Wednesday 12/04/13

Plan: * Review : Solving the Heat Equation

* Heat Eq. : Non-homogeneous B.C. (Not for Final Exam.)

* The Wave Equation. (Not for Final Exam.)

* Review: Solving the Heat Eq.

Example: Find u sol. of:

$$\left[\begin{array}{ll} \partial_t u = 2\, \partial_x^2 u, & t\in[0,\infty),\ x\in[0,1], \\ u(0,x) = f(x) = \begin{cases} 2 & x\in[0,\frac{1}{2}) \\ 0 & x\in[\frac{1}{2},1] \end{cases} & \\ \partial_x u(t,0) = 0, & u(t,1) = 0. \end{array}\right] \quad (1)$$

Sol: Decompose $u(t,x) = \sum_{n=1}^{\infty} c_n\, V_n(t)\, W_n(x)$ (2)

Introduce (2) into (1) $\sum_{n=1}^{\infty} \partial_t(c_n V_n W_n) = 2 \sum_{n=1}^{\infty} \partial_x^2 (c_n V_n W_n)$

$$\sum_{n=1}^{\infty}\Big(\partial_t(c_n V_n W_n) - 2\,\partial_x^2(c_n V_n W_n)\Big) = 0$$

$$\Uparrow$$

$$\partial_t(c_n V_n W_n) = 2\, \partial_x^2 (c_n V_n W_n)$$

$$c_n W_n(x)\, \dot V_n(t) = 2\, c_n V_n(t)\, W_n''(x)$$

$$\boxed{\frac{1}{2}\frac{\dot V_n(t)}{V_n(t)} = \frac{W_n''(x)}{W_n(x)}}$$

t x

$$\boxed{\begin{array}{l} \frac{1}{2}\frac{\dot V_n(t)}{V_n(t)} = -\lambda_n \\ V_n(0) = 1 \end{array}} \qquad \boxed{\begin{array}{ll} \frac{W_n''(x)}{W_n(x)} = -\lambda_n & \\ W_n'(0) = 0 & W_n(1) = 0 \end{array}}$$

$$\boxed{W_n(0) = 1}$$

$$\Downarrow$$

$\dot V_n = -2\lambda_n V_n$

$V_n(t) = V_n(0)\, e^{-2\lambda_n t}$

$$\boxed{V_n(t) = e^{-2\lambda_n t}}$$

$$\boxed{V_n(t) = e^{-2\left((2n-1)\frac{\pi}{2}\right)^2 t}}$$

$$\downarrow$$

$$\boxed{\begin{aligned} &W_n''(x) + \lambda_n W_n(x) = 0 \\ &W_n'(0) = 0, \quad W_n(1) = 0 \end{aligned}}, \quad \left(W_n(0) = 1\right)$$

$$W_n(x) = c_1 \cos(\sqrt{\lambda_n}\, x) + c_2 \cancel{\sin(\sqrt{\lambda_n}\, x)}$$

$$W_n'(x) = -c_1\sqrt{\lambda_n} \sin(\sqrt{\lambda_n}\, x) + c_2\sqrt{\lambda_n} \cos(\sqrt{\lambda_n}\, x)$$

$$W_n'(0) = c_2 \sqrt{\lambda_n} = 0 \implies \boxed{c_2 = 0}$$

$$\boxed{W_n(x) = c_1 \cos(\sqrt{\lambda_n}\, x)}$$

$$W_n(1) = 0 = c_1 \cos(\sqrt{\lambda_n})$$

$$\Downarrow$$

$$\boxed{\sqrt{\lambda_n} = (2n-1)\frac{\pi}{2}}$$

$$W_n(x) = c_1 \cos\left((2n-1)\frac{\pi}{2} x\right), \quad W_n(0) = 1 = c_1$$

$$\boxed{W_n(x) = \cos\left((2n-1)\frac{\pi}{2} x\right)}$$

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-2\left((2n-1)\frac{\pi}{2}\right)^2 t} \cos\left((2n-1)\frac{\pi}{2} x\right)$$

$$u(0,x) = f(x) = \sum_{n=1}^{\infty} c_n \cos\left((2n-1)\frac{\pi}{2} x\right)$$

