Real, repeated eigenvalues (Sect. 5.9)

- Review: Classification of 2×2 diagonalizable systems.
- Repeated eigenvalue diagonalizable 2×2 system.
- Repeated eigenvalue non-diagonalizable 2×2 system.
- Phase portraits for 2×2 systems.

Review: Classification of 2×2 systems.

Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1 = \overline{\mathbf{v}}_2$, (Section 5.8).

(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 5.9).

Remark:

(c-2) $\lambda_1 = \lambda_2$ real-valued with only one eigen-direction. Hence, A is not diagonalizable, (Section 5.9).

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Repeated eigenvalue diagonalizable 2×2 system.

Remark: For 2×2 systems the situation is fairly simple.

Theorem

Every 2×2 diagonalizable matrix A with repeated eigenvalue λ has the form $A = \lambda I$.

Proof: Since A is diagonalizable, exists P invertible such that

$$A = P \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} P^{-1} = P \lambda I P^{-1} = \lambda P \, P^{-1} = \lambda I.$$

□

Remark: The $\mathbf{x}$ general solution for $\mathbf{x}' = \lambda I \, \mathbf{x}$ is simple

$$\mathbf{x}(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t} \quad \Leftrightarrow \quad \mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\lambda t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{\lambda t}.$$

Remark: The solution phase portraits are always straight lines passing through the origin.

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Repeated eigenvalue non-diagonalizable 2×2 system.

Remark:
Diagonalizable 2×2 matrices A with real coefficients are classified according to their eigenvalues.

(a) $\lambda_1 \neq \lambda_2$, real-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ (eigen-directions), (Section 5.7).

(b) $\lambda_1 = \overline{\lambda}_2$, complex-valued. Hence, A has two non-proportional eigenvectors $\mathbf{v}_1 = \overline{\mathbf{v}}_2$, (Section 5.8).

(c-1) $\lambda_1 = \lambda_2$ real-valued with two non-proportional eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, (Section 5.9).

Remark:

(c-2) $\lambda_1 = \lambda_2$ real-valued with only one eigen-direction. Hence, A is not diagonalizable, (Section 5.9). Next Class.

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Show that matrix $B = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable.

Solution: We need to show that all eigenvectors of matrix B are proportional to each other. We start computing the eigenvalues,

$$p(\lambda) = \det(B - \lambda I) = \begin{vmatrix} \frac{3}{2} - \lambda & \frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} - \lambda \end{vmatrix} = \Big(\frac{3}{2} - \lambda\Big)\Big(\frac{5}{2} - \lambda\Big) + \frac{1}{4}$$

$$p(\lambda) = \lambda^2 - 4\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = 2.$$

We now compute the corresponding eigenvectors,

$$(B - 2I) = \; = \begin{bmatrix} \frac{3}{2} - 2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} - 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Hence all eigenvectors are proportional to $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ◁

Repeated eigenvalue non-diagonalizable 2×2 system.

Theorem (Repeated eigenvalue)

If λ is an eigenvalue of an $n \times n$ matrix A having algebraic multiplicity $r = 2$ and only one associated eigen-direction, then the differential equation

$$\mathbf{x}'(t) = A\,\mathbf{x}(t),$$

has a linearly independent set of solutions given by

$$\big\{\mathbf{x}^{(1)}(t) = \mathbf{v}\; e^{\lambda t}, \quad \mathbf{x}^{(2)}(t) = \big(\mathbf{v}\; t + \mathbf{w}\big)\, e^{\lambda t}\big\}.$$

where the vector $\mathbf{w}$ is solution of

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$

which always has a solution $\mathbf{w}$.

Repeated eigenvalue non-diagonalizable 2×2 system.

