The Laplace Transform (Sect. 4.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
- Properties of the Laplace Transform.
- Laplace Transform and differential equations.
The definition of the Laplace Transform.

Definition
The function \( F : D_F \to \mathbb{R} \) is the Laplace transform of a function \( f : [0, \infty) \to \mathbb{R} \) iff for all \( s \in D_F \) holds,

\[
F(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt,
\]

where \( D_F \subset \mathbb{R} \) is the set where the integral converges.

Remark: The domain \( D_F \) of \( F \) depends on the function \( f \).

Notation: We often denote: \( F(s) = \mathcal{L}[f(t)] \).

- This notation \( \mathcal{L}[\ ] \) emphasizes that the Laplace transform defines a map from a set of functions into a set of functions.
- Functions are denoted as \( t \mapsto f(t) \).
- The Laplace transform is also a function: \( f \mapsto \mathcal{L}[f] \).
Review: Improper integrals.

Recall: Improper integral are defined as a limit.

\[ \int_{t_0}^{\infty} g(t) \, dt = \lim_{N \to \infty} \int_{t_0}^{N} g(t) \, dt. \]

- The integral converges iff the limit exists.
- The integral diverges iff the limit does not exist.

Example

Compute the improper integral \( \int_{0}^{\infty} e^{-at} \, dt \), with \( a > 0 \).

Solution:

\[
\int_{0}^{\infty} e^{-at} \, dt = \lim_{N \to \infty} \int_{0}^{N} e^{-at} \, dt = \lim_{N \to \infty} \left( \frac{1}{a} \left( e^{-aN} - 1 \right) \right).
\]

Since \( \lim_{N \to \infty} e^{-aN} = 0 \) for \( a > 0 \), we conclude \( \int_{0}^{\infty} e^{-at} \, dt = \frac{1}{a}. \)

---

The Laplace Transform (Sect. 4.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- **Examples of Laplace Transforms.**
- A table of Laplace Transforms.
- Properties of the Laplace Transform.
- Laplace Transform and differential equations.
Examples of Laplace Transforms.

Example

Solution: We have to find the Laplace Transform of $f(t) = 1$. Following the definition we obtain,

$$L[1] = \int_0^{\infty} e^{-st} \, dt = \int_0^{\infty} e^{-st} \, dt$$

But $\int_0^{\infty} e^{-at} \, dt = \frac{1}{a}$ for $a > 0$, and diverges for $a \leq 0$.

Therefore $L[1] = \frac{1}{s}$ for $s > 0$, and $L[1]$ does not exists for $s \leq 0$.

In other words, $F(s) = L[1]$ is the function $F : D_F \to \mathbb{R}$ given by

$$f(t) = 1, \quad F(s) = \frac{1}{s}, \quad D_F = (0, \infty).$$

◁

Examples of Laplace Transforms.

Example
Compute $L[e^{at}]$, where $a \in \mathbb{R}$.

Solution: Following the definition of Laplace Transform,

$$L[e^{at}] = \int_0^{\infty} e^{st} e^{at} \, dt = \int_0^{\infty} e^{-(s-a)t} \, dt.$$ 

We have seen that the improper integral is given by

$$\int_0^{\infty} e^{-(s-a)t} \, dt = \frac{1}{(s-a)} \quad \text{for} \quad (s-a) > 0.$$ 

We conclude that $L[e^{at}] = \frac{1}{s-a}$ for $s > a$. In other words,

$$f(t) = e^{at}, \quad F(s) = \frac{1}{(s-a)}, \quad s > a.$$ 

◁
Examples of Laplace Transforms.

Example
Compute $\mathcal{L}[\sin(at)]$, where $a \in \mathbb{R}$.