$$\int_0^1 f(x) \cos\left((2m-1)\frac{\pi}{2} x\right) dx = \sum_{n=1}^{\infty} c_n \int_0^1 dx \cos\left((2n-1)\frac{\pi}{2} x\right) \cos\left((2m-1)\frac{\pi}{2} x\right)$$

$$\int_0^1 f(x) \cos\left((2m-1)\frac{\pi}{2} x\right) dx = c_m \int_0^1 \cos^2\left((2m-1)\frac{\pi}{2} x\right) dx = c_m \frac{1}{2}$$

$$\boxed{c_m = \frac{2}{1} \int_0^1 f(x) \cos\left((2m-1)\frac{\pi}{2} x\right) dx} = \frac{2}{1} \int_0^{1/2} 2 \cos\left((2m-1)\frac{\pi}{2} x\right) dx$$

$$c_n = 4\left(\frac{2}{(2n-1)\pi}\right) \sin\left((2n-1)\frac{\pi}{2} x\right)\Big|_0^{1/2}, \quad \boxed{c_n = \frac{8}{(2n-1)\pi} \sin\left((2n-1)\frac{\pi}{4}\right)}$$

* <u>Heat Eq.</u> : <u>Non-homogeneous</u> <u>B.C.</u>

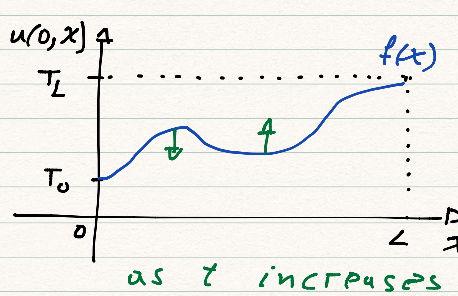
Consider the problem:

$$\left[\begin{array}{llll} \text{Find } u \text{ Sol. of:} & \partial_t u = k\,\partial_x^2 u \quad , & t\in[0,\infty)\,, & x\in[0,L]\,, \\ & u(0,x) = f(x) \quad ; & u(t,0) = T_0 \quad , & u(t,L) = T_L \end{array}\right] \quad (1)$$

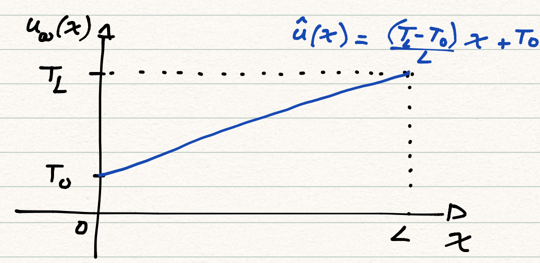
Now T_0, T_L might be non-zero.

<u>Remark</u>: A priori qualitative knowledge of the solution is needed to sole the problem.

1) Time evolution tries to smooth out high-curvature regions in the x-dependence of the solution.



$t\to\infty$



<u>Conclusion</u>: The must exist a time-independent sol. to the Heat Eq. Let's call it u_∞; It is function of x.

u_∞ must be Sol. of (1) in the limit $t\to\infty$

$$\partial_t u_\infty = k\,\partial_x^2 u_\infty \quad , \quad \text{but } u_\infty \text{ depends only on } x.$$

$$u_\infty''(x) = 0 \quad ; \quad \text{B.C.}: \quad u_\infty(0) = T_0 \quad , \quad u_\infty(L) = T_L .$$

This problem has the unique Sol:

$$\boxed{u_\infty(x) = \frac{(T_L - T_0)}{L}\,x + T_0}$$

2) <u>Idea</u>: Write: $u(t,x) = u_\infty(x) + \delta u(t,x)$

Find the problem for δu.

