linear Equations.

**Example**

Find the solution to: \( x' = Ax \), \( x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \), \( A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \).

**Solution:** Recall: \( \lambda_{\pm} = \pm 3 \), \( \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \), \( \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \).

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Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

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Systems of linear Equations.

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\]

\[
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2 + 1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\]
Systems of linear Equations.

Example
Find the solution to: \( x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \)

Solution: Recall: \( \lambda_{\pm} = \pm 3, \quad v^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad v^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \)

The general solution is \( x(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}. \)

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\begin{bmatrix} 3 \\ 2 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.
\]

\[
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2 + 1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.
\]
Systems of linear Equations.

Example

Find the solution to: \[ x' = Ax, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \]

Solution: Recall: \[ \lambda_\pm = \pm 3, \quad \mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \]

The general solution is \[ x(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}. \]

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\[
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2 + 1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 1 \end{bmatrix}.
\]

We conclude: \[ x(t) = \frac{5}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}. \]
Review for Final Exam.

- Heat Eq. and Fourier Series (Chptr. 6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).
Laplace transforms.

Summary:

- Main Properties:
Laplace transforms.

Summary:

- **Main Properties:**

  \[ \mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0); \quad (18) \]
Laplace transforms.

Summary:

- **Main Properties:**
  \[
  \mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0); \quad (18)
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  \[
  e^{-cs} \mathcal{L}[f(t)] = \mathcal{L}[u_c(t) f(t - c)]; \quad (13)
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Laplace transforms.

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- **Main Properties:**

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  \[ \mathcal{L}[f(t)] \bigg|_{(s-c)} = \mathcal{L}[e^{ct} f(t)]. \quad (14) \]
Laplace transforms.

Summary:

▶ Main Properties:

\[
\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0); \tag{18}
\]

\[
e^{-cs} \mathcal{L}[f(t)] = \mathcal{L}[u_c(t) f(t - c)]; \tag{13}
\]

\[
\mathcal{L}[f(t)] \bigg|_{s\rightarrow s-c} = \mathcal{L}[e^{ct} f(t)]. \tag{14}
\]

▶ Convolutions:
Laplace transforms.

Summary:

- **Main Properties:**
  \[ \mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0); \quad (18) \]

  \[ e^{-cs} \mathcal{L}[f(t)] = \mathcal{L}[u_c(t) f(t - c)]; \quad (13) \]

  \[ \mathcal{L}[f(t)] \bigg|_{(s-c)} = \mathcal{L}[e^{ct} f(t)]. \quad (14) \]

- **Convolutions:**
  \[ \mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)] \mathcal{L}[g(t)]. \]
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- **Partial fraction decompositions, completing the squares.**
Laplace transforms.

Example

Use L.T. to find the solution to the IVP

\[ y'' + 9y = u_5(t), \quad y(0) = 3, \quad y'(0) = 2. \]
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Laplace transforms.

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\[ \mathcal{L}[y] = 3 \frac{s}{(s^2 + 9)} + \frac{2}{3} \frac{3}{(s^2 + 9)} + e^{-5s} \frac{1}{s(s^2 + 9)}. \]
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Partial fractions on

\[ H(s) = \frac{1}{s(s^2 + 9)} \]
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\]

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Solution: So, \( \mathcal{L}[y] = 3 \mathcal{L}[\cos(3t)] + \frac{2}{3} \mathcal{L}[\sin(3t)] + e^{-5s} H(s) \), and

\[ H(s) = \frac{1}{s(s^2 + 9)} = \frac{1}{9} \left[ \frac{1}{s} - \frac{s}{s^2 + 9} \right] \]
Example

Use L.T. to find the solution to the IVP

\[y'' + 9y = u_5(t), \quad y(0) = 3, \quad y'(0) = 2.\]

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\[H(s) = \frac{1}{s(s^2 + 9)} = \frac{1}{9} \left[ \frac{1}{s} - \frac{s}{s^2 + 9} \right] = \frac{1}{9} \left( L[u(t)] - L[\cos(3t)] \right)\]
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\[
e^{-5s} H(s) = \frac{1}{9} \left( \mathcal{L}[u_5(t)] - \mathcal{L}[u_5(t) \cos(3(t - 5))] \right).
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Laplace transforms.

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**Example**

Use L.T. to find the solution to the IVP

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\]

Therefore, we conclude that,

\[
y(t) = 3 \cos(3t) + \frac{2}{3} \sin(3t) + \frac{u_5(t)}{9} \left[ 1 - \cos(3(t - 5)) \right].
\]
Review for Final Exam.

- Heat Eq. and Fourier Series (Chptr. 6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- **Second order linear equations (Chptr. 2).**
- First order differential equations (Chptr. 1).
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \),
Second order linear equations.

**Summary:** Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 r + a_0 \).
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(a) If \( r_1 \neq r_2 \), real,
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y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.
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\[
y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t} \left[ \cos(\beta t) \pm i \sin(\beta t) \right],
\]
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(a) If \( r_1 \neq r_2 \), real, then the general solution is
\[
y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.
\]

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\[
y_{\pm}(t) = e^{(\alpha \pm \beta i)t} \iff y_{\pm}(t) = e^{\alpha t} \left[ \cos(\beta t) \pm i \sin(\beta t) \right],
\]
and real-valued fundamental solutions are
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 r + a_0 \).

(a) If \( r_1 \neq r_2 \), real, then the general solution is
\[
y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.
\]

(b) If \( r_1 \neq r_2 \), complex, then denoting \( r_{\pm} = \alpha \pm \beta i \), complex-valued fundamental solutions are
\[
y_{\pm}(t) = e^{(\alpha \pm \beta i) t} \iff y_{\pm}(t) = e^{\alpha t} [\cos(\beta t) \pm i \sin(\beta t)],
\]
and real-valued fundamental solutions are
\[
y_1(t) = e^{\alpha t} \cos(\beta t),
\]
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 r + a_0 \).

(a) If \( r_1 \neq r_2 \), real, then the general solution is

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(b) If \( r_1 \neq r_2 \), complex, then denoting \( r_\pm = \alpha \pm \beta i \),
complex-valued fundamental solutions are

\[
y_\pm(t) = e^{(\alpha \pm \beta i)t} \quad \Leftrightarrow \quad y_\pm(t) = e^{\alpha t} \left[ \cos(\beta t) \pm i \sin(\beta t) \right],
\]
and real-valued fundamental solutions are

\[
y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).
\]
Second order linear equations.

Summary: Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 r + a_0 \).

(a) If \( r_1 \neq r_2 \), real, then the general solution is
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\[
y_\pm(t) = e^{(\alpha \pm \beta i)t} \iff y_\pm(t) = e^{\alpha t} \left[ \cos(\beta t) \pm i \sin(\beta t) \right],
\]
and real-valued fundamental solutions are
\[
y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).
\]
If \( r_1 = r_2 = r \), real,
Second order linear equations.

**Summary:** Solve \( y'' + a_1 y' + a_0 y = g(t) \).

First find fundamental solutions \( y(t) = e^{rt} \) to the case \( g = 0 \), where \( r \) is a root of \( p(r) = r^2 + a_1 r + a_0 \).

(a) If \( r_1 \neq r_2 \), real, then the general solution is
\[
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\]
and real-valued fundamental solutions are
\[
y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t).
\]
If \( r_1 = r_2 = r \), real, then the general solution is
\[
y(t) = (c_1 + c_2 t) e^{rt}.
\]
Remark: Case (c) is solved using the \textit{reduction of order method}.
Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook.
Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.
Second order linear equations.

Remark: Case (c) is solved using the reduction of order method. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: \( g \neq 0 \).
Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients:
Second order linear equations.

Remark: Case \( (c) \) is solved using the \textit{reduction of order method}. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: \( g \neq 0 \).

(i) Undetermined coefficients: Guess the particular solution \( y_p \).
Second order linear equations.

Remark: Case (c) is solved using the reduction of order method. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: \( g \neq 0 \).

(i) Undetermined coefficients: Guess the particular solution \( y_p \) using the guessing table, \( g \rightarrow y_p \).
Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution $y_p$ using the guessing table, $g \rightarrow y_p$.

(ii) Variation of parameters:
Second order linear equations.

Remark: Case (c) is solved using the reduction of order method. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution $y_p$ using the guessing table, $g \rightarrow y_p$.