Solution: In this case we need to compute

$$\mathcal{L}[\sin(at)] = \lim_{N \to \infty} \int_0^N e^{-st} \sin(at) \, dt.$$  

Integrating by parts twice it is not difficult to obtain:

$$\int_0^N e^{-st} \sin(at) \, dt = -\frac{1}{s} [e^{-st} \sin(at)]_0^N - \frac{a}{s^2} [e^{-st} \cos(at)]_0^N - \frac{a^2}{s^2} \int_0^N e^{-st} \sin(at) \, dt.$$  

This identity implies

$$\left(1 + \frac{a^2}{s^2}\right) \int_0^N e^{-st} \sin(at) \, dt = -\frac{1}{s} [e^{-st} \sin(at)]_0^N - \frac{a}{s^2} [e^{-st} \cos(at)]_0^N.$$

Hence, it is not difficult to see that

$$\left(\frac{s^2 + a^2}{s^2}\right) \int_0^\infty e^{-st} \sin(at) \, dt = \frac{a}{s^2}, \quad s > 0,$$

which is equivalent to

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}, \quad s > 0. \quad \triangle$$
The Laplace Transform (Sect. 4.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- **A table of Laplace Transforms.**
- Properties of the Laplace Transform.
- Laplace Transform and differential equations.

### A table of Laplace Transforms.

- $f(t) = 1 \quad F(s) = \frac{1}{s} \quad s > 0,$
- $f(t) = e^{at} \quad F(s) = \frac{1}{s - a} \quad s > \max\{a, 0\},$
- $f(t) = t^n \quad F(s) = \frac{n!}{s(n+1)} \quad s > 0,$
- $f(t) = \sin(at) \quad F(s) = \frac{a}{s^2 + a^2} \quad s > 0,$
- $f(t) = \cos(at) \quad F(s) = \frac{s}{s^2 + a^2} \quad s > 0,$
- $f(t) = \sinh(at) \quad F(s) = \frac{a}{s^2 - a^2} \quad s > |a|,$
- $f(t) = \cosh(at) \quad F(s) = \frac{s}{s^2 + a^2} \quad s > |a|,$
- $f(t) = t^n e^{at} \quad F(s) = \frac{n!}{(s - a)(n+1)} \quad s > \max\{a, 0\},$
- $f(t) = e^{at} \sin(bt) \quad F(s) = \frac{b}{(s - a)^2 + b^2} \quad s > \max\{a, 0\}.$
Properties of the Laplace Transform.

Theorem (Sufficient conditions)

If the function \( f : [0, \infty) \rightarrow \mathbb{R} \) is piecewise continuous and there exist positive constants \( k \) and \( a \) such that

\[
|f(t)| \leq k e^{at},
\]

then the Laplace Transform of \( f \) exists for all \( s > a \).

Theorem (Linear combination)

If the \( \mathcal{L}[f] \) and \( \mathcal{L}[g] \) are well-defined and \( a, b \) are constants, then

\[
\mathcal{L}[af + bg] = a \mathcal{L}[f] + b \mathcal{L}[g].
\]

Proof: Integration is a linear operation:

\[
\int [af(t) + bg(t)] \, dt = a \int f(t) \, dt + b \int g(t) \, dt.
\]
Theorem (Derivatives)

If the $\mathcal{L}[f]$ and $\mathcal{L}[f']$ are well-defined, then holds,

$$\mathcal{L}[f'] = s \mathcal{L}[f] - f(0). \quad (1)$$

Furthermore, if $\mathcal{L}[f'']$ is well-defined, then it also holds

$$\mathcal{L}[f''] = s^2 \mathcal{L}[f] - sf(0) - f'(0). \quad (2)$$

Proof of Eq (2): Use Eq. (1) twice:

$$\mathcal{L}[f''] = \mathcal{L}[(f')'] = s\mathcal{L}[(f')] - f'(0) = s(s\mathcal{L}[f] - f(0)) - f'(0),$$

that is,

$$\mathcal{L}[f''] = s^2 \mathcal{L}[f] - sf(0) - f'(0).$$

Proof of Eq (1): Recall the definition of the Laplace Transform,

$$\mathcal{L}[f'] = \int_0^\infty e^{-st}f'(t)\,dt = \lim_{n \to \infty} \int_0^n e^{-st}f'(t)\,dt$$

