Review for Exam 3.

- 6 problems, 60 minutes, in CC-415.
- 5 grading attempts per problem.
- Problems similar to homeworks.
- Integration table an LT table provided.
- No notes, no books, no calculators.
- Exam covers:
  - Chapter 4: Laplace Transform methods.
    - Definition of Laplace Transform (4.1).
    - Solving IVP using LT (4.2).
    - Solving IVP with discontinuous sources using LT, (4.3).
    - Solving IVP with generalized sources using LT (4.4).
    - Convolutions and LT (4.5).
  - Chapter 5: Systems of linear equations.
    - Systems of linear Differential Equations (5.1).
    - 2×2 systems (actual 5.7, 5.8, 5.9).
  - BVP, eigenfunction problems, (6.1).

Review for Exam 3.

- **Chapter 4: Laplace Transform methods.**
  - Definition of Laplace Transform (4.1).
  - Solving IVP using LT (4.2).
  - Solving IVP with discontinuous sources using LT, (4.3).
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  - 2×2 systems (actual 5.7, 5.8, 5.9).
- BVP, eigenfunction problems, (6.1).

Laplace transforms (Chptr. 4).

Summary:

- Main Properties:

$$\mathcal{L}\big[f^{(n)}(t)\big] = s^n\,\mathcal{L}[f(t)] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0); \quad (18)$$

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]; \quad (13)$$

$$\mathcal{L}[f(t)]\Big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]. \quad (14)$$

- Convolutions:

$$\mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)].$$

- Partial fraction decompositions, completing the squares.

Chapter 4: Laplace Transform methods.

Example

Use Laplace Transform to find y solution of

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: Compute the LT of the equation,

$$\mathcal{L}[y''] - 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[\delta(t-2)] = e^{-2s}$$

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0), \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0).$$

$$(s^2 - 2s + 2)\,\mathcal{L}[y] - s\,y(0) - y'(0) + 2\,y(0) = e^{-2s}$$

$$(s^2 - 2s + 2)\,\mathcal{L}[y] - s - 1 = e^{-2s}$$

$$\mathcal{L}[y] = \frac{(s+1)}{(s^2 - 2s + 2)} + \frac{1}{(s^2 - 2s + 2)}\,e^{-2s}.$$

Example

Use Laplace Transform to find y solution of

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s+1)}{(s^2-2s+2)} + \dfrac{1}{(s^2-2s+2)}\, e^{-2s}.$

$$s^2 - 2s + 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\big[2 \pm \sqrt{4-8}\big], \quad \text{complex roots.}$$

$$s^2 - 2s + 2 = (s^2 - 2s + 1) - 1 + 2 = (s-1)^2 + 1.$$

$$\mathcal{L}[y] = \frac{s+1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}\, e^{-2s}$$

$$\mathcal{L}[y] = \frac{(s-1+1)+1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}\, e^{-2s}$$

Example

Use Laplace Transform to find y solution of

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s-1)+2}{(s-1)^2+1} + \dfrac{1}{(s-1)^2+1}\, e^{-2s},$

$$\mathcal{L}[y] = \frac{(s-1)}{(s-1)^2+1} + 2\,\frac{1}{(s-1)^2+1} + e^{-2s}\,\frac{1}{(s-1)^2+1},$$

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2+a^2}, \qquad \mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2},$$

$$\mathcal{L}[y] = \mathcal{L}[\cos(t)]\big|_{(s-1)} + 2\,\mathcal{L}[\sin(t)]\big|_{(s-1)} + e^{-2s}\,\mathcal{L}[\sin(t)]\big|_{(s-1)}.$$

Chapter 4: Laplace Transform methods.

