Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: Eigenvalue-eigenfunction problem.

Two-point Boundary Value Problem.

Definition
A two-point BVP is the following: Given functions \( p, q, g \), and constants 
\[ x_1 < x_2, \quad y_1, y_2, \quad b_1, b_2, \quad \tilde{b}_1, \tilde{b}_2, \]
find a function \( y \) solution of the differential equation
\[ y'' + p(x) y' + q(x) y = g(x), \]
and the extra, boundary conditions,
\[ b_1 y(x_1) + b_2 y'(x_1) = y_1, \]
\[ \tilde{b}_1 y(x_2) + \tilde{b}_2 y'(x_2) = y_2. \]

Remarks:
- Both \( y \) and \( y' \) might appear in the boundary condition, evaluated at the same point.
- In this notes we only study the case of constant coefficients,
\[ y'' + a_1 y' + a_0 y = g(x). \]
Two-point Boundary Value Problem.

Example
Examples of BVP. Assume $x_1 \neq x_2$.
(1) Find $y$ solution of

$$y'' + a_1 y' + a_0 y = g(x), \quad y(x_1) = y_1, \quad y(x_2) = y_2.$$ 

(2) Find $y$ solution of

$$y'' + a_1 y' + a_0 y = g(x), \quad y'(x_1) = y_1, \quad y'(x_2) = y_2.$$ 

(3) Find $y$ solution of

$$y'' + a_1 y' + a_0 y = g(x), \quad y(x_1) = y_1, \quad y'(x_2) = y_2.$$ 

Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: Eigenvalue-eigenfunction problem.
Example from physics.

Problem: The equilibrium (time independent) temperature of a bar of length $L$ with insulated horizontal sides and the bar vertical extremes kept at fixed temperatures $T_0$, $T_L$ is the solution of the BVP:

$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$

Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- **Comparison: IVP vs BVP.**
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: Eigenvalue-eigenfunction problem.
Comparison: IVP vs BVP.

Review: IVP:
Find the function values $y(t)$ solutions of the differential equation
$$y'' + a_1 y' + a_0 y = g(t),$$
together with the initial conditions
$$y(t_0) = y_1, \quad y'(t_0) = y_2.$$

Remark: In physics:
- $y(t)$: Position at time $t$.
- Initial conditions: Position and velocity at the initial time $t_0$.

Comparison: IVP vs BVP.

Review: BVP:
Find the function values $y(x)$ solutions of the differential equation
$$y'' + a_1 y' + a_0 y = g(x),$$
together with the initial conditions
$$y(x_1) = y_1, \quad y(x_2) = y_2.$$

Remark: In physics:
- $y(x)$: A physical quantity (temperature) at a position $x$.
- Boundary conditions: Conditions at the boundary of the object under study, where $x_1 \neq x_2$. 
Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- **Existence, uniqueness of solutions to BVP.**
- Particular case of BVP: Eigenvalue-eigenfunction problem.

Existence, uniqueness of solutions to BVP.

**Review:** The initial value problem.

**Theorem (IVP)**

Consider the homogeneous initial value problem:

\[ y'' + a_1 y' + a_0 y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1, \]

and let \( r_\pm \) be the roots of the characteristic polynomial

\[ p(r) = r^2 + a_1 r + a_0. \]

If \( r_+ \neq r_- \), real or complex, then for every choice of \( y_0, y_1 \), there exists a unique solution \( y \) to the initial value problem above.

**Summary:** The IVP above always has a unique solution, no matter what \( y_0 \) and \( y_1 \) we choose.
Existence, uniqueness of solutions to BVP.

Theorem (BVP)
Consider the homogeneous boundary value problem:

\[ y'' + a_1 y' + a_0 y = 0, \quad y(0) = y_0, \quad y(L) = y_1, \]

and let \( r_\pm \) be the roots of the characteristic polynomial

\[ p(r) = r^2 + a_1 r + a_0. \]

(A) If \( r_+ \neq r_- \), real, then for every choice of \( L \neq 0 \) and \( y_0, y_1 \), there exists a unique solution \( y \) to the BVP above.