(ii) Variation of parameters: If $y_1$ and $y_2$ are fundamental solutions to the homogeneous equation,
Second order linear equations.

Remark: Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:

Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution $y_p$ using the guessing table, $g \rightarrow y_p$.

(ii) Variation of parameters: If $y_1$ and $y_2$ are fundamental solutions to the homogeneous equation, and $W$ is their Wronskian,
Second order linear equations.

**Remark:** Case (c) is solved using the *reduction of order method*. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

**Summary:**
Non-homogeneous equations: \( g \neq 0 \).

(i) **Undetermined coefficients:** Guess the particular solution \( y_p \) using the guessing table, \( g \rightarrow y_p \).

(ii) **Variation of parameters:** If \( y_1 \) and \( y_2 \) are fundamental solutions to the homogeneous equation, and \( W \) is their Wronskian, then \( y_p = u_1 y_1 + u_2 y_2 \),
Second order linear equations.

Remark: Case (c) is solved using the \textit{reduction of order method}. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution $y_p$ using the guessing table, $g \rightarrow y_p$.

(ii) Variation of parameters: If $y_1$ and $y_2$ are fundamental solutions to the homogeneous equation, and $W$ is their Wronskian, then $y_p = u_1 y_1 + u_2 y_2$, where

$$u'_1 = -\frac{y_2 g}{W},$$
Second order linear equations.

Remark: Case (c) is solved using the reduction of order method. See page 170 in the textbook. Do the extra homework problems Sect. 3.4: 23, 25, 27.

Summary:
Non-homogeneous equations: $g \neq 0$.

(i) Undetermined coefficients: Guess the particular solution $y_p$ using the guessing table, $g \rightarrow y_p$.

(ii) Variation of parameters: If $y_1$ and $y_2$ are fundamental solutions to the homogeneous equation, and $W$ is their Wronskian, then $y_p = u_1 y_1 + u_2 y_2$, where

\[
u_1' = -\frac{y_2 g}{W}, \quad u_2' = \frac{y_1 g}{W}.\]
Second order linear equations.

Example
Knowing that $y_1(x) = x^2$ solves $x^2 y'' - 4x y' + 6y = 0$, with $x > 0$, find a second solution $y_2$ not proportional to $y_1$. 

Solution:
Use the reduction of order method.

We verify that $y_1(x) = x^2$ solves the equation,

$$x^2 y'' - 4x y' + 6y = 0.$$

Look for a solution $y_2(x) = v(x)y_1(x)$, and find an equation for $v$.

$$y_2(x) = x^2 v(x),$$
$$y_2'(x) = x^2 v'(x) + 2xv(x),$$
$$y_2''(x) = x^2 v''(x) + 4xv'(x) + 2v(x).$$

Substituting into the equation,

$$x^2 (x^2 v''(x) + 4xv'(x) + 2v(x)) - 4x (x^2 v'(x) + 2xv(x)) + 6 (x^2 v(x)) = 0.$$ 

Simplify and solve for $v''$.

$$x^4 v''(x) + (4x^3 - 4x^3) v'(x) + (2x^2 - 8x^2 + 6x^2) v(x) = 0.$$ 

$$x^4 v''(x) = 0.$$ 

$$v''(x) = 0.$$

$$v(x) = c_1 + c_2 x.$$ 

$$y_2(x) = c_1 y_1(x) + c_2 x y_1(x).$$

Choose $c_1 = 0$, $c_2 = 1$.

Hence $y_2(x) = x^3$, and $y_1(x) = x^2$. 

\[ \square \]
Second order linear equations.

Example
Knowing that \( y_1(x) = x^2 \) solves \( x^2 y'' - 4x y' + 6y = 0 \), with \( x > 0 \), find a second solution \( y_2 \) not proportional to \( y_1 \).

Solution: Use the reduction of order method.
Second order linear equations.

Example
Knowing that \( y_1(x) = x^2 \) solves \( x^2 y'' - 4x y' + 6y = 0 \), with \( x > 0 \), find a second solution \( y_2 \) not proportional to \( y_1 \).

Solution: Use the reduction of order method. We verify that \( y_1 = x^2 \) solves the equation,
Second order linear equations.

Example
Knowing that $y_1(x) = x^2$ solves $x^2 y'' - 4x y' + 6y = 0$, with $x > 0$, find a second solution $y_2$ not proportional to $y_1$.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2 (2) - 4x (2x) + 6x^2 = 0.$$
Second order linear equations.

Example
Knowing that \( y_1(x) = x^2 \) solves \( x^2 y'' - 4x y' + 6y = 0 \), with \( x > 0 \), find a second solution \( y_2 \) not proportional to \( y_1 \).

Solution: Use the reduction of order method. We verify that \( y_1 = x^2 \) solves the equation,

\[
x^2 (2) - 4x (2x) + 6x^2 = 0.
\]

Look for a solution \( y_2(x) = v(x) y_1(x) \),
Second order linear equations.

Example
Knowing that $y_1(x) = x^2$ solves $x^2 y'' - 4x y' + 6y = 0$, with $x > 0$, find a second solution $y_2$ not proportional to $y_1$.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2 (2) - 4x (2x) + 6x^2 = 0.$$  

Look for a solution $y_2(x) = v(x) y_1(x)$, and find an equation for $v$. 

$$x^4 v'' + (4x^3 - 4x^3) v' + (2x^2 - 8x^2 + 6x^2) v = 0.$$
Second order linear equations.

Example
Knowing that \( y_1(x) = x^2 \) solves \( x^2 y'' - 4x y' + 6y = 0 \), with \( x > 0 \), find a second solution \( y_2 \) not proportional to \( y_1 \).

Solution: Use the reduction of order method. We verify that \( y_1 = x^2 \) solves the equation,

\[
x^2 (2) - 4x (2x) + 6x^2 = 0.
\]

Look for a solution \( y_2(x) = \nu(x) y_1(x) \), and find an equation for \( \nu \).

\[
y_2 = x^2 \nu,
\]
Second order linear equations.

Example
Knowing that $y_1(x) = x^2$ solves $x^2 y'' - 4x y' + 6y = 0$, with $x > 0$, find a second solution $y_2$ not proportional to $y_1$.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2 (2) - 4x (2x) + 6x^2 = 0.$$ 

Look for a solution $y_2(x) = v(x) y_1(x)$, and find an equation for $v$.

$$y_2 = x^2 v, \quad y'_2 = x^2 v' + 2xv,$$
Second order linear equations.

Example
Knowing that \( y_1(x) = x^2 \) solves \( x^2 y'' - 4x y' + 6y = 0 \), with \( x > 0 \), find a second solution \( y_2 \) not proportional to \( y_1 \).

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x^2 (2) - 4x (2x) + 6x^2 = 0.
\]
Look for a solution \( y_2(x) = v(x) y_1(x) \), and find an equation for \( v \).
\[
y_2 = x^2 v, \quad y'_2 = x^2 v' + 2xv, \quad y''_2 = x^2 v'' + 4xv' + 2v.
\]
Second order linear equations.

Example
Knowing that \( y_1(x) = x^2 \) solves \( x^2 y'' - 4x y' + 6y = 0 \), with \( x > 0 \), find a second solution \( y_2 \) not proportional to \( y_1 \).

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\[
y_2 = x^2 v, \quad y'_2 = x^2 v' + 2xv, \quad y''_2 = x^2 v'' + 4xv' + 2v.
\]
\[
x^2(x^2 v'' + 4xv' + 2v) - 4x(x^2 v' + 2xv) + 6(x^2 v) = 0.
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Second order linear equations.

Example
Knowing that \( y_1(x) = x^2 \) solves \( x^2 y'' - 4x y' + 6y = 0 \), with \( x > 0 \), find a second solution \( y_2 \) not proportional to \( y_1 \).

Solution: Use the reduction of order method. We verify that \( y_1 = x^2 \) solves the equation,

\[
x^2 (2) - 4x (2x) + 6x^2 = 0.
\]

Look for a solution \( y_2(x) = v(x) y_1(x) \), and find an equation for \( v \).

\[
y_2 = x^2 v, \quad y'_2 = x^2 v' + 2xv, \quad y''_2 = x^2 v'' + 4xv' + 2v.
\]

\[
x^2(x^2 v'' + 4xv' + 2v) - 4x (x^2 v' + 2xv) + 6 (x^2 v) = 0.
\]

\[
x^4 v'' + (4x^3 - 4x^3) v' + (2x^2 - 8x^2 + 6x^2) v = 0.
\]
Second order linear equations.

