

The Laplace Transform of step functions (Sect. 4.3).

- ▶ Overview and notation.
- ▶ The definition of a step function.
- ▶ Piecewise discontinuous functions.
- ▶ The Laplace Transform of discontinuous functions.
- ▶ Properties of the Laplace Transform.

Overview and notation.

Overview: The Laplace Transform method can be used to solve constant coefficients differential equations with *discontinuous source functions*.

Notation:

If $\mathcal{L}[f(t)] = F(s)$, then we denote $\mathcal{L}^{-1}[F(s)] = f(t)$.

Remark: One can show that for a particular type of functions f , that includes all functions we work with in this Section, the notation above is well-defined.

Example

From the Laplace Transform table we know that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$.

Then also holds that $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$. ◁

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The definition of a step function.

Definition

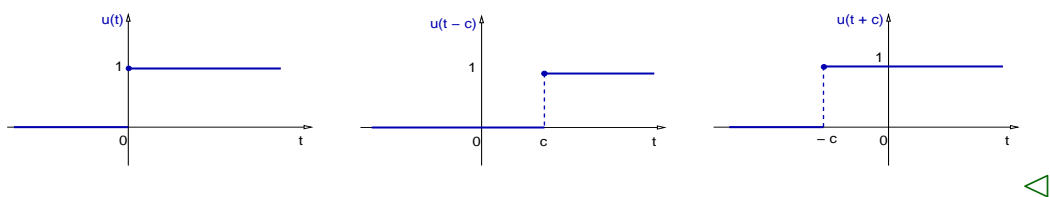
A function u is called a *step function* at $t = 0$ iff holds

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0. \end{cases}$$

Example

Graph the step function values $u(t)$ above, and the translations $u(t - c)$ and $u(t + c)$ with $c > 0$.

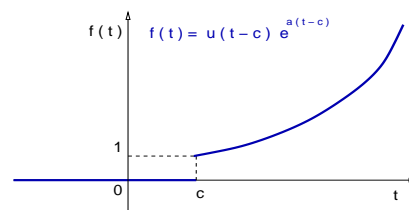
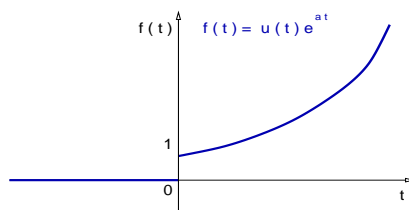
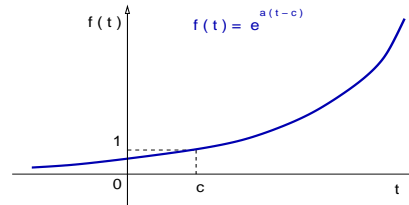
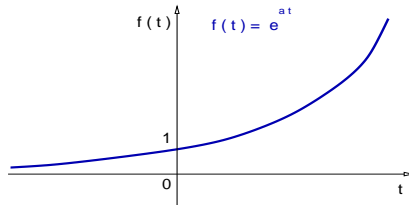
Solution:



The definition of a step function.

Remark: Given any function values $f(t)$ and $c > 0$, then $f(t - c)$ is a right translation of f and $f(t + c)$ is a left translation of f .

Example



The Laplace Transform of step functions (Sect. 4.3).

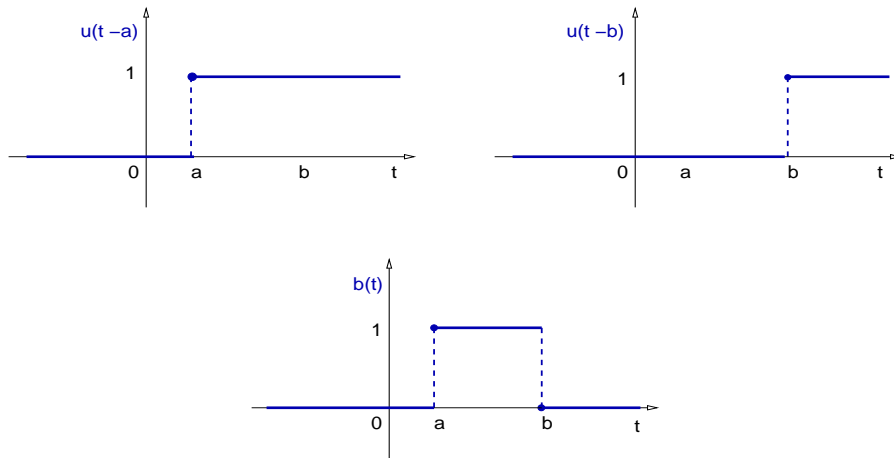
- ▶ Overview and notation.
- ▶ The definition of a step function.
- ▶ **Piecewise discontinuous functions.**
- ▶ The Laplace Transform of discontinuous functions.
- ▶ Properties of the Laplace Transform.

Piecewise discontinuous functions.

Example

Graph of the function $b(t) = u(t - a) - u(t - b)$, with $0 < a < b$.