Example

Use Laplace Transform to find y solution of

$$y'' - 2y' + 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: Recall:

$$\mathcal{L}[y] = \mathcal{L}[\cos(t)]\big|_{(s-1)} + 2\,\mathcal{L}[\sin(t)]\big|_{(s-1)} + e^{-2s}\,\mathcal{L}[\sin(t)]\big|_{(s-1)}$$

and $\mathcal{L}[f(t)]\big|_{(s-c)} = \mathcal{L}[e^{ct}\,f(t)]$. Therefore,

$$\mathcal{L}[y] = \mathcal{L}[e^t\,\cos(t)] + 2\,\mathcal{L}[e^t\,\sin(t)] + e^{-2s}\,\mathcal{L}[e^t\,\sin(t)].$$

Also recall: $e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u_c(t)\,f(t-c)]$. Therefore,

$$\mathcal{L}[y] = \mathcal{L}[e^t\,\cos(t)] + 2\,\mathcal{L}[e^t\,\sin(t)] + \mathcal{L}[u_2(t)\,e^{(t-2)}\,\sin(t-2)].$$

$$y(t) = \big[\cos(t) + 2\sin(t)\big]\,e^t + u_2(t)\,\sin(t-2)\,e^{(t-2)}. \qquad \triangleleft$$

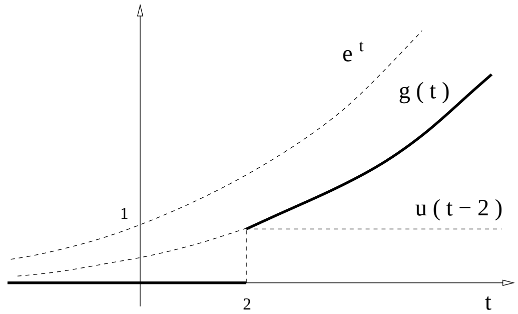
Chapter 4: Laplace Transform methods.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution:



Express g using step functions,

$$g(t) = u_2(t)\,e^{(t-2)}.$$

$$\mathcal{L}[u_c(t)\,f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)].$$

Therefore,

$$\mathcal{L}[g(t)] = e^{-2s}\mathcal{L}[e^t].$$

We obtain: $\mathcal{L}[g(t)] = \dfrac{e^{-2s}}{(s-1)}.$

Chapter 4: Laplace Transform methods.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution: Recall: $\mathcal{L}[g(t)] = \dfrac{e^{-2s}}{(s-1)}$.

$$\mathcal{L}[y''] + 3\,\mathcal{L}[y] = \mathcal{L}[g(t)] = \frac{e^{-2s}}{(s-1)}.$$

$$(s^2+3)\,\mathcal{L}[y] = \frac{e^{-2s}}{(s-1)} \quad \Rightarrow \quad \mathcal{L}[y] = e^{-2s}\,\frac{1}{(s-1)(s^2+3)}.$$

$$H(s) = \frac{1}{(s-1)(s^2+3)} = \frac{a}{(s-1)} + \frac{(bs+c)}{(s^2+3)}$$

$$1 = a(s^2+3) + (bs+c)(s-1)$$

Chapter 4: Laplace Transform methods.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution: Recall: $1 = a(s^2+3) + (bs+c)(s-1)$.

$$1 = as^2 + 3a + bs^2 + cs - bs - c$$

$$1 = (a+b)\,s^2 + (c-b)\,s + (3a-c)$$

$$a + b = 0, \quad c - b = 0, \quad 3a - c = 1.$$

$$a = -b, \quad c = b, \quad -3b - b = 1 \quad \Rightarrow \quad b = -\frac{1}{4},\ a = \frac{1}{4},\ c = -\frac{1}{4}.$$

$$H(s) = \frac{1}{4}\Big[\frac{1}{s-1} - \frac{s+1}{s^2+3}\Big].$$

Chapter 4: Laplace Transform methods.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution: Recall: $H(s) = \frac{1}{4}\Big[\frac{1}{s-1} - \frac{s+1}{s^2+3}\Big]$, $\mathcal{L}[y] = e^{-2s} H(s)$.

$$H(s) = \frac{1}{4}\Big[\frac{1}{s-1} - \frac{s}{s^2+3} - \frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{s^2+3}\Big],$$

$$H(s) = \frac{1}{4}\Big[\mathcal{L}[e^t] - \mathcal{L}\big[\cos\big(\sqrt{3}\,t\big)\big] - \frac{1}{\sqrt{3}}\,\mathcal{L}\big[\sin\big(\sqrt{3}\,t\big)\big]\Big].$$

$$H(s) = \mathcal{L}\Big[\frac{1}{4}\Big(e^t - \cos\big(\sqrt{3}\,t\big) - \frac{1}{\sqrt{3}}\,\sin\big(\sqrt{3}\,t\big)\Big)\Big].$$