Example
Knowing that $y_1(x) = x^2$ solves $x^2 y'' - 4x y' + 6y = 0$, with $x > 0$, find a second solution $y_2$ not proportional to $y_1$.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

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Look for a solution $y_2(x) = v(x) y_1(x)$, and find an equation for $v$.

$y_2 = x^2 v, \quad y'_2 = x^2 v' + 2xv, \quad y''_2 = x^2 v'' + 4xv' + 2v.$

$$x^2(x^2 v'' + 4xv' + 2v) - 4x (x^2 v' + 2xv) + 6 (x^2 v) = 0.$$ 

$$x^4 v'' + (4x^3 - 4x^3) v' + (2x^2 - 8x^2 + 6x^2) v = 0.$$ 

$v'' = 0$
Second order linear equations.

Example
Knowing that $y_1(x) = x^2$ solves $x^2 y'' - 4x y' + 6y = 0$, with $x > 0$, find a second solution $y_2$ not proportional to $y_1$.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

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Look for a solution $y_2(x) = v(x) y_1(x)$, and find an equation for $v$.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2x v, \quad y_2'' = x^2 v'' + 4x v' + 2v.$$ 

$$x^2(x^2 v'' + 4x v' + 2v) - 4x (x^2 v' + 2x v) + 6 (x^2 v) = 0.$$ 

$$x^4 v'' + (4x^3 - 4x^3) v' + (2x^2 - 8x^2 + 6x^2) v = 0.$$ 

$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x$$
Second order linear equations.

Example
Knowing that \( y_1(x) = x^2 \) solves \( x^2 y'' - 4x y' + 6y = 0 \), with \( x > 0 \), find a second solution \( y_2 \) not proportional to \( y_1 \).

Solution: Use the reduction of order method. We verify that \( y_1 = x^2 \) solves the equation,

\[
x^2 (2) - 4x (2x) + 6x^2 = 0.
\]

Look for a solution \( y_2(x) = v(x) y_1(x) \), and find an equation for \( v \).

\[
y_2 = x^2 v, \quad y'_2 = x^2 v' + 2xv, \quad y''_2 = x^2 v'' + 4xv' + 2v.
\]

\[
x^2(x^2 v'' + 4xv' + 2v) - 4x (x^2 v' + 2xv) + 6 (x^2 v) = 0.
\]

\[
x^4 v'' + (4x^3 - 4x^3) v' + (2x^2 - 8x^2 + 6x^2) v = 0.
\]

\[
v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2 x y_1.
\]
Second order linear equations.

Example
Knowing that $y_1(x) = x^2$ solves $x^2 y'' - 4x y' + 6y = 0$, with $x > 0$, find a second solution $y_2$ not proportional to $y_1$.

Solution: Use the reduction of order method. We verify that $y_1 = x^2$ solves the equation,

$$x^2 (2) - 4x (2x) + 6x^2 = 0.$$ 

Look for a solution $y_2(x) = v(x) y_1(x)$, and find an equation for $v$.

$$y_2 = x^2 v, \quad y_2' = x^2 v' + 2xv, \quad y_2'' = x^2 v'' + 4xv' + 2v.$$ 

$$x^2(x^2 v'' + 4xv' + 2v) - 4x (x^2 v' + 2xv) + 6 (x^2 v) = 0.$$ 

$$x^4 v'' + (4x^3 - 4x^2) v' + (2x^2 - 8x^2 + 6x^2) v = 0.$$ 

$$v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2 x y_1.$$ 

Choose $c_1 = 0$, $c_2 = 1$. 

$\triangleleft$
Second order linear equations.

Example
Knowing that \( y_1(x) = x^2 \) solves \( x^2 y'' - 4x y' + 6y = 0 \), with \( x > 0 \), find a second solution \( y_2 \) not proportional to \( y_1 \).

Solution: Use the reduction of order method. We verify that \( y_1 = x^2 \) solves the equation,
\[
x^2 (2) - 4x (2x) + 6x^2 = 0.
\]
Look for a solution \( y_2(x) = v(x) y_1(x) \), and find an equation for \( v \).
\[
y_2 = x^2 v, \quad y'_2 = x^2 v' + 2xv, \quad y''_2 = x^2 v'' + 4xv' + 2v.
\]
\[
x^2(x^2 v'' + 4xv' + 2v) - 4x(x^2 v' + 2xv) + 6(x^2 v) = 0.
\]
\[
x^4 v'' + (4x^3 - 4x^3) v' + (2x^2 - 8x^2 + 6x^2) v = 0.
\]
\[
v'' = 0 \quad \Rightarrow \quad v = c_1 + c_2 x \quad \Rightarrow \quad y_2 = c_1 y_1 + c_2 x y_1.
\]
Choose \( c_1 = 0, \ c_2 = 1 \). Hence \( y_2(x) = x^3 \), and \( y_1(x) = x^2 \). \( \Diamond \)
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$
Second order linear equations.

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Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: (1) Solve the homogeneous equation.
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt},$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$  

$$r_{\pm} = \frac{1}{2} [2 \pm \sqrt{4 + 12}]$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$ 

$$r_{\pm} = \frac{1}{2} \left[ 2 \pm \sqrt{4 + 12} \right] = \frac{1}{2} \left[ 2 \pm \sqrt{16} \right]$$
Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$ 

$$r_{\pm} = \frac{1}{2} \left[ 2 \pm \sqrt{4 + 12} \right] = \frac{1}{2} \left[ 2 \pm \sqrt{16} \right] = 1 \pm 2$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$ 

$$r_{\pm} = \frac{1}{2} \left[ 2 \pm \sqrt{4 + 12} \right] = \frac{1}{2} \left[ 2 \pm \sqrt{16} \right] = 1 \pm 2 \implies \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}. $$

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0. $$

$$r_{\pm} = \frac{1}{2} \left[ 2 \pm \sqrt{4 + 12} \right] = \frac{1}{2} \left[ 2 \pm \sqrt{16} \right] = 1 \pm 2 \Rightarrow \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases} $$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$. 
Second order linear equations.

Example
Find the solution \( y \) to the initial value problem
\[
y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.
\]

Solution:

(1) Solve the homogeneous equation.
\[
y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.
\]
\[
r_{\pm} = \frac{1}{2} [2 \pm \sqrt{4 + 12}] = \frac{1}{2} [2 \pm \sqrt{16}] = 1 \pm 2 \Rightarrow \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}
\]

Fundamental solutions: \( y_1(t) = e^{3t} \) and \( y_2(t) = e^{-t} \).

(2) Guess \( y_p \).
Second order linear equations.

Example
Find the solution \( y \) to the initial value problem

\[
y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.
\]

Solution: (1) Solve the homogeneous equation.

\[
y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.
\]

\[
r_{\pm} = \frac{1}{2} \left[ 2 \pm \sqrt{4 + 12} \right] = \frac{1}{2} \left[ 2 \pm \sqrt{16} \right] = 1 \pm 2 \Rightarrow \begin{cases} r_+ = 3, \\
r_- = -1. \end{cases}
\]

Fundamental solutions: \( y_1(t) = e^{3t} \) and \( y_2(t) = e^{-t} \).

(2) Guess \( y_p \). Since \( g(t) = 3 e^{-t} \)
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$ 

$$r_{\pm} = \frac{1}{2} [2 \pm \sqrt{4 + 12}] = \frac{1}{2} [2 \pm \sqrt{16}] = 1 \pm 2 \Rightarrow \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$ 

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$. 

(2) Guess $y_p$. Since $g(t) = 3 e^{-t} \Rightarrow y_p(t) = k e^{-t}$. 

Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$ 

$$r_{\pm} = \frac{1}{2} \left[ 2 \pm \sqrt{4 + 12} \right] = \frac{1}{2} \left[ 2 \pm \sqrt{16} \right] = 1 \pm 2 \quad \Rightarrow \quad \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$. 

(2) Guess $y_p$. Since $g(t) = 3e^{-t} \quad \Rightarrow \quad y_p(t) = ke^{-t}$. 