* Problem for δu:

The Eq.:
$$\partial_t (u_\infty + \delta u) = k\, \partial_x^2 (u_\infty + \delta u)$$
$$\underbrace{\partial_t u_\infty}_{=0} + \partial_t \delta u = k \underbrace{\partial_x^2 u_\infty}_{=0} + k\, \partial_x \delta u$$
$$\boxed{\partial_t (\delta u) = k\, \partial_x^2 (\delta u)}$$

The I.C.:
$$f(x) = u(0,x) = u_\infty(x) + \delta u(0,x)$$
$$u_\infty(x) = \frac{(T_L - T_0)}{L} x + T_0$$
$$\boxed{\delta u(0,x) = f(x) - \frac{(T_L - T_0)}{L} x - T_0}$$

The B.C.:
$$T_0 = u(t,0) = u_\infty(0) + \delta u(t,0) = T_0 + \delta u(t,0)$$
$$\boxed{\delta u(t,0) = 0}$$
$$T_L = u(t,L) = u_\infty(L) + \delta u(t,L) = T_L + \delta u(t,L)$$
$$\boxed{\delta u(t,L) = 0}$$

Thrm: If u is sol. of $\partial_t u = k\, \partial_x^2 u$,
with I.C. $u(0,x) = f(x)$
B.C. $u(t,0) = T_0$, $u(t,L) = T_L$

Then $u(t,x) = u_\infty(x) + \delta u(t,x)$
with $u_\infty(x) = \frac{(T_0 - T_L)}{L} x + T_0$

and $\partial_t \delta u = k\, \partial_x^2 \delta u$
with I.C. $\delta u(0,x) = f(x) - u_\infty(x)$
and B.C. $\delta u(t,0) = 0$, $\delta u(t,L) = 0.$

* Solving the Wave Eq.

The IBVP :

Find u Sol. of: $\quad \partial_t^2 u = a^2 \, \partial_x^2 u \,, \quad t \in [0,\infty) \,, \quad x \in [0, L] \qquad (1)$

I.C. $\quad u(0,x) = f(x) \,, \quad \partial_t u(0,x) = g(x)$

B.C. $\quad u(t,0) = 0 \,, \quad u(t,L) = 0 .$

Sol.: consider the Vector Space :

$$V = \{ \tilde{v} \text{ , differentiable on } [0,L] \text{ s.t. } \tilde{v}(0) = \tilde{v}(L) = 0 \} .$$

Let $\{ w_n \}_{n=1}^{\infty}$ be an orthogonal basis of V

Decompose: $\quad u(t,x) = \sum_{n=1}^{\infty} v_n(t) \, w_n(x) \qquad (2)$

Introduce (2) in (1)

$$\sum_{n=1}^{\infty} \left[\left(\partial_t^2 v_n \right) w_n - a^2 \, v_n \left(\partial_x^2 w_n \right) \right] = 0 \quad \Longleftarrow \quad \left(\partial_t^2 v_n \right) w_n = a^2 \, v_n \left(\partial_x^2 w_n \right)$$

We focus on: $\quad \dfrac{1}{a^2} \dfrac{\ddot{v}_n(t)}{v_n(t)} = \dfrac{w_n''(x)}{w_n(x)}$

depends only on t $\qquad$ depends only on x

So each side must be a constant.

$$\boxed{\frac{1}{a^2} \frac{\ddot{v}_n(t)}{v_n(t)} = -\lambda_n} \quad ; \quad \boxed{\begin{array}{l} \dfrac{w_n''(x)}{w_n(x)} = -\lambda_n \\ \\ w_n(0) = w_n(L) = 0 . \end{array}}$$

We start solving for w_n : $\quad w_n(x) = \sin\left(\frac{n\pi}{L} x \right) \quad ; \quad \lambda_n = \left(\frac{n\pi}{L} \right)^2$

Recall : $\quad \int_0^L \sin^2\left(\frac{n\pi}{L} x \right) dx = \frac{L}{2}$

We now solve for V_n : $\ddot{V}_n(t) + a^2 \lambda_n V_n(t) = 0$

The general sol. is: $\boxed{V_n(t) = c_n \cos\left(a \frac{n\pi}{L} t\right) + d_n \sin\left(a \frac{n\pi}{L} t\right)}$

Therefore: $u(t,x) = \sum_{n=1}^{\infty} \left[c_n \cos\left(a \frac{n\pi}{L} t\right) + d_n \sin\left(a \frac{n\pi}{L} t\right) \right] \sin\left(\frac{n\pi}{L} x\right)$

c_n, d_n are obtained from the I.C.s

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right)$$

$$g(x) = \sum_{n=1}^{\infty} d_n \frac{L}{a n\pi} \sin\left(\frac{n\pi}{L} x\right)$$

Hence: $c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$

$$d_n = \frac{2}{L} \, a \frac{n\pi}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

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