Generalized sources (Sect. 4.4).

▶ The Dirac delta generalized function.
▶ Properties of Dirac's delta.
▶ Relation between deltas and steps.
▶ Dirac's delta in Physics.
▶ The Laplace Transform of Dirac's delta.
▶ Differential equations with Dirac's delta sources.
The Dirac delta generalized function.

Definition
Consider the sequence of functions for $n \geq 1$,

$$\delta_n(t) = \begin{cases} 
0, & t < 0 \\
n, & 0 \leq t \leq \frac{1}{n} \\
0, & t > \frac{1}{n}.
\end{cases}$$

The Dirac delta generalized function is given by

$$\lim_{n \to \infty} \delta_n(t) = \delta(t), \quad t \in \mathbb{R}. \quad \text{(Remarks:)}$$

(a) There exist infinitely many sequences $\delta_n$ that define the same generalized function $\delta$.

(b) For example, compare with the sequences $\delta_n$ in the literature.

The Dirac delta generalized function.

Remarks:
(a) The Dirac $\delta$ is a function on the domain $\mathbb{R} - \{0\}$, and $\delta(t) = 0$ for $t \in \mathbb{R} - \{0\}$.

(b) $\delta$ at $t = 0$ is not defined, since $\delta(0) = \lim_{n \to \infty} n = +\infty$.

(c) $\delta$ is not a function on $\mathbb{R}$. 
Generalized sources (Sect. 4.4).

- The Dirac delta generalized function.
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**Properties of Dirac’s delta.**

**Remark:** The Dirac $\delta$ is not a function on $\mathbb{R}$.

We define operations on Dirac’s $\delta$ as limits $n \to \infty$ of the operation on the sequence elements $\delta_n$.

**Definition**

\[
\delta(t - c) = \lim_{n \to \infty} \delta_n(t - c),
\]

\[
a \delta(t) + b \delta(t) = \lim_{n \to \infty} [a \delta_n(t) + b \delta_n(t)],
\]

\[
f(t) \delta(t) = \lim_{n \to \infty} [f(t) \delta_n(t)],
\]

\[
\int_a^b \delta(t) \, dt = \lim_{n \to \infty} \int_a^b \delta_n(t) \, dt,
\]

\[
\mathcal{L}[\delta] = \lim_{n \to \infty} \mathcal{L}[\delta_n].
\]
Properties of Dirac's delta.

Theorem

\[ \int_{-a}^{a} \delta(t) \, dt = 1, \quad a > 0. \]

Proof:

\[
\int_{-a}^{a} \delta(t) \, dt = \lim_{n \to \infty} \int_{-a}^{a} \delta_n(t) \, dt = \lim_{n \to \infty} \int_{0}^{1/n} n \, dt
\]

\[
\int_{-a}^{a} \delta(t) \, dt = \lim_{n \to \infty} \left[ n \left( t \bigg|_{0}^{1/n} \right) \right] = \lim_{n \to \infty} \left[ n \left( 1/n \right) \right].
\]

We conclude: \( \int_{-a}^{a} \delta(t) \, dt = 1. \)

Properties of Dirac’s delta.

Theorem

If \( f : \mathbb{R} \to \mathbb{R} \) is continuous, \( t_0 \in \mathbb{R} \) and \( a > 0 \), then

\[ \int_{t_0-a}^{t_0+a} \delta(t-t_0) f(t) \, dt = f(t_0). \]

Proof: Introduce the change of variable \( \tau = t - t_0 \),

\[
l = \int_{t_0-a}^{t_0+a} \delta(t-t_0) f(t) \, dt = \int_{-a}^{a} \delta(\tau) f(\tau + t_0) \, d\tau,
\]

\[
l = \lim_{n \to \infty} \int_{-a}^{a} \delta_n(\tau) f(\tau + t_0) \, d\tau = \lim_{n \to \infty} \int_{0}^{1/n} n f(\tau + t_0) \, d\tau
\]

Therefore, \( l = \lim_{n \to \infty} n \int_{0}^{1/n} F'(\tau + t_0) \, d\tau \), where we introduced the primitive \( F(t) = \int f(t) \, dt \), that is, \( f(t) = F'(t) \).
Properties of Dirac's delta.

