Non-homogeneous equations (Sect. 2.7).

- We study: \( y'' + p(t) y' + q(t) y = f(t) \).
- Method of variation of parameters.
- Using the method in an example.
- The proof of the variation of parameter method.
- Using the method in another example.

Method of variation of parameters.

Remarks:
- This is a general method to find solutions to equations having **variable coefficients** and **non-homogeneous** with a continuous but otherwise **arbitrary source** function,

\[
y'' + p(t) y' + q(t) y = f(t).
\]

- The variation of parameter method can be applied to more general equations than the undetermined coefficients method.
- The variation of parameter method usually takes more time to implement than the simpler method of undetermined coefficients.
Method of variation of parameters.

Theorem (Variation of parameters)

A particular solution to the equation

\[ y'' + p(t) y' + q(t) y = f(t) \]

with \( p, q, f : (t_1, t_2) \to \mathbb{R} \) continuous functions, is given by

\[ y_p = u_1 y_1 + u_2 y_2, \]

where \( y_1, y_2 : (t_1, t_2) \to \mathbb{R} \) are fundamental solutions of the homogeneous equation

\[ y'' + p(t) y' + q(t) y = 0, \]

and functions \( u_1 \) and \( u_2 \) are defined by

\[ u_1(t) = \int -\frac{y_2(t)f(t)}{W_{y_1y_2}(t)} \, dt, \quad u_2(t) = \int \frac{y_1(t)f(t)}{W_{y_1y_2}(t)} \, dt, \]

with \( W_{y_1y_2} \) the Wronskian of \( y_1, y_2 \).

Non-homogeneous equations (Sect. 2.7).

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Using the method in an example.

Example
Find the general solution of the inhomogeneous equation
\[ y'' - 5y' + 6y = 2e^t. \]

Solution:
First: Find fundamental solutions to the homogeneous equation.
The characteristic equation is
\[ r^2 - 5r + 6 = 0 \Rightarrow r = \frac{1}{2} (5 \pm \sqrt{25 - 24}) \Rightarrow \begin{cases} r_1 = 3, \\ r_2 = 2. \end{cases} \]

Hence, \( y_1(t) = e^{3t} \) and \( y_2(t) = e^{2t} \). Compute their Wronskian,
\[ W_{y_1y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t}) \Rightarrow W_{y_1y_2}(t) = -e^{5t}. \]

Second: We compute the functions \( u_1 \) and \( u_2 \). By definition,
\[ u_1' = -\frac{y_2f}{W_{y_1y_2}}, \quad u_2' = \frac{y_1f}{W_{y_1y_2}}. \]

Third: The particular solution is
\[ y_p = (-e^{-2t})(e^{3t}) + (2e^{-t})(e^{2t}) \Rightarrow y_p = e^t. \]

The general solution is \( y(t) = c_1e^{3t} + c_2e^{2t} + e^t, \) \( c_1, c_2 \in \mathbb{R}. \) \( \langle \)
Non-homogeneous equations (Sect. 2.7).

- We study: $y'' + p(t) y' + q(t) y = f(t)$.
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- Using the method in an example.
- **The proof of the variation of parameter method.**
- Using the method in another example.

The proof of the variation of parameter method.

**Proof:** Denote $L(y) = y'' + p(t) y' + q(t) y$.

We need to find $y_p$ solution of $L(y_p) = f$.

We know $y_1$ and $y_2$ solutions of $L(y_1) = 0$ and $L(y_2) = 0$.

Idea: The reduction of order method: Find $y_2$ proposing $y_2 = uy_1$.

First idea: Propose that $y_p$ is given by $y_p = u_1 y_1 + u_2 y_2$.

We hope that the equation for $u_1$ and $u_2$ will be simpler than the original equation for $y_p$, since $y_1$ and $y_2$ are solutions to the homogeneous equation. Compute:

$$y'_p = u'_1 y_1 + u_1 y'_1 + u'_2 y_2 + u_2 y'_2,$$

$$y''_p = u''_1 y_1 + 2u'_1 y'_1 + u_1 y''_1 + u''_2 y_2 + 2u'_2 y'_2 + u_2 y''_2.$$
The proof of the variation of parameter method.

Proof: Then \( L(y_p) = f \) is given by

\[
\left[ u''_1 y_1 + 2u'_1 y'_1 + u_1 y'''_1 + u''_2 y_2 + 2u'_2 y'_2 + u_2 y''_2 \right] \\
p(t) \left[ u'_1 y_1 + u_1 y'_1 + u'_2 y_2 + u_2 y'_2 \right] + q(t) \left[ u_1 y_1 + u_2 y_2 \right] = f(t).
\]

\[
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u_1 y'_2) + p (u'_1 y_1 + u'_2 y_2) \\
+ u_1 (y''_1 + p y'_1 + q y_1) + u_2 (y''_2 + p y'_2 + q y_2) = f
\]

Recall: \( y''_1 + p y'_1 + q y_1 = 0 \) and \( y''_2 + p y'_2 + q y_2 = 0 \). Hence,

\[
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) + p (u'_1 y_1 + u'_2 y_2) = f
\]

Second idea: Look for \( u_1 \) and \( u_2 \) that satisfy the extra equation

\[
u'_1 y_1 + u'_2 y_2 = 0.
\]

The proof of the variation of parameter method.