Chapter 4: Laplace Transform methods.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' + 3y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < 2, \\ e^{(t-2)}, & t \geqslant 2. \end{cases}$$

Solution: Recall: $H(s) = \mathcal{L}\Big[\frac{1}{4}\Big(e^t - \cos\big(\sqrt{3}\,t\big) - \frac{1}{\sqrt{3}}\,\sin\big(\sqrt{3}\,t\big)\Big)\Big]$.

$$h(t) = \frac{1}{4}\Big(e^t - \cos\big(\sqrt{3}\,t\big) - \frac{1}{\sqrt{3}}\,\sin\big(\sqrt{3}\,t\big)\Big), \quad H(s) = \mathcal{L}[h(t)].$$

$$\mathcal{L}[y] = e^{-2s}\,H(s) = e^{-2s}\,\mathcal{L}[h(t)] = \mathcal{L}[u_2(t)\,h(t-2)].$$

We conclude: $y(t) = u_2(t)\,h(t-2)$. Equivalently,

$$y(t) = \frac{u_2(t)}{4}\,\Big[e^{(t-2)} - \cos\big(\sqrt{3}\,(t-2)\big) - \frac{1}{\sqrt{3}}\,\sin\big(\sqrt{3}\,(t-2)\big)\Big]. \quad \triangleleft$$

Example

Use convolutions to find f satisfying $\mathcal{L}[f(t)] = \dfrac{e^{-2s}}{(s-1)(s^2+3)}$.

Solution: One way to solve this is with the splitting

$$\mathcal{L}[f(t)] = e^{-2s}\,\frac{1}{(s^2+3)}\,\frac{1}{(s-1)} = e^{-2s}\,\frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{(s^2+3)}\,\frac{1}{(s-1)},$$

$$\mathcal{L}[f(t)] = e^{-2s}\,\frac{1}{\sqrt{3}}\,\mathcal{L}[\sin(\sqrt{3}\,t)]\,\mathcal{L}[e^t]$$

$$\mathcal{L}[f(t)] = \frac{1}{\sqrt{3}}\mathcal{L}[u_2(t)\,\sin(\sqrt{3}\,(t-2))]\,\mathcal{L}[e^t].$$

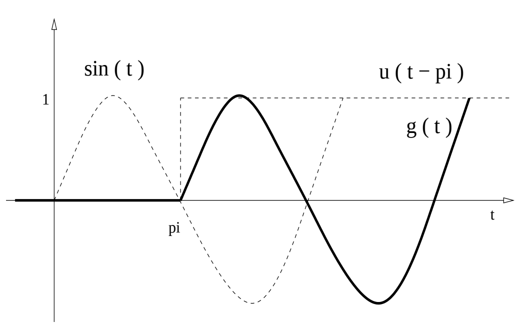
$$f(t) = \frac{1}{\sqrt{3}}\int_0^t u_2(\tau)\,\sin(\sqrt{3}\,(\tau-2))\,e^{(t-\tau)}\,d\tau. \qquad \triangleleft$$

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t-\pi), & t \geqslant \pi. \end{cases}$$

Solution:



Express g using step functions,

$$g(t) = u_\pi(t)\,\sin(t-\pi).$$

$$\mathcal{L}[u_c(t)\,f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)].$$

Therefore,

$$\mathcal{L}[g(t)] = e^{-\pi s}\mathcal{L}[\sin(t)].$$

We obtain: $\mathcal{L}[g(t)] = \dfrac{e^{-\pi s}}{s^2+1}$.