**Theorem**

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) \, dt = f(t_0).$$

**Proof:** So, $I = \lim_{n \to \infty} n \int_0^{1/n} F'(\tau + t_0) \, d\tau$, with $f(t) = F'(t)$.

$$I = \lim_{n \to \infty} n \left[ F(\tau + t_0) \bigg|_0^{1/n} \right] = \lim_{n \to \infty} n \left[ F(t_0 + \frac{1}{n}) - F(t_0) \right].$$

$$I = \lim_{n \to \infty} \frac{F(t_0 + \frac{1}{n}) - F(t_0)}{\frac{1}{n}} = F'(t_0) = f(t_0).$$

We conclude: $\int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) \, dt = f(t_0)$. \qed

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Relation between deltas and steps.

**Theorem**

The sequence of functions for $n \geq 1$,

$$u_n(t) = \begin{cases} 
0, & t < 0 \\
nt, & 0 \leq t \leq \frac{1}{n} \\
1, & t > \frac{1}{n}.
\end{cases}$$

satisfies, for $t \in (-\infty, 0) \cup (0, 1/n) \cup (1/n, \infty)$, both equations,

$$u_n'(t) = \delta_n(t), \quad \lim_{n \to \infty} u_n(t) = u(t), \quad t \in \mathbb{R}.$$

**Remark:**

- If we generalize the notion of derivative as $u'(t) = \lim_{n \to \infty} u_n'(t)$, then holds $u'(t) = \delta(t)$.
- Dirac’s delta is a generalized derivative of the step function.

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**Generalized sources (Sect. 4.4).**

- The Dirac delta generalized function.
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Dirac’s delta in Physics.

Remarks:
(a) Dirac’s delta generalized function is useful to describe *impulsive forces* in mechanical systems.

(b) An impulsive force transmits a finite momentum in an infinitely short time.

(c) For example: The momentum transmitted to a pendulum when hit by a hammer. Newton’s law of motion says,

\[ m v'(t) = F(t), \quad \text{with} \quad F(t) = F_0 \delta(t - t_0). \]

The momentum transfer is:

\[
\Delta I = \lim_{\Delta t \to 0} mv(t)\bigg|_{t_0-\Delta t}^{t_0+\Delta t} = \lim_{\Delta t \to 0} \int_{t_0-\Delta t}^{t_0+\Delta t} F(t) \, dt = F_0.
\]

That is, \( \Delta I = F_0. \)

Generalized sources (Sect. 4.4).

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Recall: The Laplace Transform can be generalized to Dirac's delta function, as follows:

\[ \mathcal{L}[\delta(t - c)] = \lim_{n \to \infty} \mathcal{L}[\delta_n(t - c)]. \]

**Theorem**

\[ \mathcal{L}[\delta(t - c)] = e^{-cs}. \]

**Proof:**

\[ \mathcal{L}[\delta(t - c)] = \lim_{n \to \infty} \mathcal{L}[\delta_n(t - c)], \quad \delta_n(t) = n \left[ u(t) - u(t - \frac{1}{n}) \right]. \]

\[ \mathcal{L}[\delta(t - c)] = \lim_{n \to \infty} n \left( \mathcal{L}[u(t - c)] - \mathcal{L}[u(t - c - \frac{1}{n})] \right) \]

\[ \mathcal{L}[\delta(t - c)] = \lim_{n \to \infty} n \left( \frac{e^{-cs}}{s} - \frac{e^{-s(c + \frac{1}{n})}}{s} \right) = e^{-cs} \lim_{n \to \infty} \frac{1 - e^{-\frac{s}{n}}}{\left( \frac{s}{n} \right)}. \]

This is a singular limit, \( \frac{0}{0} \). Use l'Hôpital's rule.

