Non-homogeneous equations (Sect. 2.6).

- We study: \( y'' + a_1 y' + a_0 y = b(t) \).
- Review and preliminary results.
- Summary of the undetermined coefficients method.
- Using the method in few examples.
- The guessing solution table.

Review and preliminary results.

Operator notation: The differential equation

\[
y'' + p(t) y' + q(t) y = f(t)
\]

will be written as \( L(y) = f \), with

\[
L(y) = y'' + p(t) y' + q(t) y.
\]

The homogeneous equation is \( L(y) = 0 \).

Remark: The operator \( L \) is a linear function of \( y \).

Theorem

For every continuously differentiable functions \( y_1, y_2 \) and every constants \( c_1, c_2 \) holds that

\[
L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2).
\]
Theorem

Let \( L(y) = y'' + p(t) y' + q(t) y \) with \( p \) and \( q \) given functions. If \( y_1 \) and \( y_2 \) are fundamental solutions of

\[
L(y) = 0,
\]

and \( y_p \) is any solution of the non-homogeneous equation

\[
L(y_p) = f, \quad (1)
\]

for a given source \( f \), then any other solution \( y \) of the non-homogeneous equation above is given by

\[
y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t), \quad (2)
\]

where \( c_1, c_2 \in \mathbb{R} \).

Notation: The expression for \( y \) in Eq. (2) is called the general solution of the non-homogeneous Eq. (1).
Non-homogeneous equations (Sect. 2.6).

- We study: \( y'' + a_1 y' + a_0 y = b(t) \).
- Operator notation and preliminary results.
- **Summary of the undetermined coefficients method.**
- Using the method in few examples.
- The guessing solution table.

**Summary of the undetermined coefficients method.**

**Problem:** Given a constant coefficients linear operator
\[ L(y) = y'' + a_1 y' + a_0 y, \] with \( a_1, a_2 \in \mathbb{R} \), find every solution of the non-homogeneous differential equation
\[ L(y) = f. \]

**Remarks:**
- The undetermined coefficients is a method to find solutions to linear, non-homogeneous, constant coefficients, differential equations.
- It consists in guessing the solution \( y_p \) of the non-homogeneous equation
\[ L(y_p) = f, \]
for particularly simple source functions \( f \).
Summary of the undetermined coefficients method.

Summary:

1. Find the general solution of the homogeneous equation
   \[ L(y_h) = 0. \]

2. If \( f \) has the form \( f = f_1 + \cdots + f_n \), with \( n \geq 1 \), then look for solutions \( y_{p_i} \), with \( i = 1, \cdots, n \) to the equations
   \[ L(y_{p_i}) = f_i. \]

   Once the functions \( y_{p_i} \) are found, then construct
   \[ y_p = y_{p_1} + \cdots + y_{p_n}. \]

3. Given the source functions \( f_i \), guess the solutions functions \( y_{p_i} \) following the Table below.

---

Summary (cont.):

<table>
<thead>
<tr>
<th>( f_i(t) ) ((K, m, a, b, \text{ given.}))</th>
<th>( y_{p_i}(t) ) ((\text{Guess})) ((k \text{ not given.}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Ke^{at} )</td>
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Summary of the undetermined coefficients method.

Summary (cont.):

(4) If any guessed function $y_{pi}$ satisfies the homogeneous equation $L(y_{pi}) = 0$, then change the guess to the function

$$t^s y_{pi}, \quad \text{with} \quad s \geq 1,$$

and $s$ sufficiently large such that $L(t^s y_{pi}) \neq 0$.

(5) Impose the equation $L(y_{pi}) = f_i$ to find the undetermined constants $k_1, \cdots, k_m$, for the appropriate $m$, given in the table above.

(6) The general solution to the original differential equation $L(y) = f$ is then given by

$$y(t) = y_h(t) + y_{p1} + \cdots + y_{pn}.$$

Non-homogeneous equations (Sect. 2.6).

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- **Using the method in few examples.**
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Using the method in few examples.

Example
Find all solutions to the non-homogeneous equation

\[ y'' - 3y' - 4y = 3e^{2t}. \]

Solution: Notice: \( L(y) = y'' - 3y' - 4y \) and \( f(t) = 3e^{2t} \).

1. Find all solutions \( y_h \) to the homogeneous equation \( L(y_h) = 0 \).
   The characteristic equation is
   \[ r^2 - 3r - 4 = 0 \Rightarrow \begin{cases} r_1 = 4, \\ r_2 = -1. \end{cases} \]
   \[ y_h(t) = c_1 e^{4t} + c_2 e^{-t}. \]

2. Trivial in our case. The source function \( f(t) = 3e^{2t} \) cannot be simplified into a sum of simpler functions.

3. Table says: For \( f(t) = 3e^{2t} \) guess \( y_p(t) = k e^{2t} \)

Using the method in few examples.

Example
Find all solutions to the non-homogeneous equation

\[ y'' - 3y' - 4y = 3e^{2t}. \]

Solution: Recall: \( y_p(t) = k e^{2t} \). We need to find \( k \).

4. Trivial here, since \( L(y_p) \neq 0 \), we do not modify our guess.
   (Recall: \( L(y_h) = 0 \) iff \( y_h(t) = c_1 e^{4t} + c_2 e^{-t} \).)

5. Introduce \( y_p \) into \( L(y_p) = f \) and find \( k \).
   \[ (2^2 - 6 - 4)ke^{2t} = 3e^{2t} \Rightarrow -6k = 3 \Rightarrow k = -\frac{1}{2}. \]
   We have obtained that \( y_p(t) = -\frac{1}{2} e^{2t} \).