Integrating by parts,

$$\lim_{n \to \infty} \int_0^n e^{-st}f'(t)\,dt = \lim_{n \to \infty} \left[\left(e^{-st}f(t)\right)|_0^n - \int_0^n (-s)e^{-st}f(t)\,dt\right]$$

$$\mathcal{L}[f'] = \lim_{n \to \infty} \left[e^{-sn}f(n) - f(0)\right] + s \int_0^\infty e^{-st}f(t)\,dt = -f(0) + s \mathcal{L}[f],$$

where we used that $\lim_{n \to \infty} e^{-sn}f(n) = 0$ for $s$ big enough, and we also used that $\mathcal{L}[f]$ is well-defined.

We then conclude that $\mathcal{L}[f'] = s \mathcal{L}[f] - f(0)$. 

Proof of Eq (2): Use Eq. (1) twice:

$$\mathcal{L}[f''] = \mathcal{L}[(f')'] = s\mathcal{L}[(f')] - f'(0) = s(s\mathcal{L}[f] - f(0)) - f'(0),$$

that is,

$$\mathcal{L}[f''] = s^2 \mathcal{L}[f] - sf(0) - f'(0).$$
Laplace Transform and differential equations.

**Remark:** Laplace Transforms can be used to find solutions to differential equations with **constant coefficients**.

**Idea of the method:**

\[
\mathcal{L} \left[ \begin{array}{c}
\text{Differential Eq. for } y(t).
\end{array} \right] \quad \overset{(1)}{\rightarrow} \quad \text{Algebraic Eq. for } \mathcal{L}[y(t)].
\]

\[
\overset{(2)}{\rightarrow} \quad \text{Solve the Algebraic Eq. for } \mathcal{L}[y(t)].
\]

\[
\overset{(3)}{\rightarrow} \quad \text{Transform back to obtain } y(t). \quad (\text{Using the table.})
\]
Example

Use the Laplace transform to find the solution \( y(t) \) to the IVP

\[ y' + 2y = 0, \quad y(0) = 3. \]

Solution: We know the solution: \( y(t) = 3e^{-2t} \).

(1): Compute the Laplace transform of the differential equation,

\[ \mathcal{L}[y' + 2y] = \mathcal{L}[0] \quad \Rightarrow \quad \mathcal{L}[y' + 2y] = 0. \]

Find an algebraic equation for \( \mathcal{L}[y] \). Recall linearity:

\[ \mathcal{L}[y'] + 2 \mathcal{L}[y] = 0. \]

Also recall the property: \( \mathcal{L}[y'] = s \mathcal{L}[y] - y(0) \), that is,

\[ \left[ s \mathcal{L}[y] - y(0) \right] + 2 \mathcal{L}[y] = 0 \quad \Rightarrow \quad (s + 2) \mathcal{L}[y] = y(0). \]

(2): Solve the algebraic equation for \( \mathcal{L}[y] \).

\[ \mathcal{L}[y] = \frac{y(0)}{s + 2}, \quad y(0) = 3, \quad \Rightarrow \quad \mathcal{L}[y] = \frac{3}{s + 2}. \]

(3): Transform back to \( y(t) \). From the table:

\[ \mathcal{L}[e^{at}] = \frac{1}{s - a} \quad \Rightarrow \quad \frac{3}{s + 2} = 3 \mathcal{L}[e^{-2t}] \quad \Rightarrow \quad \frac{3}{s + 2} = \mathcal{L}[3e^{-2t}]. \]

Hence, \( \mathcal{L}[y] = \mathcal{L}[3e^{-2t}] \quad \Rightarrow \quad y(t) = 3e^{-2t}. \) \( \triangleright \)