Proof: Recall: \( u'_1 y_1 + u'_2 y_2 = 0 \) and

\[
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) + p (u'_1 y_1 + u'_2 y_2) = f.
\]

These two equations imply that \( L(y_p) = f \) is

\[
u''_1 y_1 + u''_2 y_2 + 2(u'_1 y'_1 + u'_2 y'_2) = f.
\]

From \( u'_1 y_1 + u'_2 y_2 = 0 \) we get \( \left[ u'_1 y_1 + u'_2 y_2 \right]' = 0 \), that is

\[
u''_1 y_1 + u''_2 y_2 + (u'_1 y'_1 + u'_2 y'_2) = 0.
\]

This information in \( L(y_p) = f \) implies

\[
u'_1 y'_1 + u'_2 y'_2 = f.
\]

Summary: If \( u_1 \) and \( u_2 \) satisfy \( u'_1 y_1 + u'_2 y_2 = 0 \) and \( u'_1 y'_1 + u'_2 y'_2 = f \),
then \( y_p = u_1 y_1 + u_2 y_2 \) satisfies \( L(y_p) = f \).
The proof of the variation of parameter method.

Proof: Summary: If \( u_1 \) and \( u_2 \) satisfy
\[
\begin{align*}
  u_1'y_1 + u_2'y_2 &= 0, \\
  u_1'y_1' + u_2'y_2' &= f,
\end{align*}
\]
then \( y_p = u_1y_1 + u_2y_2 \) satisfies \( L(y_p) = f \).

The equations above are simple to solve for \( u_1 \) and \( u_2 \),
\[
u_1' = -\frac{y_1}{y_2} u_1' \quad \Rightarrow \quad u_1'y_1' - \frac{y_1y_2'}{y_2} u_1' = f \quad \Rightarrow \quad u_1'\left(\frac{y_1'y_2 - y_1y_2'}{y_2}\right) = f.
\]
Since \( W_{y_1y_2} = y_1y_2' - y_1'y_2 \), then \( u_1' = -\frac{y_2f}{W_{y_1y_2}} \quad \Rightarrow \quad u_2' = \frac{y_1f}{W_{y_1y_2}}.\)

Integrating in the variable \( t \) we obtain
\[
u_1(t) = \int -\frac{y_2(t)f(t)}{W_{y_1y_2}(t)} \, dt, \quad u_2(t) = \int \frac{y_1(t)f(t)}{W_{y_1y_2}(t)} \, dt,
\]
This establishes the Theorem. \(\square\)

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Using the method in another example.

Example
Find a particular solution to the differential equation
\[ t^2 y'' - 2y = 3t^2 - 1, \]
knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: First, write the equation in the form of the Theorem. That is, divide the whole equation by \( t^2 \),

\[ y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2} \quad \Rightarrow \quad f(t) = 3 - \frac{1}{t^2}. \]

We know that \( y_1 = t^2 \) and \( y_2 = 1/t \). Their Wronskian is

\[ W_{y_1 y_2}(t) = (t^2) \left( -\frac{1}{t^2} \right) - (2t) \left( \frac{1}{t} \right) \quad \Rightarrow \quad W_{y_1 y_2}(t) = -3. \]

Using the method in another example.

Example
Find a particular solution to the differential equation
\[ t^2 y'' - 2y = 3t^2 - 1, \]
knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: \( y_1 = t^2, \ y_2 = 1/t, \ f(t) = 3 - \frac{1}{t^2}, \ W_{y_1 y_2}(t) = -3. \)

We now compute \( y_1 \) and \( u_2 \),

\[ u'_1 = -\frac{1}{t} \left( 3 - \frac{1}{t^2} \right) \frac{1}{3} - \frac{1}{3} t^{-3} \quad \Rightarrow \quad u_1 = \ln(t) + \frac{1}{6} t^{-2}, \]

\[ u'_2 = (t^2) \left( 3 - \frac{1}{t^2} \right) \frac{1}{3} = -t^2 + \frac{1}{3} \quad \Rightarrow \quad u_2 = -\frac{1}{3} t^3 + \frac{1}{3} t. \]
Using the method in another example.

Example
Find a particular solution to the differential equation

\[ t^2 y'' - 2y = 3t^2 - 1, \]

knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: The particular solution \( \tilde{y}_p = u_1 y_1 + u_2 y_2 \) is

\[
\tilde{y}_p = \left[ \ln(t) + \frac{1}{6} t^{-2} \right] (t^2) + \frac{1}{3} (-t^3 + t) (t^{-1})
\]

\[
\tilde{y}_p = t^2 \ln(t) + \frac{1}{6} - \frac{1}{3} t^2 + \frac{1}{3} = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3} t^2
\]

\[
\tilde{y}_p = t^2 \ln(t) + \frac{1}{2} - \frac{1}{3} y_1(t).
\]

A simpler expression is \( y_p = t^2 \ln(t) + \frac{1}{2} \).  

Using the method in another example.

Example
Find a particular solution to the differential equation

\[ t^2 y'' - 2y = 3t^2 - 1, \]

knowing that the functions \( y_1 = t^2 \) and \( y_2 = 1/t \) are solutions to the homogeneous equation \( t^2 y'' - 2y = 0 \).

Solution: If we do not remember the formulas for \( u_1, u_2 \), we can always solve the system

\[ u'_1 y_1 + u'_2 y_2 = 0 \]

\[ u'_1 y'_1 + u'_2 y'_2 = f. \]

\[ t^2 u'_1 + u'_2 \frac{1}{t} = 0, \quad 2t u'_1 + u'_2 \frac{(-1)}{t^2} = 3 - \frac{1}{t^2}. \]

\[ u'_2 = -t^3 u'_1 \Rightarrow 2t u'_1 + t u'_1 = 3 - \frac{1}{t^2} \Rightarrow \left\{ \begin{array}{l} u'_1 = \frac{1}{t} - \frac{1}{3t^3} \\ u'_2 = -t^2 + \frac{1}{3}. \end{array} \right. \]