(B) If \( r_\pm = \alpha \pm i\beta \), with \( \beta \neq 0 \), and \( \alpha, \beta \in \mathbb{R} \), then the solutions to the BVP above belong to one of these possibilities:

1. There exists a unique solution.
2. There exists no solution.
3. There exist infinitely many solutions.

Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case \( r_+ \neq r_- \). The general solution is

\[ y(t) = c_1 e^{r_- t} + c_2 e^{r_+ t}, \quad c_1, c_2 \in \mathbb{R}. \]

The initial conditions determine \( c_1 \) and \( c_2 \) as follows:

\[ y_0 = y(t_0) = c_1 e^{r_- t_0} + c_2 e^{r_+ t_0} \]
\[ y_1 = y'(t_0) = c_1 r_- e^{r_- t_0} + c_2 r_+ e^{r_+ t_0} \]

Using matrix notation,

\[
\begin{bmatrix}
  e^{r_- t_0} & e^{r_+ t_0} \\
  r_- e^{r_- t_0} & r_+ e^{r_+ t_0}
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} = \begin{bmatrix}
  y_0 \\
  y_1
\end{bmatrix}.
\]

The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff the \( \det(Z) \neq 0 \), where

\[
Z = \begin{bmatrix}
  e^{r_- t_0} & e^{r_+ t_0} \\
  r_- e^{r_- t_0} & r_+ e^{r_+ t_0}
\end{bmatrix} \Rightarrow Z \begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} = \begin{bmatrix}
  y_0 \\
  y_1
\end{bmatrix}.
\]
Existence, uniqueness of solutions to BVP.

Proof of IVP:
Recall: \( Z = \begin{bmatrix} e^{r_+ t_0} & e^{r_- t_0} \\ r_+ e^{r_- t_0} & r_- e^{r_+ t_0} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \).

A simple calculation shows
\[
\det(Z) = (r_+ - r_-) e^{(r_+ + r_-) t_0} \neq 0 \Leftrightarrow r_+ \neq r_-.
\]
Since \( r_+ \neq r_- \), the matrix \( Z \) is invertible and so

\[
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Z^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.
\]

We conclude that for every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the IVP above has a unique solution. \( \square \)

---

Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is
\[
y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}.
\]

The boundary conditions determine \( c_1 \) and \( c_2 \) as follows:
\[
y_0 = y(0) = c_1 + c_2.
\]
\[
y_1 = y(L) = c_1 e^{r_- L} + c_2 e^{r_+ L}.
\]

Using matrix notation,
\[
\begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.
\]

The linear system above has a unique solution \( c_1 \) and \( c_2 \) for every constants \( y_0 \) and \( y_1 \) iff the \( \det(Z) \neq 0 \), where
\[
Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.
\]
Existence, uniqueness of solutions to BVP.

**Proof of IVP:** Recall: 
\[ Z = \begin{bmatrix} 1 & 1 \\ e^{-r}L & e^{r}L \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \]

A simple calculation shows
\[ \det(Z) = e^{r}L - e^{-r}L \neq 0 \iff e^{r}L \neq e^{-r}L. \]

**(A)** If \( r_+ \neq r_- \) and real-valued, then \( \det(Z) \neq 0 \).

We conclude: For every choice of \( y_0 \) and \( y_1 \), there exist a unique value of \( c_1 \) and \( c_2 \), so the BVP in (A) above has a unique solution.

**(B)** If \( r_\pm = \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \), then
\[ \det(Z) = e^{\alpha L}(e^{i\beta L} - e^{-i\beta L}) \Rightarrow \det(Z) = 2ie^{\alpha L}\sin(\beta L). \]

Since \( \det(Z) = 0 \) iff \( \beta L = n\pi \), with \( n \) integer,

(1) If \( \beta L \neq n\pi \), then BVP has a unique solution.

(2) If \( \beta L = n\pi \) then BVP either has no solutions or it has infinitely many solutions. \( \square \)

---

**Example**

Find \( y \) solution of the BVP
\[ y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1. \]

**Solution:** The characteristic polynomial is
\[ p(r) = r^2 + 1 \Rightarrow r_\pm = \pm i. \]

The general solution is
\[ y(x) = c_1 \cos(x) + c_2 \sin(x). \]

The boundary conditions are
\[ 1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1 \Rightarrow c_1 = 1, \quad c_2 \text{ free}. \]

We conclude: \( y(x) = \cos(x) + c_2 \sin(x) \), with \( c_2 \in \mathbb{R} \).

