Overview of Fourier Series (Sect. 6.2).

- Periodic functions.
- Orthogonality of Sines and Cosines.
- The Fourier Theorem: Continuous case.
- Example: Using the Fourier Theorem.
- The Fourier Theorem: Piecewise continuous case.
- Example: Using the Fourier Theorem.

Periodic functions.

Definition
A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is called \textit{periodic} iff there exists \( \tau > 0 \) such that for all \( x \in \mathbb{R} \) holds

\[
f(x + \tau) = f(x).
\]

Remark: \( f \) is invariant under translations by \( \tau \).

Definition
A \textit{period} \( T \) of a periodic function \( f \) is the smallest value of \( \tau \) such that \( f(x + \tau) = f(x) \) holds.

Notation:
A periodic function with period \( T \) is also called \( T \)-periodic.
Periodic functions.

Example
The following functions are periodic, with period $T$,
\[
    f(x) = \sin(x), \quad T = 2\pi.
\]
\[
    f(x) = \cos(x), \quad T = 2\pi.
\]
\[
    f(x) = \tan(x), \quad T = \pi.
\]
\[
    f(x) = \sin(ax), \quad T = \frac{2\pi}{a}.
\]

The proof of the latter statement is the following:
\[
    f\left(x + \frac{2\pi}{a}\right) = \sin\left(ax + a\frac{2\pi}{a}\right) = \sin(ax + 2\pi) = \sin(ax) = f(x).
\]

Periodic functions.

Example
Show that the function below is periodic, and find its period,
\[
    f(x) = e^x, \quad x \in [0, 2), \quad f(x - 2) = f(x).
\]

Solution: We just graph the function,
\[
    \text{So the function is periodic with period } T = 2.
\]
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Orthogonality of Sines and Cosines.

**Remark:**
From now on we work on the following domain: $[-L, L]$. 

\[ T = 2L \]

\[
\cos \left( \frac{\pi x}{L} \right) \quad \sin \left( \frac{\pi x}{L} \right)
\]
Orthogonality of Sines and Cosines.

Theorem (Orthogonality)

The following relations hold for all \( n, m \in \mathbb{N} \),

\[
\int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) \cos \left( \frac{m\pi x}{L} \right) \, dx = \begin{cases} 
0 & n \neq m, \\
L & n = m \neq 0, \\
2L & n = m = 0,
\end{cases}
\]

\[
\int_{-L}^{L} \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{m\pi x}{L} \right) \, dx = \begin{cases} 
0 & n \neq m, \\
L & n = m,
\end{cases}
\]

\[
\int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) \sin \left( \frac{m\pi x}{L} \right) \, dx = 0.
\]

Remark:

- The operation \( f \cdot g = \int_{-L}^{L} f(x)g(x) \, dx \) is an inner product in the vector space of functions. Like the dot product is in \( \mathbb{R}^2 \).
- Two functions \( f, g \), are orthogonal iff \( f \cdot g = 0 \).

Orthogonality of Sines and Cosines.

Recall:

\[
\cos(\theta) \cos(\phi) = \frac{1}{2} \left[ \cos(\theta + \phi) + \cos(\theta - \phi) \right];
\]

\[
\sin(\theta) \sin(\phi) = \frac{1}{2} \left[ \cos(\theta - \phi) - \cos(\theta + \phi) \right];
\]

\[
\sin(\theta) \cos(\phi) = \frac{1}{2} \left[ \sin(\theta + \phi) + \sin(\theta - \phi) \right].
\]

Proof: First formula: If \( n = m = 0 \), it is simple to see that

\[
\int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) \cos \left( \frac{m\pi x}{L} \right) \, dx = \int_{-L}^{L} \, dx = 2L.
\]

In the case where one of \( n \) or \( m \) is non-zero, use the relation

\[
\int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) \cos \left( \frac{m\pi x}{L} \right) \, dx = \frac{1}{2} \int_{-L}^{L} \cos \left( \frac{(n + m)\pi x}{L} \right) \, dx
\]

\[
+ \frac{1}{2} \int_{-L}^{L} \cos \left( \frac{(n - m)\pi x}{L} \right) \, dx.
\]
Orthogonality of Sines and Cosines.

Proof: Since one of $n$ or $m$ is non-zero, holds

$$\frac{1}{2} \int_{-L}^{L} \cos\left(\frac{(n+m)\pi x}{L}\right) dx = \frac{L}{2(n+m)\pi} \sin\left[\frac{(n+m)\pi x}{L}\right] \bigg|_{-L}^{L} = 0.$$ 

We obtain that

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \frac{1}{2} \int_{-L}^{L} \cos\left[\frac{(n-m)\pi x}{L}\right] dx.$$ 

If we further restrict $n \neq m$, then

$$\frac{1}{2} \int_{-L}^{L} \cos\left[\frac{(n-m)\pi x}{L}\right] dx = \frac{L}{2(n-m)\pi} \sin\left[\frac{(n-m)\pi x}{L}\right] \bigg|_{-L}^{L} = 0.$$ 

If $n = m \neq 0$, we have that

$$\frac{1}{2} \int_{-L}^{L} \cos\left[\frac{(n-m)\pi x}{L}\right] dx = \frac{1}{2} \int_{-L}^{L} dx = L.$$ 

This establishes the first equation in the Theorem. The remaining equations are proven in a similar way. \qed

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- Example: Using the Fourier Theorem.
The Fourier Theorem: Continuous case.