**Remarks:**

- (a) This result is consistent with a previous result:
  \[ \int_{t_0-a}^{t_0+a} \delta(t - t_0) f(t) \, dt = f(t_0). \]

- (b) \( \mathcal{L}[\delta(t - c)] = \int_0^\infty \delta(t - c) e^{-st} \, dt = e^{-cs}. \)

- (c) \( \mathcal{L}[\delta(t - c) f(t)] = \int_0^\infty \delta(t - c) e^{-st} f(t) \, dt = e^{-cs} f(c). \)
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Differential equations with Dirac's delta sources.

**Example**

Find the solution $y$ to the initial value problem

$$y'' - y = -20 \delta(t - 3), \quad y(0) = 1, \quad y'(0) = 0.$$  

**Solution:** Compute: $\mathcal{L}[y''] - \mathcal{L}[y] = -20 \mathcal{L}[\delta(t - 3)].$

$$\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy(0) - y'(0) \quad \Rightarrow \quad (s^2 - 1) \mathcal{L}[y] - s = -20 e^{-3s},$$

We arrive to the equation

$$\mathcal{L}[y] = \frac{s}{(s^2 - 1)} - 20 e^{-3s} \frac{1}{(s^2 - 1)},$$

$$\mathcal{L}[y] = \mathcal{L}[\cosh(t)] - 20 \mathcal{L}[u(t - 3) \sinh(t - 3)],$$

We conclude: $y(t) = \cosh(t) - 20 u(t - 3) \sinh(t - 3). \quad \triangle$
Differential equations with Dirac’s delta sources.

Example
Find the solution to the initial value problem
\[ y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Compute:
\[ \mathcal{L}[y''] + 4 \mathcal{L}[y] = \mathcal{L}[\delta(t - \pi)] - \mathcal{L}[\delta(t - 2\pi)], \]
\( (s^2 + 4) \mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s} \implies \mathcal{L}[y] = \frac{e^{-\pi s}}{(s^2 + 4)} - \frac{e^{-2\pi s}}{(s^2 + 4)}, \]
that is, \( \mathcal{L}[y] = \frac{e^{-\pi s}}{2} \frac{2}{(s^2 + 4)} - \frac{e^{-2\pi s}}{2} \frac{2}{(s^2 + 4)}. \)

Recall: \( e^{-cs} \mathcal{L}[f(t)] = \mathcal{L}[u(t - c)f(t - c)]. \) Therefore,
\[ \mathcal{L}[y] = \frac{1}{2} \mathcal{L}
\left[u(t - \pi) \sin(2(t - \pi))\right] - \frac{1}{2} \mathcal{L}
\left[u(t - 2\pi) \sin(2(t - 2\pi))\right]. \]

This implies that,
\[ y(t) = \frac{1}{2} u(t - \pi) \sin(2(t - \pi)) - \frac{1}{2} u(t - 2\pi) \sin(2(t - 2\pi)), \]
We conclude: \[ y(t) = \frac{1}{2} \left[u(t - \pi) - u(t - 2\pi)\right] \sin(2t). \]

Differential equations with Dirac’s delta sources.

Example
Find the solution to the initial value problem
\[ y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0. \]

Solution: Recall:
\[ \mathcal{L}[y] = \frac{1}{2} \mathcal{L}
\left[u(t - \pi) \sin(2(t - \pi))\right] - \frac{1}{2} \mathcal{L}
\left[u(t - 2\pi) \sin(2(t - 2\pi))\right]. \]

This implies that,
\[ y(t) = \frac{1}{2} u(t - \pi) \sin(2(t - \pi)) - \frac{1}{2} u(t - 2\pi) \sin(2(t - 2\pi)), \]
We conclude: \[ y(t) = \frac{1}{2} \left[u(t - \pi) - u(t - 2\pi)\right] \sin(2t). \]