The BVP has infinitely many solutions. \( \triangleleft \)
Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP
$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$  

Solution: The characteristic polynomial is
$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$  

The general solution is
$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$  

The boundary conditions are
$$1 = y(0) = c_1, \quad 0 = y(\pi) = -c_1$$  

The BVP has no solution.  

Existence, uniqueness of solutions to BVP.

Example
Find $y$ solution of the BVP
$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$  

Solution: The characteristic polynomial is
$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$  

The general solution is
$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$  

The boundary conditions are
$$1 = y(0) = c_1, \quad 0 = y(\pi/2) = -c_2 \quad \Rightarrow \quad c_1 = c_2 = 1.$$  

We conclude: $y(x) = \cos(x) + \sin(x)$.  

The BVP has a unique solution.
Boundary Value Problems (Sect. 6.1).

- Two-point BVP.
- Example from physics.
- Comparison: IVP vs BVP.
- Existence, uniqueness of solutions to BVP.
- Particular case of BVP: **Eigenvalue-eigenfunction problem.**

---

Particular case of BVP: Eigenvalue-eigenfunction problem.

**Problem:**
Find a number $\lambda$ and a non-zero function $y$ solutions to the boundary value problem

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

**Remark:** This problem is similar to the eigenvalue-eigenvector problem in Linear Algebra: Given an $n \times n$ matrix $A$, find $\lambda$ and a non-zero $n$-vector $v$ solutions of

$$Av - \lambda v = 0.$$

**Differences:**

- $A \rightarrow \{\text{computing a second derivative and}
  \text{applying the boundary conditions.}\}$
- $v \rightarrow \{\text{a function } y\}$. 
Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP
$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$ 

Remarks: We will show that:

1. If $\lambda \leq 0$, then the BVP has no solution.
2. If $\lambda > 0$, then there exist infinitely many eigenvalues $\lambda_n$ and eigenfunctions $y_n$, with $n$ any positive integer, given by
   $$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right),$$
3. Analogous results can be proven for the same equation but with different types of boundary conditions. For example, for $y(0) = 0$, $y'(L) = 0$; or for $y'(0) = 0$, $y'(L) = 0$.

Solution: Case $\lambda = 0$. The equation is
$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2x.$$ 

The boundary conditions imply
$$0 = y(0) = c_1, \quad 0 = c_1 + c_2L \quad \Rightarrow \quad c_1 = c_2 = 0.$$ 

Since $y = 0$, there are NO non-zero solutions for $\lambda = 0.$
Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$  

Solution: Case $\lambda < 0$. Introduce the notation $\lambda = -\mu^2$. The characteristic equation is

$$p(r) = r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu.$$

The general solution is

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$  

The boundary condition are

$$0 = y(0) = c_1 + c_2,$$

$$0 = y(L) = c_1 e^{\mu L} + c_2 e^{-\mu L}.$$  

Since $\det(Z) = e^{-\mu L} - e^{\mu L} \neq 0$ for $L \neq 0$, matrix $Z$ is invertible, so the linear system above has a unique solution $c_1 = 0$ and $c_2 = 0$. Since $y = 0$, there are NO non-zero solutions for $\lambda < 0$. 

Particular case of BVP: Eigenvalue-eigenfunction problem.

Example
Find every $\lambda \in \mathbb{R}$ and non-zero functions $y$ solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$ 

Solution: Case $\lambda > 0$. Introduce the notation $\lambda = \mu^2$. The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm \mu i.$$ 

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$ 

The boundary condition are

$$0 = y(0) = c_1, \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(L) = c_2 \sin(\mu L), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu L) = 0.$$ 

Recall: $c_1 = 0$, $c_2 \neq 0$, and $\sin(\mu L) = 0$.

The non-zero solution condition is the reason for $c_2 \neq 0$. Hence

$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L}.$$ 

Recalling that $\lambda_n = \mu_n^2$, and choosing $c_2 = 1$, we conclude

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right). \quad \triangleq$$