Recall: The case of a single second order equation

$$y'' + a_1 y' + a_0 y = 0$$

with characteristic polynomial

$$p(r) = r^2 + a_1 r + a_0 = (r - r_1)^2.$$

In this case a fundamental set of solutions is

$$\{y_1(t) = e^{r_1 t}, \quad y_2(t) = t\, e^{r_1 t}\}.$$

This is not the case with systems of first order linear equations,

$$\{\mathbf{x}^{(1)}(t) = \mathbf{v}\, e^{\lambda t}, \quad \mathbf{x}^{(2)}(t) = (\mathbf{v}\, t + \mathbf{w})\, e^{\lambda t}\}.$$

In general, $\mathbf{w} \neq \mathbf{0}$.

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Find fundamental solutions of $\mathbf{x}' = A\,\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Find the eigenvalues of A. Its characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} \left(-\frac{3}{2} - \lambda\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} - \lambda\right) \end{vmatrix} = \left(\lambda + \frac{3}{2}\right)\left(\lambda + \frac{1}{2}\right) + \frac{1}{4}.$$

So $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$. The roots and multiplicity are

$$\lambda = -1, \qquad r = 2.$$

The corresponding eigenvectors are the solutions of $(A + I)\mathbf{v} = \mathbf{0}$,

$$\begin{bmatrix} \left(-\frac{3}{2} + 1\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} + 1\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Find fundamental solutions of $\mathbf{x}' = A\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Recall: $\lambda = -1$, with $r = 2$, and $(A + I) \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

The eigenvector components satisfy: $v_1 = 2v_2$. We obtain,

$$\lambda = -1, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_2.$$

We conclude that this eigenvalue has only one eigen-direction.
Matrix A is not diagonalizable.
Theorem above says we need to find $\mathbf{w}$ solution of $(A + I)\mathbf{w} = \mathbf{v}$.

$$\left[\begin{array}{cc|c} -\frac{1}{2} & 1 & 2 \\ -\frac{1}{4} & \frac{1}{2} & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 1 & -2 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Find fundamental solutions of $\mathbf{x}' = A\mathbf{x}$, with $A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$.

Solution: Recall that:
$\lambda = -1$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_2$, and $(A + I)\mathbf{w} = \mathbf{v} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 0 & 0 \end{array}\right]$.

We obtain $w_1 = 2w_2 - 4$. That is, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

Given a solution $\mathbf{w}$, then $c\mathbf{v} + \mathbf{w}$ is also a solution, $c \in \mathbb{R}$.

We choose the simplest solution, $\mathbf{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$. We conclude,

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}, \qquad \mathbf{x}^{(2)}(t) = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) e^{-t}. \qquad \triangleleft$$

Repeated eigenvalue non-diagonalizable 2×2 system.

Example

Find the solution $\mathbf{x}$ to the IVP

$$\mathbf{x}' = A\,\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution: The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$$

The initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix}.$$

We conclude: $\mathbf{x}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{4}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$ $\lhd$

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Phase portraits for 2 × 2 systems.

Example

Sketch a phase portrait for solutions of

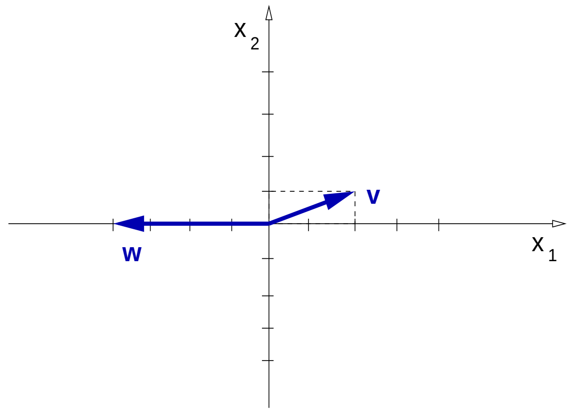
$$\mathbf{x}' = A\mathbf{x}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution:

We start plotting the vectors

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\mathbf{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$$



Phase portraits for 2 × 2 systems.