6. The general solution to the inhomogeneous equation is
   \[ y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2} e^{2t}. \]
Using the method in few examples.

Example
Find all solutions to the non-homogeneous equation
\[ y'' - 3y' - 4y = 3e^{4t}. \]

Solution: We know that the general solution to homogeneous equation is
\[ y_h(t) = c_1 e^{4t} + c_2 e^{-t}. \]
Following the table we guess \( y_p \) as \( y_p = k e^{4t} \).
However, this guess satisfies \( L(y_p) = 0 \).
So we modify the guess to \( y_p = kt e^{4t} \).
Introduce the guess into \( L(y_p) = f \). We need to compute
\[
y_p' = k e^{4t} + 4kt e^{4t}, \quad y_p'' = 8k e^{4t} + 16kt e^{4t}.
\]
\[
(8k + 16kt) - 3(k + 4kt) - 4kt e^{4t} = 3e^{4t}.
\]
\[
[(8+16t) - 3(1+4t) - 4t] k = 3 \quad \Rightarrow \quad [5 + (16-12-4) t] k = 3
\]
We obtain that \( k = \frac{3}{5} \). Therefore, \( y_p(t) = \frac{3}{5} t e^{4t} \), and
\[
y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{3}{5} t e^{4t}. \]

Using the method in few examples.

Example
Find all solutions to the non-homogeneous equation
\[ y'' - 3y' - 4y = 3e^{4t}. \]

Solution: Recall:
\[
y_p = kt e^{4t}, \quad y_p' = k e^{4t} + 4kt e^{4t}, \quad y_p'' = 8k e^{4t} + 16kt e^{4t}.
\]
\[
(8k + 16kt) - 3(k + 4kt) - 4kt e^{4t} = 3e^{4t}.
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Example
Find all the solutions to the inhomogeneous equation
\[ y'' - 3y' - 4y = 2 \sin(t). \]

Solution: We know that the general solution to homogeneous equation is
\[ y(t) = c_1 e^{4t} + c_2 e^{-t}. \]

Following the table: Since \( f = 2 \sin(t) \), then we guess
\[ y_p = k_1 \sin(t) + k_2 \cos(t). \]

This guess satisfies \( L(y_p) \neq 0 \).

Compute:
\[ y'_p = k_1 \cos(t) - k_2 \sin(t), \quad y''_p = -k_1 \sin(t) - k_2 \cos(t). \]

\[ L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)] - 4[k_1 \sin(t) + k_2 \cos(t)] = 2 \sin(t), \]

Using the method in few examples.

Example
Find all the solutions to the inhomogeneous equation
\[ y'' - 3y' - 4y = 2 \sin(t). \]

Solution: Recall:
\[ L(y_p) = [-k_1 \sin(t) - k_2 \cos(t)] - 3[k_1 \cos(t) - k_2 \sin(t)] - 4[k_1 \sin(t) + k_2 \cos(t)] = 2 \sin(t), \]
\[ (-5k_1 + 3k_2) \sin(t) + (-3k_1 - 5k_2) \cos(t) = 2 \sin(t). \]

This equation holds for all \( t \in \mathbb{R} \). In particular, at \( t = \frac{\pi}{2}, \ t = 0 \).

\[ \begin{align*}
-5k_1 + 3k_2 &= 2, \\
-3k_1 - 5k_2 &= 0,
\end{align*} \]
\[ \Rightarrow \begin{cases}
    k_1 = -\frac{5}{17}, \\
    k_2 = \frac{3}{17}.
\end{cases} \]
Using the method in few examples.

Example
Find all the solutions to the inhomogeneous equation
\[ y'' - 3y' - 4y = 2 \sin(t). \]

Solution: Recall: \( k_1 = -\frac{5}{17} \) and \( k_2 = \frac{3}{17} \).
So the particular solution to the inhomogeneous equation is
\[ y_p(t) = \frac{1}{17} [-5 \sin(t) + 3 \cos(t)]. \]
The general solution is
\[ y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{1}{17} [-5 \sin(t) + 3 \cos(t)]. \]

Using the method in few examples.

Example
Find all the solutions to the inhomogeneous equation
\[ y'' - 3y' - 4y = 3e^{2t} + 2 \sin(t). \]

Solution: We know that the general solution \( y \) is given by
\[ y(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t), \]
where \( y_h(t) = c_1 e^{4t} + c_2 e^{2t}, \) \( L(y_{p_1}) = 3e^{2t}, \) and \( L(y_{p_2}) = 2 \sin(t). \)
We have just found out that
\[ y_{p_1}(t) = -\frac{1}{2} e^{2t}, \quad y_{p_2}(t) = \frac{1}{17} [-5 \sin(t) + 3 \cos(t)]. \]
We conclude that
\[ y(t) = c_1 e^{4t} + c_2 e^{2t} - \frac{1}{2} e^{2t} + \frac{1}{17} [-5 \sin(t) + 3 \cos(t)]. \]
Using the method in few examples.

Example

- For $y'' - 3y' - 4y = 3e^{2t} \sin(t)$, guess
  \[ y_p(t) = [k_1 \sin(t) + k_2 \cos(t)] \ e^{2t}. \]

- For $y'' - 3y' - 4y = 2t^2 e^{3t}$, guess
  \[ y_p(t) = (k_0 + k_1 t + k_2 t^2) \ e^{3t}. \]

- For $y'' - 3y' - 4y = 3t \sin(t)$, guess
  \[ y_p(t) = (1 + k_1 t) \ [k_2 \sin(t) + k_3 \cos(t)]. \]

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**Guessing Solution Table.**

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