Theorem (Fourier Series)

If the function \( f : [-L, L] \subset \mathbb{R} \rightarrow \mathbb{R} \) is continuous, then \( f \) can be expressed as an infinite series

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]
\]

(1)

with the constants \( a_n \) and \( b_n \) given by

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 0,
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 1.
\]

Furthermore, the Fourier series in Eq. (1) provides a 2L-periodic extension of function \( f \) from the domain \([-L, L] \subset \mathbb{R} \) to \( \mathbb{R} \).

The Fourier Theorem: Continuous case.

Sketch of the Proof:

- Define the partial sum functions

\[
f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]
\]

with \( a_n \) and \( b_n \) given by

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 0,
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 1.
\]

- Express \( f_N \) as a convolution of Sine, Cosine, functions and the original function \( f \).

- Use the convolution properties to show that

\[
\lim_{N \to \infty} f_N(x) = f(x), \quad x \in [-L, L].
\]
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Example: Using the Fourier Theorem.

**Example**

Find the Fourier series expansion of the function

\[
f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases}
\]

**Solution:** In this case \( L = 1 \). The Fourier series expansion is

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(n\pi x) + b_n \sin(n\pi x) \right],
\]

where the \( a_n, b_n \) are given in the Theorem. We start with \( a_0 \),

\[
a_0 = \int_{-1}^{1} f(x) \, dx = \int_{-1}^{0} (1 + x) \, dx + \int_{0}^{1} (1 - x) \, dx.
\]

\[
a_0 = \left( x + \frac{x^2}{2} \right) \bigg|_{-1}^{0} + \left( x - \frac{x^2}{2} \right) \bigg|_{0}^{1} = \left( 1 - \frac{1}{2} \right) + (1 - \frac{1}{2})
\]

We obtain: \( a_0 = 1 \).
Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: Recall: \( a_0 = 1 \). Similarly, the rest of the \( a_n \) are given by,

\[
a_n = \int_{-1}^{1} f(x) \cos(n\pi x) \, dx \]

\[
a_n = \int_{-1}^{0} (1 + x) \cos(n\pi x) \, dx + \int_{0}^{1} (1 - x) \cos(n\pi x) \, dx.
\]

Recall the integrals \( \int \cos(n\pi x) \, dx = \frac{1}{n\pi} \sin(n\pi x) \), and

\[
\int x \cos(n\pi x) \, dx = \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x).
\]

Example: Using the Fourier Theorem.

Example

Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: It is not difficult to see that

\[
a_n = \frac{1}{n\pi} \sin(n\pi x) \bigg|_{-1}^{0} + \left[ \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x) \right] \bigg|_{0}^{1} \]

\[
+ \frac{1}{n\pi} \sin(n\pi x) \bigg|_{1}^{0} - \left[ \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x) \right] \bigg|_{0}^{1}
\]

\[
a_n = \left[ \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} \cos(-n\pi) \right] - \left[ \frac{1}{n^2\pi^2} \cos(n\pi) - \frac{1}{n^2\pi^2} \right].
\]

We then conclude that \( a_n = \frac{2}{n^2\pi^2} [1 - \cos(n\pi)] \).
Example: Using the Fourier Theorem.

Example
Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: Recall: \( a_0 = 1 \), and \( a_n = \frac{2}{n^2 \pi^2} \left[ 1 - \cos(n\pi) \right] \).

Finally, we must find the coefficients \( b_n \).

A similar calculation shows that \( b_n = 0 \).

Then, the Fourier series of \( f \) is given by

\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \left[ 1 - \cos(n\pi) \right] \cos(n\pi x). \]

\[ \triangle \]

Example: Using the Fourier Theorem.

Example
Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: Recall: \( f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \left[ 1 - \cos(n\pi) \right] \cos(n\pi x). \)

We can obtain a simpler expression for the Fourier coefficients \( a_n \).

Recall the relations \( \cos(n\pi) = (-1)^n \), then

\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \left[ 1 - (-1)^n \right] \cos(n\pi x). \]

\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \left[ 1 + (-1)^{n+1} \right] \cos(n\pi x). \]
Example: Using the Fourier Theorem.