But this $y_p = ke^{-t}$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: (1) Solve the homogeneous equation.

$$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$ 

$$r_{\pm} = \frac{1}{2} [2 \pm \sqrt{4 + 12}] = \frac{1}{2} [2 \pm \sqrt{16}] = 1 \pm 2 \Rightarrow \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

(2) Guess $y_p$. Since $g(t) = 3e^{-t} \Rightarrow y_p(t) = ke^{-t}$.

But this $y_p = ke^{-t}$ is solution of the homogeneous equation.
Second order linear equations.

**Example**
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

**Solution:**

1. Solve the homogeneous equation.

   $$y(t) = e^{rt}, \quad p(r) = r^2 - 2r - 3 = 0.$$  

   $$r_{\pm} = \frac{1}{2} [2 \pm \sqrt{4 + 12}] = \frac{1}{2} [2 \pm \sqrt{16}] = 1 \pm 2 \Rightarrow \begin{cases} r_+ = 3, \\ r_- = -1. \end{cases}$$

   Fundamental solutions: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}.$

2. Guess $y_p$. Since $g(t) = 3e^{-t}$ \quad $y_p(t) = ke^{-t}.$

But this $y_p = ke^{-t}$ is solution of the homogeneous equation. Then propose $y_p(t) = kte^{-t}.$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y_p(t) = kt e^{-t}$.
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not solution of the homogeneous equation.
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not solution of the homogeneous equation.

(3) Find the undetermined coefficient $k$. 

Second order linear equations.

Example

Find the solution \( y \) to the initial value problem

\[
  y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.
\]

Solution: Recall: \( y_p(t) = kt e^{-t} \). This is correct, since \( te^{-t} \) is not solution of the homogeneous equation.

(3) Find the undetermined coefficient \( k \).

\[
y_p' = k e^{-t} - kt e^{-t},
\]
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not solution of the homogeneous equation.

(3) Find the undetermined coefficient $k$.

$$y'_p = ke^{-t} - kt e^{-t}, \quad y''_p = -2k e^{-t} + kt e^{-t}.$$
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not solution of the homogeneous equation.

(3) Find the undetermined coefficient $k$.

$$y_p' = k e^{-t} - kt e^{-t}, \quad y_p'' = -2k e^{-t} + kt e^{-t}.$$  

$$(-2k e^{-t} + kt e^{-t}) - 2(k e^{-t} - kt e^{-t}) - 3(kt e^{-t}) = 3 e^{-t}$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not solution of the homogeneous equation.

(3) Find the undetermined coefficient $k$.

$$y_p' = k e^{-t} - kt e^{-t}, \quad y_p'' = -2k e^{-t} + kt e^{-t}.$$  

$$(-2k e^{-t} + kt e^{-t}) - 2(k e^{-t} - kt e^{-t}) - 3(kt e^{-t}) = 3 e^{-t}$$  

$$(-2 + t - 2 + 2t - 3t) k e^{-t} = 3 e^{-t}$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}. $$

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not
solution of the homogeneous equation.

(3) Find the undetermined coefficient $k$.

$$y'_p = k e^{-t} - kt e^{-t}, \quad y''_p = -2k e^{-t} + kt e^{-t}. $$

$$(-2k e^{-t} + kt e^{-t}) - 2(k e^{-t} - kt e^{-t}) - 3(kt e^{-t}) = 3 e^{-t}$$

$$(-2 + t - 2 + 2t - 3t)k e^{-t} = 3 e^{-t} \quad \Rightarrow \quad -4k = 3$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not solution of the homogeneous equation.

(3) Find the undetermined coefficient $k$.

$$y'_p = k e^{-t} - kt e^{-t}, \quad y''_p = -2k e^{-t} + kt e^{-t}.$$

$$(−2k e^{-t} + kt e^{-t}) − 2(k e^{-t} − kt e^{-t}) − 3(k t e^{-t}) = 3 e^{-t}$$

$$(-2 + t - 2 + 2t - 3t) k e^{-t} = 3 e^{-t} \Rightarrow -4k = 3 \Rightarrow k = -\frac{3}{4}.$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y_p(t) = kt e^{-t}$. This is correct, since $te^{-t}$ is not solution of the homogeneous equation.

(3) Find the undetermined coefficient $k$.

$$y_p' = k e^{-t} - kt e^{-t}, \quad y_p'' = -2k e^{-t} + kt e^{-t}.$$ 

$$(-2k e^{-t} + kt e^{-t}) - 2(k e^{-t} - kt e^{-t}) - 3(kt e^{-t}) = 3e^{-t}$$

$$(-2 + t - 2 + 2t - 3t) k e^{-t} = 3 e^{-t} \Rightarrow -4k = 3 \Rightarrow k = -\frac{3}{4}.$$ 

We obtain: $y_p(t) = -\frac{3}{4} t e^{-t}$. 
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y_p(t) = -\frac{3}{4}t e^{-t}$. 
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y_p(t) = -\frac{3}{4}t e^{-t}$.

(4) Find the general solution:
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

Solution: Recall: $y_p(t) = -\frac{3}{4}t e^{-t}$.

(4) Find the general solution: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4}t e^{-t}$. 
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y_p(t) = -\frac{3}{4} t e^{-t}$.

(4) Find the general solution: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t}$.
(5) Impose the initial conditions.
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y_p(t) = -\frac{3}{4} t e^{-t}$.

(4) Find the general solution: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t}$.

(5) Impose the initial conditions. The derivative function is
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}. $$

Solution: Recall: $y_p(t) = -\frac{3}{4}t e^{-t}$.

(4) Find the general solution: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4}t e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1 e^{3t} - c_2 e^{-t} - \frac{3}{4}(e^{-t} - t e^{-t}).$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

Solution: Recall: $y_p(t) = -\frac{3}{4} t e^{-t}$.

(4) Find the general solution: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1 e^{3t} - c_2 e^{-t} - \frac{3}{4} (e^{-t} - t e^{-t}).$$

$$1 = y(0)$$
Second order linear equations.

Example
Find the solution \( y \) to the initial value problem

\[
y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.
\]

Solution: Recall: \( y_p(t) = -\frac{3}{4} t e^{-t} \).

(4) Find the general solution: \( y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t} \).

(5) Impose the initial conditions. The derivative function is

\[
y'(t) = 3c_1 e^{3t} - c_2 e^{-t} - \frac{3}{4} (e^{-t} - t e^{-t}).
\]

\[
1 = y(0) = c_1 + c_2,
\]
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y_p(t) = -\frac{3}{4} t e^{-t}$.

(4) Find the general solution: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t}$.

(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1 e^{3t} - c_2 e^{-t} - \frac{3}{4} (e^{-t} - t e^{-t}).$$

$$1 = y(0) = c_1 + c_2, \quad \frac{1}{4} = y'(0).$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y_p(t) = -\frac{3}{4} t e^{-t}$.

(4) Find the general solution: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t}$.

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$$y'(t) = 3c_1 e^{3t} - c_2 e^{-t} - \frac{3}{4}(e^{-t} - t e^{-t}).$$ 

$$1 = y(0) = c_1 + c_2, \quad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}.$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

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$$y'(t) = 3c_1 e^{3t} - c_2 e^{-t} - \frac{3}{4}(e^{-t} - t e^{-t}).$$

\[ 1 = y(0) = c_1 + c_2, \quad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}. \]

\[
\begin{align*}
    c_1 + c_2 &= 1, \\
    3c_1 - c_2 &= 1,
\end{align*}
\]
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}. $$

Solution: Recall: $y_p(t) = -\frac{3}{4} t e^{-t}$.

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(5) Impose the initial conditions. The derivative function is

$$y'(t) = 3c_1 e^{3t} - c_2 e^{-t} - \frac{3}{4}(e^{-t} - t e^{-t}).$$

$$1 = y(0) = c_1 + c_2, \quad \frac{1}{4} = y'(0) = 3c_1 - c_2 - \frac{3}{4}.$$

$$c_1 + c_2 = 1, \quad 3c_1 - c_2 = 1 \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. $$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

Solution: Recall: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t},$ and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3 e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Solution: Recall: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$
Second order linear equations.

Example
Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$ 

Solution: Recall: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$ 

Since $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$,
Second order linear equations.