Chapter 4: Laplace Transform methods.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t-\pi), & t \geqslant \pi. \end{cases}$$

Solution: $\mathcal{L}[g(t)] = \dfrac{e^{-\pi s}}{s^2+1}$.

$$\mathcal{L}[y''] - 6\,\mathcal{L}[y] = \mathcal{L}[g(t)] = \frac{e^{-\pi s}}{s^2+1}.$$

$$(s^2-6)\,\mathcal{L}[y] = \frac{e^{-\pi s}}{s^2+1} \quad \Rightarrow \quad \mathcal{L}[y] = e^{-\pi s}\,\frac{1}{(s^2+1)(s^2-6)}.$$

$$H(s) = \frac{1}{(s^2+1)(s^2-6)} = \frac{1}{(s^2+1)(s+\sqrt{6})(s-\sqrt{6})}$$

$$H(s) = \frac{a}{(s+\sqrt{6})} + \frac{b}{(s-\sqrt{6})} + \frac{(cs+d)}{(s^2+1)}.$$

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t-\pi), & t \geqslant \pi. \end{cases}$$

Solution: $H(s) = \dfrac{a}{(s+\sqrt{6})} + \dfrac{b}{(s-\sqrt{6})} + \dfrac{(cs+d)}{(s^2+1)}$.

$$\frac{1}{(s^2+1)(s+\sqrt{6})(s-\sqrt{6})} = \frac{a}{(s+\sqrt{6})} + \frac{b}{(s-\sqrt{6})} + \frac{(cs+d)}{(s^2+1)}$$

$$1 = a(s-\sqrt{6})(s^2+1) + b(s+\sqrt{6})(s^2+1) + (cs+d)(s^2-6).$$

The solution is: $a = -\dfrac{1}{14\sqrt{6}}, \quad b = \dfrac{1}{14\sqrt{6}}, \quad c = 0, \quad d = -\dfrac{1}{7}$.

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geqslant \pi. \end{cases}$$

Solution: $H(s) = \dfrac{1}{14\sqrt{6}} \Big[-\dfrac{1}{(s + \sqrt{6})} + \dfrac{1}{(s - \sqrt{6})} - \dfrac{2\sqrt{6}}{(s^2 + 1)} \Big].$

$$H(s) = \frac{1}{14\sqrt{6}} \Big[-\mathcal{L}\big[e^{-\sqrt{6}\,t}\big] + \mathcal{L}\big[e^{\sqrt{6}\,t}\big] - 2\sqrt{6}\,\mathcal{L}[\sin(t)] \Big]$$

$$H(s) = \mathcal{L}\Big[\frac{1}{14\sqrt{6}} \Big(-e^{-\sqrt{6}\,t} + e^{\sqrt{6}\,t} - 2\sqrt{6}\,\sin(t) \Big) \Big].$$

$$h(t) = \frac{1}{14\sqrt{6}} \Big[-e^{-\sqrt{6}\,t} + e^{\sqrt{6}\,t} - 2\sqrt{6}\,\sin(t) \Big] \;\Rightarrow\; H(s) = \mathcal{L}[h(t)].$$

Example

Sketch the graph of g and use LT to find y solution of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0, \quad g(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geqslant \pi. \end{cases}$$

Solution: Recall: $\mathcal{L}[y] = e^{-\pi s}\,H(s)$, where $H(s) = \mathcal{L}[h(t)]$, and

$$h(t) = \frac{1}{14\sqrt{6}} \Big[-e^{-\sqrt{6}\,t} + e^{\sqrt{6}\,t} - 2\sqrt{6}\,\sin(t) \Big].$$

$$\mathcal{L}[y] = e^{-\pi s}\,\mathcal{L}[h(t)] = \mathcal{L}[u_\pi(t)\,h(t - \pi)] \;\Rightarrow\; y(t) = u_\pi(t)\,h(t - \pi).$$

Equivalently:

$$y(t) = \frac{u_\pi(t)}{14\sqrt{6}} \Big[-e^{-\sqrt{6}\,(t-\pi)} + e^{\sqrt{6}\,(t-\pi)} - 2\sqrt{6}\,\sin(t - \pi) \Big]. \quad \triangleleft$$

Review for Exam 3.

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- **Chapter 5: Systems of linear equations.**
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Second order equations and first order systems.