Example
Find the Fourier series expansion of the function

\[ f(x) = \begin{cases} 
1 + x & x \in [-1, 0), \\
1 - x & x \in [0, 1]. 
\end{cases} \]

Solution: Recall:

\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \left[ 1 + (-1)^{n+1} \right] \cos(n \pi x). \]

If \( n = 2k \), so \( n \) is even, so \( n + 1 = 2k + 1 \) is odd, then

\[ a_{2k} = \frac{2}{(2k)^2 \pi^2} (1 - 1) \quad \Rightarrow \quad a_{2k} = 0. \]

If \( n = 2k - 1 \), so \( n \) is odd, so \( n + 1 = 2k \) is even, then

\[ a_{2k-1} = \frac{2}{(2k-1)^2 \pi^2} (1 + 1) \quad \Rightarrow \quad a_{2k-1} = \frac{4}{(2k-1)^2 \pi^2}. \]

We conclude:

\[ f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2 \pi^2} \cos((2k - 1) \pi x). \]
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- **The Fourier Theorem: Piecewise continuous case.**
- Example: Using the Fourier Theorem.

Recall:

**Definition**

A function $f : [a, b] \rightarrow \mathbb{R}$ is called *piecewise continuous* iff holds,

(a) $[a, b]$ can be partitioned in a finite number of sub-intervals such that $f$ is continuous on the interior of these sub-intervals.

(b) $f$ has finite limits at the endpoints of all sub-intervals.
The Fourier Theorem: Piecewise continuous case.

Theorem (Fourier Series)
If \( f : [-L, L] \subset \mathbb{R} \to \mathbb{R} \) is piecewise continuous, then the function

\[
f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]
\]

where \( a_n \) and \( b_n \) given by

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 0,
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx, \quad n \geq 1.
\]

satisfies that:
(a) \( f_F(x) = f(x) \) for all \( x \) where \( f \) is continuous;
(b) \( f_F(x_0) = \frac{1}{2} \left[ \lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x) \right] \) for all \( x_0 \) where \( f \) is discontinuous.

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Example: Using the Fourier Theorem.

Example

Find the Fourier series of \( f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1) \end{cases} \)

and periodic with period \( T = 2 \).

Solution: We start computing the Fourier coefficients \( b_n \):

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx, \quad L = 1,
\]

\[
b_n = \int_{-1}^{0} (-1) \sin(n \pi x) \, dx + \int_{0}^{1} (1) \sin(n \pi x) \, dx,
\]

\[
b_n = \frac{(-1)}{n \pi} [-\cos(n \pi x)]_{-1}^{0} + \frac{1}{n \pi} [-\cos(n \pi x)]_{0}^{1},
\]

\[
b_n = \frac{(-1)}{n \pi} [-1 + \cos(-n \pi)] + \frac{1}{n \pi} [-\cos(n \pi) + 1].
\]

Example: Using the Fourier Theorem.

Example

Find the Fourier series of \( f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1) \end{cases} \)

and periodic with period \( T = 2 \).

Solution: \( b_n = \frac{(-1)}{n \pi} [-1 + \cos(-n \pi)] + \frac{1}{n \pi} [-\cos(n \pi) + 1] \).

\[
b_n = \frac{1}{n \pi} [1 - \cos(-n \pi) - \cos(n \pi) + 1] = \frac{2}{n \pi} [1 - \cos(n \pi)],
\]

We obtain: \( b_n = \frac{2}{n \pi} [1 - (-1)^n] \).

If \( n = 2k \), then \( b_{2k} = \frac{2}{2k \pi} [1 - (-1)^{2k}] \), hence \( b_{2k} = 0 \).

If \( n = 2k - 1 \), then \( b_{2k-1} = \frac{2}{(2k-1) \pi} [1 - (-1)^{2k-1}] \),

hence \( b_{2k} = \frac{4}{(2k-1) \pi} \).
Example: Using the Fourier Theorem.

Example

Find the Fourier series of \( f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1) \end{cases} \)
and periodic with period \( T = 2 \).

Solution: Recall: \( b_{2k} = 0 \), and \( b_{2k} = \frac{4}{(2k-1)\pi} \).

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx, \quad L = 1,
\]
\[
a_n = \int_{-1}^{0} (-1) \cos(n\pi x) \, dx + \int_{0}^{1} (1) \cos(n\pi x) \, dx,
\]
\[
a_n = \frac{(-1)}{n\pi} \left[ \sin(n\pi x) \right]_{-1}^{0} + \frac{1}{n\pi} \left[ \sin(n\pi x) \right]_{0}^{1},
\]
\[
a_n = \frac{(-1)}{n\pi} [0 - \sin(-n\pi)] + \frac{1}{n\pi} [\sin(n\pi) - 0] \quad \Rightarrow \quad a_n = 0.
\]

Example: Using the Fourier Theorem.

Example

Find the Fourier series of \( f(x) = \begin{cases} -1 & x \in [-1, 0), \\ 1 & x \in [0, 1) \end{cases} \)
and periodic with period \( T = 2 \).

Solution: Recall: \( b_{2k} = 0 \), \( b_{2k} = \frac{4}{(2k-1)\pi} \), and \( a_n = 0 \).

Therefore, we conclude that

\[
f_{F}(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin((2k-1)\pi x).
\]

\( \triangle \)