Example

Find the solution $y$ to the initial value problem

$$y'' - 2y' - 3y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = \frac{1}{4}.$$  

Solution: Recall: $y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4}t e^{-t}$, and

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$  

Since $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$, we obtain,

$$y(t) = \frac{1}{2} \left( e^{3t} + e^{-t} \right) - \frac{3}{4}t e^{-t}.$$  

\[\triangle\]
Review for Final Exam.

- Heat Eq. and Fourier Series (Chptr. 6).
- Eigenvalue-Eigenfunction BVP (Chptr. 6).
- Systems of linear Equations (Chptr. 5).
- Laplace transforms (Chptr. 4).
- Second order linear equations (Chptr. 2).
- First order differential equations (Chptr. 1).
First order differential equations.

Summary:

- **Linear, first order equations**: \( y' + p(t) y = q(t) \).

- **Separable, non-linear equations**: \( h(y) y' = g(t) \).
  
  Integrate with the substitution: \( u = y(t), \) \( du = y'(t) \, dt \), that is, \( \int h(u) \, du = \int g(t) \, dt + c \).

- **Homogeneous equations** can be converted into separable equations.

- Read page 49 in the textbook.

- No modeling problems from Sect. 2.3.
First order differential equations.

Summary:

- **Linear**, first order equations: \( y' + p(t) y = q(t) \).
  
  Use the integrating factor method: \( \mu(t) = e^{\int p(t) \, dt} \).
First order differential equations.

Summary:

- **Linear**, first order equations: \( y' + p(t) y = q(t) \).
  
  Use the integrating factor method: \( \mu(t) = e^{\int p(t) \, dt} \).

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First order differential equations.

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- **Linear**, first order equations: \( y' + p(t) y = q(t) \).
  
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  that is,
  \[
  \int h(u) \, du = \int g(t) \, dt + c.
  \]
First order differential equations.

Summary:

- **Linear**, first order equations: \( y' + p(t) y = q(t) \).
  Use the integrating factor method: \( \mu(t) = e^{\int p(t) \, dt} \).

- **Separable**, non-linear equations: \( h(y) y' = g(t) \).
  Integrate with the substitution: \( u = y(t), \ du = y'(t) \, dt \),
  that is,
  \[
  \int h(u) \, du = \int g(t) \, dt + c.
  \]
  The solution can be found in implicit or explicit form.

Homogeneous equations can be converted into separable equations. Read page 49 in the textbook.

No modeling problems from Sect. 2.3.
First order differential equations.

Summary:

- **Linear**, first order equations: \( y' + p(t) y = q(t) \).
  
  Use the integrating factor method: \( \mu(t) = e^{\int p(t) \, dt} \).

- **Separable**, non-linear equations: \( h(y) \, y' = g(t) \).
  
  Integrate with the substitution: \( u = y(t), \ du = y'(t) \, dt \), that is,
  \[
  \int h(u) \, du = \int g(t) \, dt + c.
  \]

  The solution can be found in implicit or explicit form.

- **Homogeneous equations** can be converted into separable equations.

Read page 49 in the textbook.

No modeling problems from Sect. 2.3.
First order differential equations.

Summary:

- **Linear**, first order equations: \( y' + p(t) y = q(t) \).
  
  Use the integrating factor method: \( \mu(t) = e^{\int p(t) \, dt} \).

- **Separable**, non-linear equations: \( h(y) y' = g(t) \).
  
  Integrate with the substitution: \( u = y(t), \, du = y'(t) \, dt \), that is,
  \[
  \int h(u) \, du = \int g(t) \, dt + c.
  \]

  The solution can be found in implicit or explicit form.

- **Homogeneous equations** can be converted into separable equations.

Read page 49 in the textbook.
First order differential equations.

Summary:

▶ Linear, first order equations: \( y' + p(t) y = q(t) \).

Use the integrating factor method: \( \mu(t) = e^{\int p(t) \, dt} \).

▶ Separable, non-linear equations: \( h(y) \, y' = g(t) \).

Integrate with the substitution: \( u = y(t), \, du = y'(t) \, dt \), that is,
\[
\int h(u) \, du = \int g(t) \, dt + c.
\]

The solution can be found in implicit or explicit form.

▶ Homogeneous equations can be converted into separable equations.

Read page 49 in the textbook.

▶ No modeling problems from Sect. 2.3.
First order differential equations.

Summary:
- Bernoulli equations: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).
First order differential equations.

Summary:

- Bernoulli equations: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).

Read page 77 in the textbook,
First order differential equations.

Summary:
- Bernoulli equations: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).

Read page 77 in the textbook, page 11 in the Lecture Notes.
First order differential equations.

Summary:

- Bernoulli equations: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).

Read page 77 in the textbook, page 11 in the Lecture Notes.

A Bernoulli equation for \( y \) can be converted into a linear equation for

\[ v = y^{n-1} \]
First order differential equations.

Summary:
- Bernoulli equations: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).

Read page 77 in the textbook, page 11 in the Lecture Notes.

A Bernoulli equation for \( y \) can be converted into a linear equation for \( v = \frac{1}{y^{n-1}} \).
First order differential equations.

Summary:

▶ Bernoulli equations: \( y' + p(t)y = q(t)y^n \), with \( n \in \mathbb{R} \).

Read page 77 in the textbook, page 11 in the Lecture Notes.

A Bernoulli equation for \( y \) can be converted into a linear equation for \( \nu = \frac{1}{y^{n-1}} \).

▶ Exact equations and integrating factors.
First order differential equations.

Summary:

- **Bernoulli equations**: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for \( y \) can be converted into a linear equation for \( v = \frac{1}{y^{n-1}} \).

- **Exact equations** and integrating factors.

  \[
  N(x, y) y' + M(x, y) = 0.
  \]
First order differential equations.

Summary:

- **Bernoulli equations**: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).

Read page 77 in the textbook, page 11 in the Lecture Notes.

A Bernoulli equation for \( y \) can be converted into a linear equation for \( v = \frac{1}{y^{n-1}} \).

- **Exact equations** and integrating factors.

\[ N(x, y) y' + M(x, y) = 0. \]

The equation is exact iff \( \partial_x N = \partial_y M \).
First order differential equations.

Summary:

- **Bernoulli equations**: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for \( y \) can be converted into a linear equation for \( \nu = \frac{1}{y^{n-1}} \).

- **Exact equations** and integrating factors.

  \[ N(x, y) y' + M(x, y) = 0. \]

  The equation is exact iff \( \partial_x N = \partial_y M \).

  If the equation is exact, then there is a potential function \( \psi \),
First order differential equations.

Summary:
- **Bernoulli equations**: \( y' + p(t) y = q(t) y^n \), with \( n \in \mathbb{R} \).

Read page 77 in the textbook, page 11 in the Lecture Notes.

A Bernoulli equation for \( y \) can be converted into a linear equation for \( \nu = \frac{1}{y^{n-1}} \).

- **Exact equations** and integrating factors.

\[ N(x, y) y' + M(x, y) = 0. \]

The equation is exact iff \( \partial_x N = \partial_y M \).

If the equation is exact, then there is a potential function \( \psi \), such that \( N = \partial_y \psi \).
First order differential equations.

Summary:

- **Bernoulli equations**: $y' + p(t) y = q(t) y^n$, with $n \in \mathbb{R}$.

  Read page 77 in the textbook, page 11 in the Lecture Notes.

  A Bernoulli equation for $y$ can be converted into a linear equation for $v = \frac{1}{y^{n-1}}$.

- **Exact equations** and integrating factors.

  $N(x, y) y' + M(x, y) = 0$.

  The equation is exact iff $\partial_x N = \partial_y M$.

  If the equation is exact, then there is a potential function $\psi$, such that $N = \partial_y \psi$ and $M = \partial_x \psi$. 
First order differential equations.

Summary:

- **Bernoulli equations**: \( y' + p(t)y = q(t)y^n \), with \( n \in \mathbb{R} \).

Read page 77 in the textbook, page 11 in the Lecture Notes.

A Bernoulli equation for \( y \) can be converted into a linear equation for \( v = \frac{1}{y^{n-1}} \).

- **Exact equations** and integrating factors.

\[
N(x, y)y' + M(x, y) = 0.
\]

The equation is exact iff \( \partial_x N = \partial_y M \).

If the equation is exact, then there is a potential function \( \psi \), such that \( N = \partial_y \psi \) and \( M = \partial_x \psi \).