Example

Sketch a phase portrait for solutions of

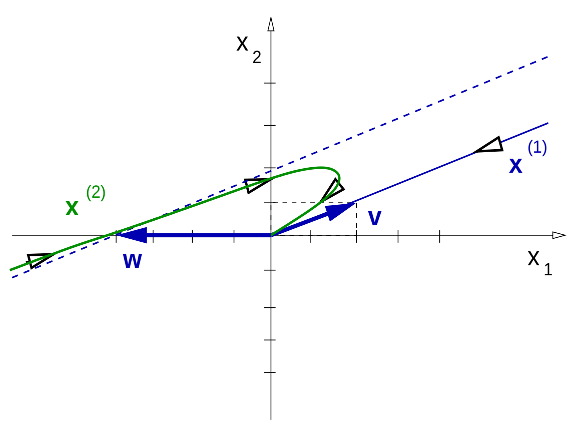
$$\mathbf{x}' = A\mathbf{x}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution:

Now plot the solutions

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$$

$$\mathbf{x}^{(2)} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}.$$



Phase portraits for 2 × 2 systems.

Example

Sketch a phase portrait for solutions of

$\mathbf{x}' = A\mathbf{x}, \quad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$

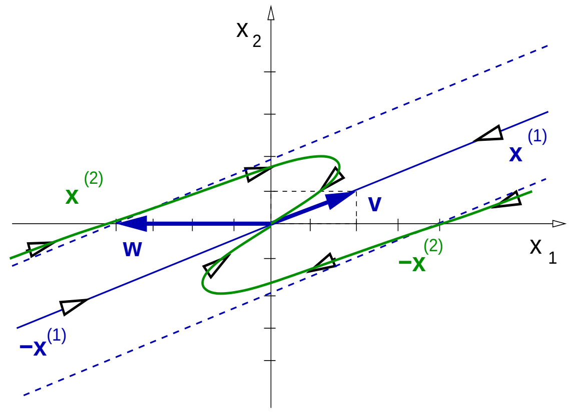
Solution:

Now plot the solutions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

$$\mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)},$$

This is the case $\lambda < 0$.



Phase portraits for 2 × 2 systems.

Example

Given any vectors **v** and **w**, and any constant λ, plot the phase portraits of the functions

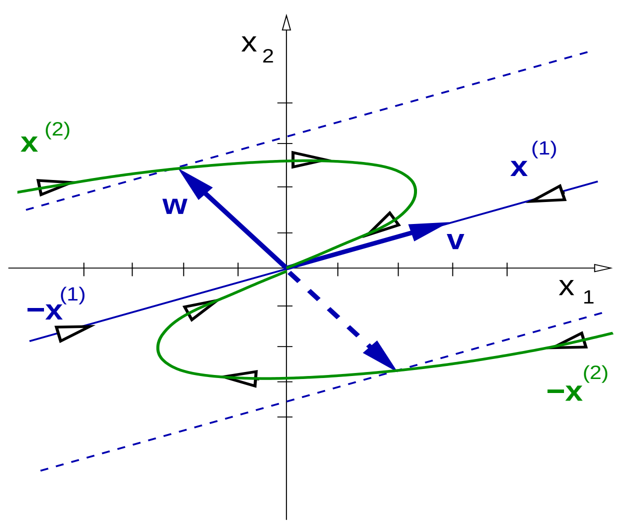
$$\mathbf{x}^{(1)}(t) = \mathbf{v}\, e^{\lambda t}, \qquad \mathbf{x}^{(2)}(t) = \big(\mathbf{v}\, t + \mathbf{w}\big)\, e^{\lambda t},$$

Solution:

The case $\lambda < 0$. We plot the functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

$$\mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)}.$$



Phase portraits for 2×2 systems.

Example

Given any vectors $\mathbf{v}$ and $\mathbf{w}$, and any constant λ, plot the phase portraits of the functions

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Solution:
The case $\lambda > 0$. We plot the functions

$$\mathbf{x}^{(1)}, \quad -\mathbf{x}^{(1)},$$

$$\mathbf{x}^{(2)}, \quad -\mathbf{x}^{(2)}.$$