Solution: The bump function b can be graphed as follows:

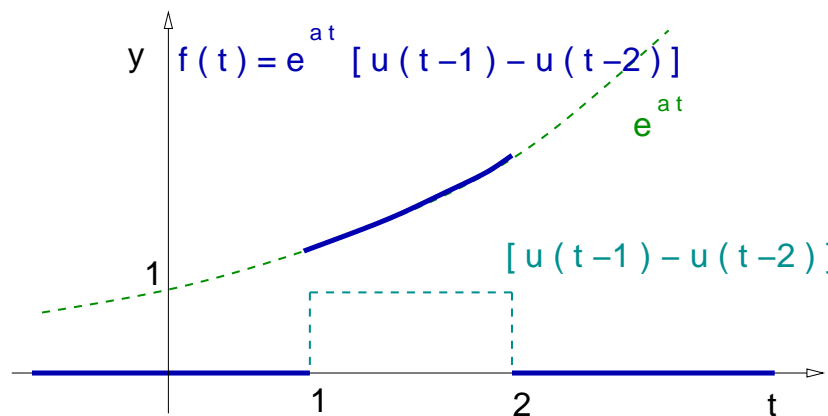


Piecewise discontinuous functions.

Example

Graph of the function $f(t) = e^{at} [u(t - 1) - u(t - 2)]$.

Solution:



Notation: It is common in the literature to denote the function values $u(t - c)$ as $u_c(t)$.

The Laplace Transform of step functions (Sect. 4.3).

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- ▶ **The Laplace Transform of discontinuous functions.**
- ▶ Properties of the Laplace Transform.

The Laplace Transform of discontinuous functions.

Theorem

Given any real number $c \geq 0$, the following equation holds,

$$\mathcal{L}[u(t - c)] = \frac{e^{-cs}}{s}, \quad s > 0.$$

Proof:

$$\mathcal{L}[u(t - c)] = \int_0^{\infty} e^{-st} u(t - c) dt = \int_c^{\infty} e^{-st} dt,$$

$$\mathcal{L}[u(t - c)] = \lim_{N \rightarrow \infty} -\frac{1}{s} (e^{-Ns} - e^{-cs}) = \frac{e^{-cs}}{s}, \quad s > 0.$$

We conclude that $\mathcal{L}[u(t - c)] = \frac{e^{-cs}}{s}$.

□

The Laplace Transform of discontinuous functions.

Example

Compute $\mathcal{L}[3u(t-2)]$.

Solution: $\mathcal{L}[3u(t-2)] = 3\mathcal{L}[u(t-2)] = 3\frac{e^{-2s}}{s}$.

We conclude: $\mathcal{L}[3u(t-2)] = \frac{3e^{-2s}}{s}$. \triangleleft

Example

Compute $\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s}\right]$.

Solution: Since $\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}$, for $c=3$ we get

$\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right] = u(t-3)$. Therefore, $\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s}\right] = 2u(t-3)$. \triangleleft

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- ▶ **Properties of the Laplace Transform.**

Properties of the Laplace Transform.

Theorem (Translations)

If $F(s) = \mathcal{L}[f(t)]$ exists for $s > a \geq 0$ and $c \geq 0$, then holds

$$\mathcal{L}[u(t-c)f(t-c)] = e^{-cs} F(s), \quad s > a.$$

Furthermore,

$$\mathcal{L}[e^{ct}f(t)] = F(s-c), \quad s > a+c.$$

Remark:

- ▶ $\mathcal{L}[\text{translation}(uf)] = (\text{exp})(\mathcal{L}[f])$.
- ▶ $\mathcal{L}[(\text{exp})(f)] = \text{translation}(\mathcal{L}[f])$.

Equivalent notation:

- ▶ $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs} \mathcal{L}[f(t)]$,
- ▶ $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f](s-c)$.

Properties of the Laplace Transform.

Example

Compute $\mathcal{L}[u(t-2) \sin(a(t-2))]$.

Solution: $\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}$, $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs} \mathcal{L}[f(t)]$.

$$\mathcal{L}[u(t-2) \sin(a(t-2))] = e^{-2s} \mathcal{L}[\sin(at)] = e^{-2s} \frac{a}{s^2 + a^2}.$$

We conclude: $\mathcal{L}[u(t-2) \sin(a(t-2))] = e^{-2s} \frac{a}{s^2 + a^2}$. ◁

Example

Compute $\mathcal{L}[e^{3t} \sin(at)]$.

Solution: Recall: $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f](s-c)$, $\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}$.

We conclude: $\mathcal{L}[e^{3t} \sin(at)] = \frac{a}{(s-3)^2 + a^2}$, with $s > 3$. ◁

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geq 1. \end{cases}$

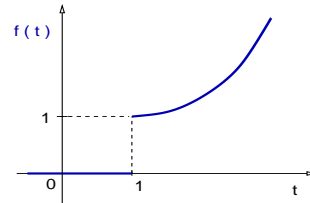
Solution: Using step function notation,

$$f(t) = u(t-1)(t^2 - 2t + 2).$$

Completing the square we obtain,

$$t^2 - 2t + 2 = (t^2 - 2t + 1) - 1 + 2 = (t-1)^2 + 1.$$

This is a parabola t^2 translated to the right by 1 and up by one. Because of the step function, this is a discontinuous function at $t = 1$.



Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geq 1. \end{cases}$

Solution: Recall: $f(t) = u(t-1) [(t-1)^2 + 1]$.

This is equivalent to

$$f(t) = u(t-1)(t-1)^2 + u(t-1).$$

Since $\mathcal{L}[t^2] = 2/s^3$, and $\mathcal{L}[u(t-c)g(t-c)] = e^{-cs} \mathcal{L}[g(t)]$, then

$$\mathcal{L}[f(t)] = \mathcal{L}[u(t-1)(t-1)^2] + \mathcal{L}[u(t-1)] = e^{-s} \frac{2}{s^3} + e^{-s} \frac{1}{s}.$$

We conclude: $\mathcal{L}[f(t)] = \frac{e^{-s}}{s^3} (2 + s^2)$.

◁

Properties of the Laplace Transform.

Remark: The inverse of the formulas in the Theorem above are:

$$\mathcal{L}^{-1}[e^{-cs} F(s)] = u(t - c) f(t - c),$$

$$\mathcal{L}^{-1}[F(s - c)] = e^{ct} f(t).$$

Example

Find $\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2 + 9}\right]$.

Solution: $\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2 + 9}\right] = \frac{1}{3} \mathcal{L}^{-1}\left[e^{-4s} \frac{3}{s^2 + 9}\right]$.

Recall: $\mathcal{L}^{-1}\left[\frac{a}{s^2 + a^2}\right] = \sin(at)$. Then, we conclude that

$$\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2 + 9}\right] = \frac{1}{3} u(t - 4) \sin(3(t - 4)). \quad \triangleleft$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\frac{(s - 2)}{(s - 2)^2 + 9}\right]$.

Solution: $\mathcal{L}^{-1}\left[\frac{s}{s^2 + a^2}\right] = \cos(at)$, $\mathcal{L}^{-1}[F(s - c)] = e^{ct} f(t)$.

We conclude: $\mathcal{L}^{-1}\left[\frac{(s - 2)}{(s - 2)^2 + 9}\right] = e^{2t} \cos(3t). \quad \triangleleft$

Example

Find $\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right]$.

Solution: Recall: $\mathcal{L}^{-1}\left[\frac{a}{s^2 - a^2}\right] = \sinh(at)$

and $\mathcal{L}^{-1}[e^{-cs} F(s)] = u(t - c) f(t - c)$.

Properties of the Laplace Transform.

Example

$$\text{Find } \mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right].$$

Solution: Recall:

$$\mathcal{L}^{-1}\left[\frac{a}{s^2 - a^2}\right] = \sinh(at), \quad \mathcal{L}^{-1}[e^{-cs} F(s)] = u(t - c) f(t - c).$$

$$\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right] = \mathcal{L}^{-1}\left[e^{-3s} \frac{2}{s^2 - 4}\right].$$

$$\text{We conclude: } \mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s^2 - 4}\right] = u(t - 3) \sinh(2(t - 3)). \quad \triangleleft$$

Properties of the Laplace Transform.

Example

$$\text{Find } \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right].$$

Solution: Find the roots of the denominator:

$$s_{\pm} = \frac{1}{2}[-1 \pm \sqrt{1 + 8}] \Rightarrow \begin{cases} s_+ = 1, \\ s_- = -2. \end{cases}$$

Therefore, $s^2 + s - 2 = (s - 1)(s + 2)$.

Use partial fractions to simplify the rational function:

$$\frac{1}{s^2 + s - 2} = \frac{1}{(s - 1)(s + 2)} = \frac{a}{s - 1} + \frac{b}{s + 2},$$

$$\frac{1}{s^2 + s - 2} = a(s + 2) + b(s - 1) = \frac{(a + b)s + (2a - b)}{(s - 1)(s + 2)}.$$

Properties of the Laplace Transform.

Example

$$\text{Find } \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right].$$

$$\text{Solution: Recall: } \frac{1}{s^2 + s - 2} = \frac{(a+b)s + (2a-b)}{(s-1)(s+2)}$$

$$a + b = 0, \quad 2a - b = 1, \quad \Rightarrow \quad a = \frac{1}{3}, \quad b = -\frac{1}{3}.$$

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right] = \frac{1}{3} \mathcal{L}^{-1}\left[e^{-2s} \frac{1}{s-1}\right] - \frac{1}{3} \mathcal{L}^{-1}\left[e^{-2s} \frac{1}{s+2}\right].$$

$$\text{Recall: } \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}, \quad \mathcal{L}^{-1}[e^{-cs} F(s)] = u(t-c) f(t-c),$$

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right] = \frac{1}{3} u(t-2) e^{(t-2)} - \frac{1}{3} u(t-2) e^{-2(t-2)}.$$

$$\text{Hence: } \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + s - 2}\right] = \frac{1}{3} u(t-2) \left[e^{(t-2)} - e^{-2(t-2)}\right]. \quad \triangleleft$$