The solution of the differential equation is

\[
\psi(x, y(x)) = c.
\]
First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations. (Just by looking at it: $y' + a(t)y = b(t)$.)

2. Bernoulli equations. (Just by looking at it: $y' + a(t)y = b(t)y^n$.)

3. Separable equations. (Few manipulations: $h(y)y' = g(t)$.)

4. Homogeneous equations. (Several manipulations: $y' = F(y/t)$.)

5. Exact equations. (Check one equation: $N y' + M = 0$, and $\partial_t N = \partial_y M$.)

6. Exact equation with integrating factor. (Very complicated to check.)
First order differential equations.

**Advice:** In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. **Linear equations.**
   (Just by looking at it: \( y' + a(t) y = b(t) \).)

2. **Bernoulli equations.**
   (Just by looking at it: \( y' + a(t) y = b(t) y^n \).)

3. **Separable equations.**
   (Few manipulations: \( h(y) y' = g(t) \).)

4. **Homogeneous equations.**
   (Several manipulations: \( y' = F(y/t) \).)

5. **Exact equations.**
   (Check one equation: \( Ny' + M = 0 \), and \( \frac{\partial}{\partial t} N = \frac{\partial}{\partial y} M \).)

6. **Exact equation with integrating factor.**
   (Very complicated to check.)
First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: \( y' + a(t) y = b(t) \).)

2. Bernoulli equations.
   (Just by looking at it: \( y' + a(t) y = b(t) y^n \).)
First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: \( y' + a(t)y = b(t) \).)

2. Bernoulli equations.
   (Just by looking at it: \( y' + a(t)y = b(t)y^n \).)

   (Few manipulations: \( h(y)y' = g(t) \).)
First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
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   (Just by looking at it: \( y' + a(t) y = b(t) y^n \).)

   (Few manipulations: \( h(y) y' = g(t) \).)

   (Several manipulations: \( y' = F(y/t) \).)
First order differential equations.

**Advice:** In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

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   (Few manipulations: \( h(y) y' = g(t) \).)

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   (Several manipulations: \( y' = F(y/t) \).)

5. **Exact equations.**
   (Check one equation: \( Ny' + M = 0 \), and \( \partial_t N = \partial_y M \).)
First order differential equations.

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: \( y' + a(t) y = b(t) \).)

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   (Just by looking at it: \( y' + a(t) y = b(t) y^n \).)

   (Few manipulations: \( h(y) y' = g(t) \).)

   (Several manipulations: \( y' = F(y/t) \).)

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   (Check one equation: \( N y' + M = 0 \), and \( \partial_t N = \partial_y M \).)

6. Exact equation with integrating factor.
   (Very complicated to check.)
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).
First order differential equations.

**Example**

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

**Solution:** The sum of the powers in \( x \) and \( y \) on every term is the same number,
First order differential equations.

Example

Find all solutions of $y' = \frac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in $x$ and $y$ on every term is the same number, two in this example.
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: The sum of the powers in \( x \) and \( y \) on every term is the same number, two in this example. The equation is homogeneous.
First order differential equations.

Example

Find all solutions of $y' = \frac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in $x$ and $y$ on every term is the same number, two in this example. The equation is homogeneous.

$y' = \frac{x^2 + xy + y^2}{xy} \left(\frac{1}{x^2}\right) \left(\frac{1}{x^2}\right)$
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: The sum of the powers in \( x \) and \( y \) on every term is the same number, two in this example. The equation is homogeneous.

\[
y' = \frac{x^2 + xy + y^2}{xy} \left(\frac{1}{x^2}\right) \quad \Rightarrow \quad y' = \frac{1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}.
\]
First order differential equations.

Example

Find all solutions of $y' = \frac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in $x$ and $y$ on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \cdot \frac{1}{x^2} \Rightarrow y' = \frac{1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}.$$

$$v(x) = \frac{y}{x}$$
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: The sum of the powers in \( x \) and \( y \) on every term is the same number, two in this example. The equation is homogeneous.

\[
y' = \frac{x^2 + xy + y^2}{xy} \cdot \frac{(1/x^2)}{((1/x^2))} \quad \Rightarrow \quad y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.
\]

\[
v(x) = \frac{y}{x} \quad \Rightarrow \quad y' = \frac{1 + v + v^2}{v}.
\]
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: The sum of the powers in \( x \) and \( y \) on every term is the same number, two in this example. The equation is homogeneous.

\[
y' = \frac{x^2 + xy + y^2}{xy} \frac{(1/x^2)}{(1/x^2)} \implies y' = \frac{1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}.
\]

\[
v(x) = \frac{y}{x} \implies y' = \frac{1 + v + v^2}{v}.
\]

\[
y = x \ v,
\]
First order differential equations.

Example
Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: The sum of the powers in \( x \) and \( y \) on every term is the same number, two in this example. The equation is homogeneous.

\[
y' = \frac{x^2 + xy + y^2}{xy} \left(\frac{1}{x^2}\right) = \frac{1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}.
\]

Let \( v(x) = \frac{y}{x} \) then

\[
y' = 1 + v + v^2.
\]

\( y = x \, v \), \( y' = x \, v' + v \)
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: The sum of the powers in \( x \) and \( y \) on every term is the same number, two in this example. The equation is homogeneous.

\[
y' = \frac{x^2 + xy + y^2}{xy} \left(\frac{1}{x^2}\right) \left(\frac{1}{x^2}\right) \Rightarrow y' = \frac{1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}.
\]

\[
v(x) = \frac{y}{x} \Rightarrow y' = \frac{1 + v + v^2}{v}.
\]

\[
y = x \cdot v, \quad y' = x \cdot v' + v \quad x \cdot v' + v = \frac{1 + v + v^2}{v}.
\]
First order differential equations.

Example

Find all solutions of $y' = \frac{x^2 + xy + y^2}{xy}$.

Solution: The sum of the powers in $x$ and $y$ on every term is the same number, two in this example. The equation is homogeneous.

$$y' = \frac{x^2 + xy + y^2}{xy} \left(\frac{1}{x^2}\right) \Rightarrow y' = \frac{1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}.$$

$$v(x) = \frac{y}{x} \Rightarrow y' = \frac{1 + v + v^2}{v}.$$

$$y = x\ v, \quad y' = x\ v' + v \quad x\ v' + v = \frac{1 + v + v^2}{v}.$$

$$x\ v' = \frac{1 + v + v^2}{v} - v.$$
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: The sum of the powers in \( x \) and \( y \) on every term is the same number, two in this example. The equation is homogeneous.

\[
y' = \frac{x^2 + xy + y^2}{xy} \left( \frac{1}{x^2} \right) \left( \frac{1}{x^2} \right) \Rightarrow y' = \frac{1 + \left( \frac{y}{x} \right) + \left( \frac{y}{x} \right)^2}{\left( \frac{y}{x} \right)}.
\]

\[
v(x) = \frac{y}{x} \Rightarrow y' = \frac{1 + v + v^2}{v}.
\]

\[
y = x v, \quad y' = x v' + v \quad x v' + v = \frac{1 + v + v^2}{v}.
\]

\[
x v' = \frac{1 + v + v^2}{v} - v = \frac{1 + v + v^2 - v^2}{v}
\]
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: The sum of the powers in \( x \) and \( y \) on every term is the same number, two in this example. The equation is homogeneous.

\[
y' = \frac{x^2 + xy + y^2}{xy} \left(\frac{1}{x^2}\right) \left(\frac{1}{x^2}\right) \Rightarrow y' = \frac{1 + (\frac{y}{x}) + (\frac{y}{x})^2}{(\frac{y}{x})}.
\]

\[
v(x) = \frac{y}{x} \Rightarrow y' = \frac{1 + v + v^2}{v}.
\]

\[
y = xv, \quad y' = xv' + v \quad xv' + v = \frac{1 + v + v^2}{v}.
\]

\[
xv' = \frac{1 + v + v^2}{v} - v = \frac{1 + v + v^2 - v^2}{v} \Rightarrow xv' = \frac{1 + v}{v}.
\]
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: Recall: \( \frac{dv}{dx} = \frac{1}{v} + v \).