Theorem (Reduction to first order)

Every solution y to the second order linear equation

$$y'' + p(t)\,y' + q(t)\,y = g(t), \tag{1}$$

defines a solution $x_1 = y$ and $x_2 = y'$ of the 2×2 first order linear differential system

$$x_1' = x_2, \tag{2}$$

$$x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t). \tag{3}$$

Conversely, every solution x_1, x_2 of the 2×2 first order linear system in Eqs. (2)-(3) defines a solution $y = x_1$ of the second order differential equation in (1).

Second order equations and first order systems.

Remark: Systems of first order equations can, sometimes, be transformed into a second order single equation.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Compute x_1 from the second equation: $x_1 = x_2' + x_2$. Introduce this expression into the first equation,

$$(x_2' + x_2)' = -(x_2' + x_2) + 3x_2,$$

$$x_2'' + x_2' = -x_2' - x_2 + 3x_2,$$

$$x_2'' + 2x_2' - 2x_2 = 0.$$

Second order equations and first order systems.

Example

Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Recall: $x_2'' + 2x_2' - 2x_2 = 0.$

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4+8}\right] \quad \Rightarrow \quad r_\pm = -1 \pm \sqrt{3}.$$

Therefore, $x_2 = c_1\, e^{r_+ t} + c_2\, e^{r_- t}$. Since $x_1 = x_2' + x_2$,

$$x_1 = \left(c_1 r_+\, e^{r_+ t} + c_2 r_-\, e^{r_- t}\right) + \left(c_1\, e^{r_+ t} + c_2\, e^{r_- t}\right),$$

We conclude: $x_1 = c_1(1 + r_+)\, e^{r_+ t} + c_2(1 + r_-)\, e^{r_- t}.$ $\lhd$

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(a) If $\lambda_1 \neq \lambda_2$, real, then $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, and the general solution is $\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda_2 t}$.

(b) If $\lambda_1 \neq \lambda_2$, complex, then denoting $\lambda_\pm = \alpha \pm \beta i$ and $\mathbf{v}^{(\pm)} = \mathbf{a} \pm \mathbf{b}i$, the complex-valued fundamental solutions

$$\mathbf{x}^{(\pm)} = (\mathbf{a} \pm \mathbf{b}i)\,e^{(\alpha \pm \beta i)t}$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,(\mathbf{a} \pm \mathbf{b}i)\,\big[\cos(\beta t) + i\sin(\beta t)\big].$$

$$\mathbf{x}^{(\pm)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big] \pm ie^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Real-valued fundamental solutions are

$$\mathbf{x}^{(1)} = e^{\alpha t}\,\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big],$$

$$\mathbf{x}^{(2)} = e^{\alpha t}\,\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Systems of linear Equations.

Summary: Find solutions of $\mathbf{x}' = A\,\mathbf{x}$, with A a 2×2 matrix.

First find the eigenvalues λ_i and the eigenvectors $\mathbf{v}^{(i)}$ of A.

(c) If $\lambda_1 = \lambda_2 = \lambda$, real, and their eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ are linearly independent, then the general solution is

$$\mathbf{x}(x) = c_1\,\mathbf{v}^{(1)}\,e^{\lambda t} + c_2\,\mathbf{v}^{(2)}\,e^{\lambda t}.$$

(d) If $\lambda_1 = \lambda_2 = \lambda$, real, and there is only one eigendirection $\mathbf{v}$, then find $\mathbf{w}$ solution of $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then fundamental solutions to the differential equation are given by

$$\mathbf{x}^{(1)} = \mathbf{v}\,e^{\lambda t}, \quad \mathbf{x}^{(2)} = (\mathbf{v}\,t + \mathbf{w})\,e^{\lambda t}.$$

Then, the general solution is

$$\mathbf{x} = c_1\,\mathbf{v}\,e^{\lambda t} + c_2\,(\mathbf{v}\,t + \mathbf{w})\,e^{\lambda t}.$$

Exam: November 12, 2008. Problem 4.

Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Eigenvalues of A:

$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & \sqrt{2} \\ \sqrt{2} & (-2-\lambda) \end{vmatrix} = (\lambda+2)(\lambda+3) - 2 = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{1}{2}\left[-5 \pm \sqrt{25-16}\right] = \frac{1}{2}\left[-5 \pm 3\right]$$

Hence $\lambda_+ = -1$, $\lambda_- = -4$. Eigenvector for λ_+.

$$(A + I) = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \to \begin{bmatrix} 2 & -\sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 2 & -\sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

$2v_1 = \sqrt{2}\,v_2$. Choosing $v_1 = \sqrt{2}$ and $v_2 = 2$, we get $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.

Exam: November 12, 2008. Problem 4.

Example

Find the general solution of $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$.

Solution: Recall: $\lambda_+ = -1$, $\lambda_- = -4$, and $\mathbf{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$.
Eigenvector for λ_-.

$$(A + 4I) = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \to \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

$v_1 = -\sqrt{2}\,v_2$. Choosing $v_1 = -\sqrt{2}$ and $v_2 = 1$, so, $\mathbf{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Fundamental solutions: $\mathbf{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}$, $\mathbf{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}$.

General solution: $\mathbf{x} = c_1 \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}$. $\triangleleft$

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Eigenvalues of A:

$$p(\lambda) = \begin{vmatrix} (-3-\lambda) & 4 \\ -1 & (1-\lambda) \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_\pm = \frac{1}{2}\big[-2 \pm \sqrt{4-4}\big] = -1.$$

Hence $\lambda_+ = \lambda_- = -1$. Eigenvector for $\lambda_\pm$.

$$(A + I) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

$v_1 = 2\,v_2$. Choosing $v_1 = 2$ and $v_2 = 1$, we get $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_\pm = -1$, and $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Find $\mathbf{w}$ solution of $(A + I)\mathbf{w} = \mathbf{v}$.

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $w_1 = 2w_2 - 1$, that is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Choose $w_2 = 0$, so $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\lambda_{\pm} = -1$, $\mathbf{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Fundamental sol: $\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$, $\mathbf{x}^{(2)} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

General sol: $\mathbf{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

Extra problem.

Example

Find $\mathbf{x}$ solution of the IVP

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}.$$

Solution: Recall: $\mathbf{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$.

Initial condition: $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$,

that is, $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, also, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

The solution is $\mathbf{x} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + 5 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}$. ◁

Review for Exam 3.

- Chapter 4: Laplace Transform methods.
  - Definition of Laplace Transform (4.1).
  - Solving IVP using LT (4.2).
  - Solving IVP with discontinuous sources using LT, (4.3).
  - Solving IVP with generalized sources using LT (4.4).
  - Convolutions and LT (4.5).
- Chapter 5: Systems of linear equations.
  - Systems of linear Differential Equations (5.1).
  - 2×2 systems (actual 5.7, 5.8, 5.9).
- **BVP, eigenfunction problems, (6.1).**

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y(8) = 0.$$

Solution: Since $\lambda > 0$, introduce $\lambda = \mu^2$, with $\mu > 0$.

$y(x) = e^{rx}$ implies that r is solution of

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(8) = c_2 \sin(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0.$$

$$\mu = \frac{n\pi}{8}, \quad \lambda = \left(\frac{n\pi}{8}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{8}\right), \quad n = 1, 2, \cdots \triangleleft$$

Eigenvalue-Eigenfunction BVP.

Example

Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda\, y = 0, \quad y(0) = 0, \quad y'(8) = 0.$$

Solution: The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions imply:

$$0 = y(0) = c_1 \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y'(8) = c_2 \mu \cos(\mu 8), \quad c_2 \neq 0 \quad \Rightarrow \quad \cos(\mu 8) = 0.$$

$$8\mu = (2n+1)\frac{\pi}{2}, \quad \Rightarrow \quad \mu = \frac{(2n+1)\pi}{16}.$$

Then, for $n = 1, 2, \cdots$ holds

$$\lambda = \Big[\frac{(2n+1)\pi}{16}\Big]^2, \quad y_n(x) = \sin\Big(\frac{(2n+1)\pi x}{16}\Big). \qquad \triangleleft$$