\( \frac{dv}{dx} = \frac{1}{v} + v \Rightarrow \int (v - 1) v \, dv = \int dx \Rightarrow 
\int (1 - 1) v \, dv = \int dx \Rightarrow 
\frac{v^2}{2} = x + c \Rightarrow 
\frac{1 + v}{v} = x + c \Rightarrow 
v = y \Rightarrow 
1 + y(x) = x + c \Rightarrow 
\boxed{y(x) = x + c.}
First order differential equations.

Example

Find all solutions of $y' = \frac{x^2 + xy + y^2}{xy}$.

**Solution:** Recall: $v' = \frac{1 + v}{v}$. This is a separable equation.
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: Recall: \( v' = \frac{1 + v}{v} \). This is a separable equation.

\[
\frac{v(x)}{1 + v(x)} v'(x) = \frac{1}{x}
\]
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: Recall: \( v' = \frac{1 + v}{v} \). This is a separable equation.

\[
\frac{v(x)}{1 + v(x)} v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1 + v(x)} v'(x) \, dx = \int \frac{dx}{x} + c.
\]
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: Recall: \( v' = \frac{1 + v}{v} \). This is a separable equation.

\[
\frac{v(x)}{1 + v(x)} v'(x) = \frac{1}{x} \Rightarrow \int \frac{v(x)}{1 + v(x)} v'(x) \, dx = \int \frac{dx}{x} + c.
\]

Use the substitution \( u = 1 + v \),
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: Recall: \( v' = \frac{1 + v}{v} \). This is a separable equation.

\[
\frac{v(x)}{1 + v(x)} \cdot v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1 + v(x)} \, v'(x) \, dx = \int \frac{dx}{x} + c.
\]

Use the substitution \( u = 1 + v \), hence \( du = v'(x) \, dx \).
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: Recalling \( v' = \frac{1 + v}{v} \). This is a separable equation.

\[
\frac{v(x)}{1 + v(x)} v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1 + v(x)} v'(x) \, dx = \int \frac{dx}{x} + c.
\]

Use the substitution \( u = 1 + v \), hence \( du = v'(x) \, dx \).

\[
\int \frac{(u - 1)}{u} \, du = \int \frac{dx}{x} + c
\]
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: Recall: \( v' = \frac{1 + v}{v} \). This is a separable equation.

\[
\frac{v(x)}{1 + v(x)} v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1 + v(x)} v'(x) \, dx = \int \frac{dx}{x} + c.
\]

Use the substitution \( u = 1 + v \), hence \( du = v'(x) \, dx \).

\[
\int \frac{(u - 1)}{u} \, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \left(1 - \frac{1}{u}\right) \, du = \int \frac{dx}{x} + c
\]
First order differential equations.

Example

Find all solutions of $y' = \frac{x^2 + xy + y^2}{xy}$.

Solution: Recall: $v' = \frac{1 + v}{v}$. This is a separable equation.

$$\frac{v(x)}{1 + v(x)} v'(x) = \frac{1}{x} \implies \int \frac{v(x)}{1 + v(x)} v'(x) \, dx = \int \frac{dx}{x} + c.$$  

Use the substitution $u = 1 + v$, hence $du = v'(x) \, dx$.

$$\int \frac{(u - 1)}{u} \, du = \int \frac{dx}{x} + c \implies \int \left(1 - \frac{1}{u}\right) \, du = \int \frac{dx}{x} + c$$

$$u - \ln |u| = \ln |x| + c$$
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: Recall: \( v' = \frac{1 + v}{v} \). This is a separable equation.

\[
\frac{v(x)}{1 + v(x)} \cdot v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1 + v(x)} \cdot v'(x) \, dx = \int \frac{dx}{x} + c.
\]

Use the substitution \( u = 1 + v \), hence \( du = v'(x) \, dx \).

\[
\int \frac{u - 1}{u} \, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \left(1 - \frac{1}{u}\right) \, du = \int \frac{dx}{x} + c
\]

\[
u - \ln |u| = \ln |x| + c \quad \Rightarrow \quad 1 + v - \ln |1 + v| = \ln |x| + c.
\]
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: Recall: \( v' = \frac{1 + v}{v} \). This is a separable equation.

\[
\frac{v(x)}{1 + v(x)} \frac{v'(x)}{x} \Rightarrow \int \frac{v(x)}{1 + v(x)} \frac{v'(x)}{x} \, dx = \int \frac{dx}{x} + c.
\]

Use the substitution \( u = 1 + v \), hence \( du = v'(x) \, dx \).

\[
\int \frac{(u - 1)}{u} \, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \left(1 - \frac{1}{u}\right) \, du = \int \frac{dx}{x} + c
\]

\[
u - \ln |u| = \ln |x| + c \quad \Rightarrow \quad 1 + v - \ln |1 + v| = \ln |x| + c.
\]

\( v = \frac{y}{x} \)
First order differential equations.

Example

Find all solutions of \( y' = \frac{x^2 + xy + y^2}{xy} \).

Solution: Recall: \( v' = \frac{1 + v}{v} \). This is a separable equation.

\[
\frac{v(x)}{1 + v(x)} v'(x) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{v(x)}{1 + v(x)} v'(x) \, dx = \int \frac{dx}{x} + c.
\]

Use the substitution \( u = 1 + v \), hence \( du = v'(x) \, dx \).

\[
\int \frac{(u - 1)}{u} \, du = \int \frac{dx}{x} + c \quad \Rightarrow \quad \int \left( 1 - \frac{1}{u} \right) \, du = \int \frac{dx}{x} + c
\]

\[
u - \ln |u| = \ln |x| + c \quad \Rightarrow \quad 1 + v - \ln |1 + v| = \ln |x| + c.
\]

\[
v = \frac{y}{x} \quad \Rightarrow \quad 1 + \frac{y(x)}{x} - \ln \left| 1 + \frac{y(x)}{x} \right| = \ln |x| + c. \quad \text{\( \blacksquare \)}
\]
Example

Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$
First order differential equations.

Example
Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$ 

Solution: This is a Bernoulli equation,
First order differential equations.

Example
Find the solution $y$ to the initial value problem

\[ y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}. \]

Solution: This is a Bernoulli equation, $y' + y = -e^{2x} y^n$, 

\[ \mu(x) = e^{-2x}. \]

\[ e^{-2x} v' - 2 e^{-2x} v = 2 e^{2x}. \]

\[ (e^{-2x} v)' = 2. \]
First order differential equations.

Example
Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$  

Solution: This is a Bernoulli equation, $y' + y = -e^{2x} y^n$, $n = 3$. 
First order differential equations.

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First order differential equations.

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First order differential equations.

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First order differential equations.

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First order differential equations.

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First order differential equations.

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Let \( v = \frac{1}{y^2} \). Since \( v' = -2 \frac{y'}{y^3} \), we obtain \[
- \frac{1}{2} v' + v = -e^{2x}.
\]

We obtain the linear equation \( v' - 2v = 2e^{2x} \).
First order differential equations.

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Find the solution $y$ to the initial value problem

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Use the integrating factor method.
First order differential equations.

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$$\mu(x) = e^{-2x}. $$

$$e^{-2x} v' - 2 e^{-2x} v = 2$$
First order differential equations.

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First order differential equations.

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Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$

Solution: Recall: $v = \frac{1}{y^2}$ and $(e^{-2x} v)' = 2$. 

The initial condition $y(0) = \frac{1}{3}$ implies: Choose $y +$. 

$$1 = y + (0) = 1 \sqrt{2x + \frac{9}{2}} \Rightarrow c = 9 \Rightarrow y(\frac{2x}{2x + 9} = e^{-x} \sqrt{2x + 9}.$$
First order differential equations.

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$e^{-2x} v = 2x + c$
First order differential equations.

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First order differential equations.

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$$y^2 = \frac{1}{e^{2x} (2x + c)}$$
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$$y^2 = \frac{1}{e^{2x}(2x + c)} \Rightarrow y_+(x) = \pm \frac{e^{-x}}{\sqrt{2x + c}}.$$
First order differential equations.

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The initial condition $y(0) = 1/3$
First order differential equations.

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The initial condition \( y(0) = 1/3 > 0 \) implies:
First order differential equations.

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Find the solution $y$ to the initial value problem

$$y' + y + e^{2x} y^3 = 0, \quad y(0) = \frac{1}{3}.$$

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The initial condition $y(0) = 1/3 > 0$ implies: Choose $y_+.$
First order differential equations.

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The initial condition $y(0) = 1/3 > 0$ implies: Choose $y_+.$

$$\frac{1}{3} = y_{+}(0) = \frac{1}{\sqrt{c}} \Rightarrow c = 9 \Rightarrow y(x) = \frac{e^{-x}}{\sqrt{2x + 9}}. \quad \triangleq$$
First order differential equations.

Example
Find all solutions of $2xy^2 + 2y + 2x^2y' + 2x y' = 0$. 

Solution:
Rewrite the equation in a more organized way,

$\left[2x^2y + 2x\right]y' + \left[2xy^2 + 2y\right] = 0$.

$N = \left[2x^2y + 2x\right] \Rightarrow \partial_x N = 4xy + 2$.

$M = \left[2xy^2 + 2y\right] \Rightarrow \partial_y M = 4xy + 2$.

$\Rightarrow \partial_x N = \partial_y M$.

The equation is exact.

There exists a potential function $\psi$ with

$\partial_y \psi = N$, $\partial_x \psi = M$.

$\partial_y \psi = 2x^2y^2 + 2xy + g'(x) \Rightarrow \psi(x,y) = x^2y^2 + 2xy + c$.

$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y \Rightarrow g'(x) = 0$.

$\Rightarrow \psi(x,y) = x^2y^2 + 2xy + c$. 

$\triangleright$
First order differential equations.

Example
Find all solutions of \( 2xy^2 + 2y + 2x^2 y y' + 2x y' = 0 \).

Solution: Re-write the equation in a more organized way,
First order differential equations.

Example
Find all solutions of \(2xy^2 + 2y + 2x^2y y' + 2x y' = 0\).

Solution: Re-write the equation is a more organized way,
\[
[2x^2y + 2x] y' + [2xy^2 + 2y] = 0.
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Example
Find all solutions of \( 2xy^2 + 2y + 2x^2y \ y' + 2x \ y' = 0 \).

Solution: Re-write the equation is a more organized way,

\[
[2x^2y + 2x] \ y' + [2xy^2 + 2y] = 0.
\]

\[
N = [2x^2y + 2x]
\]
First order differential equations.

Example
Find all solutions of \(2xy^2 + 2y + 2x^2y \, y' + 2x \, y' = 0\).

Solution: Re-write the equation is a more organized way,

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[2x^2y + 2x] \, y' + [2xy^2 + 2y] = 0.
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\[N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.\]
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**Example**

Find all solutions of \(2xy^2 + 2y + 2x^2y \ y' + 2x \ y' = 0\).

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\(N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.
\)

\(M = [2xy^2 + 2y] \quad \Rightarrow \quad \partial_y M = 4xy + 2.\)

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[2x^2y + 2x] y' + [2xy^2 + 2y] = 0.
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\[
N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2. \\
M = [2xy^2 + 2y] \quad \Rightarrow \quad \partial_y M = 4xy + 2.
\]

\[
\begin{array}{l}
\partial_x N = \partial_y M.
\end{array}
\]

The equation is exact.
First order differential equations.

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Find all solutions of \(2xy^2 + 2y + 2x^2y \ y' + 2x \ y' = 0.\)

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[2x^2y + 2x] \ y' + [2xy^2 + 2y] = 0.
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N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.
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The equation is exact. There exists a potential function \(\psi\) with
First order differential equations.

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Find all solutions of \(2xy^2 + 2y + 2x^2 y y' + 2x y' = 0\).

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\]
\[
M = [2xy^2 + 2y] \quad \Rightarrow \quad \partial_y M = 4xy + 2.
\]

The equation is exact. There exists a potential function \(\psi\) with

\[
\partial_y \psi = N,
\]

\[
\partial_x N = \partial_y M.
\]
First order differential equations.

Example
Find all solutions of \(2xy^2 + 2y + 2x^2y \cdot y' + 2x \cdot y' = 0\).

Solution: Re-write the equation in a more organized way,

\[
[2x^2y + 2x] \cdot y' + [2xy^2 + 2y] = 0.
\]

\[
N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.
\]
\[
M = [2xy^2 + 2y] \quad \Rightarrow \quad \partial_y M = 4xy + 2.
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\[
\begin{aligned}
\Rightarrow \quad \partial_x N &= \partial_y M.
\end{aligned}
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The equation is exact. There exists a potential function \(\psi\) with

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Find all solutions of \(2xy^2 + 2y + 2x^2y \ y' + 2xy' = 0\).

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\]
\[
\left\{ \begin{array}{c}
\partial_x N = \partial_y M.
\end{array} \right.
\]

The equation is exact. There exists a potential function \(\psi\) with

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\partial_y \psi = N, \quad \partial_x \psi = M.
\]

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\partial_y \psi = 2x^2y + 2x
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First order differential equations.

Example

Find all solutions of $2xy^2 + 2y + 2x^2y\, y' + 2x\, y' = 0$.

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$$N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.$$

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$$\begin{cases} 
N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2. \\
M = [2xy^2 + 2y] \quad \Rightarrow \quad \partial_y M = 4xy + 2.
\end{cases} \quad \Rightarrow \quad \partial_x N = \partial_y M.$$

The equation is exact. There exists a potential function $\psi$ with

$$\partial_y \psi = N, \quad \partial_x \psi = M.$$ 

$$\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x, y) = x^2y^2 + 2xy + g(x).$$
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Find all solutions of \( 2xy^2 + 2y + 2x^2y \, y' + 2x \, y' = 0 \).

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[2x^2y + 2x] \, y' + [2xy^2 + 2y] = 0.
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N = [2x^2y + 2x] \quad \Rightarrow \quad \partial_x N = 4xy + 2.
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\]

\[
\Rightarrow \quad \partial_x N = \partial_y M.
\]

The equation is exact. There exists a potential function \( \psi \) with

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\partial_y \psi = N, \quad \partial_x \psi = M.
\]

\[
\partial_y \psi = 2x^2y + 2x \quad \Rightarrow \quad \psi(x, y) = x^2y^2 + 2xy + g(x).
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\[
2xy^2 + 2y + g'(x) = \partial_x \psi
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Find all solutions of \( 2xy^2 + 2y + 2x^2y \, y' + 2xy' = 0 \).

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M = [2xy^2 + 2y] \quad \Rightarrow \quad \partial_y M = 4xy + 2.
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\[
\begin{align*}
\partial_x N &= \partial_y M. \\
\partial_y \psi &= 2x^2y + 2x \quad \Rightarrow \quad \psi(x, y) = x^2y^2 + 2xy + g(x).
\end{align*}
\]

\[
2xy^2 + 2y + g'(x) = \partial_x \psi = M
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First order differential equations.

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Find all solutions of $2xy^2 + 2y + 2x^2y \ y' + 2x \ y' = 0$.

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$$2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y$$
First order differential equations.

Example
Find all solutions of \( 2xy^2 + 2y + 2x^2y \frac{y'}{y} + 2x \frac{y'}{y} = 0. \)

Solution: Re-write the equation in a more organized way,

\[
[2x^2y + 2x] \frac{y'}{y} + [2xy^2 + 2y] = 0.
\]

\[
N = [2x^2y + 2x] \Rightarrow \partial_x N = 4xy + 2.
\]

\[
M = [2xy^2 + 2y] \Rightarrow \partial_y M = 4xy + 2.
\]

\[
\Rightarrow \partial_x N = \partial_y M.
\]

The equation is exact. There exists a potential function \( \psi \) with

\[
\partial_y \psi = N, \quad \partial_x \psi = M.
\]

\[
\partial_y \psi = 2x^2y + 2x \Rightarrow \psi(x, y) = x^2y^2 + 2xy + g(x).
\]

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\psi(x, y) = x^2y^2 + 2xy + c,
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Example
Find all solutions of \( 2xy^2 + 2y + 2x^2y \, y' + 2x \, y' = 0 \).

Solution: Re-write the equation in a more organized way,
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[2x^2y + 2x] \, y' + [2xy^2 + 2y] = 0.
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2xy^2 + 2y + g'(x) = \partial_x \psi = M = 2xy^2 + 2y \quad \Rightarrow \quad g'(x) = 0.
\]
\[
\psi(x, y) = x^2y^2 + 2xy + c, \quad x^2 \, y^2(x) + 2x \, y(x) + c = 0. \]