Second order equations: Repeated roots (Sect. 2.4).

- Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.
- Guessing solutions: Repeated roots case.
- Main result for repeated roots.
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Review: On solutions of $y'' + a_1\, y' + a_0\, y = 0$.

Summary:
Given constants $a_1, a_0 \in \mathbb{R}$, consider the differential equation

$$y'' + a_1 y' + a_0 y = 0$$

with characteristic polynomial having roots

$$r_\pm = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0}.$$

(1) If $a_1^2 - 4a_0 > 0$, then $y_1(t) = e^{r_+ t}$ and $y_2(t) = e^{r_- t}$.

(2) If $a_1^2 - 4a_0 < 0$, then introducing $\alpha = -\frac{a_1}{2}$, $\beta = \frac{1}{2}\sqrt{4a_0 - a_1^2}$,

$$y_1(t) = e^{\alpha t}\cos(\beta t), \qquad y_2(t) = e^{\alpha t}\sin(\beta t).$$

(3) If $a_1^2 - 4a_0 = 0$, then $y_1(t) = e^{-\frac{a_1}{2}t}$.

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Guessing solutions: Repeated roots case.

Remark:

- Case (3), where $4a_0 - a_1^2 = 0$ guessed as $\beta \to 0$ in case (2).
- Study solutions in the case (2) as $\beta \to 0$ for fixed t.
- Since $\cos(\beta t) \to 1$ as $\beta \to 0$, we conclude that
$$y_{1\beta}(t) = e^{-\frac{a_1}{2}\,t} \cos(\beta t) \to e^{-\frac{a_1}{2}\,t} = y_1(t).$$
- Since $\dfrac{\sin(\beta t)}{\beta t} \to 1$ as $\beta \to 0$, that is, $\sin(\beta t) \to \beta t$,
$$y_{2\beta}(t) = e^{-\frac{a_1}{2}\,t} \sin(\beta t) \to \beta t\, e^{-\frac{a_1}{2}\,t} \to 0.$$
- Guessed solution: $y_2(t) = t\, y_1(t)$.
  Introducing y_2 in the differential equation one obtains: Yes.
- Hence, y_2 and y_1 are a fundamental solutions in case (3).

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Main result for repeated roots.

Theorem

If a_1, $a_0 \in R$ satisfy that $a_1^2 = 4a_0$, then the functions

$$y_1(t) = e^{-\frac{a_1}{2}\,t}, \qquad y_2(t) = t\, e^{-\frac{a_1}{2}\,t},$$

are a fundamental solution set for the differential equation

$$y'' + a_1 y' + a_0 y = 0.$$

Example

Find the general solution of $9y'' + 6y' + y = 0$.

Solution: The characteristic equation is $9r^2 + 6r + 1 = 0$, so

$$r_\pm = \frac{1}{(2)(9)}\left[-6 \pm \sqrt{36 - 36}\right] \quad \Rightarrow \quad r_\pm = -\frac{1}{3}.$$

Theorem (2.1.7) in LN implies that the general solution is

$$y(t) = c_1\, e^{-t/3} + c_2\, te^{-t/3}. \qquad \triangleleft$$

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Application: Reduction of the order method.

Proof case $a_1^2 - 4a_0 = 0$:

Recall: The characteristic equation is $r^2 + a_1 r + a_0 = 0$, and its solutions are $r_{\pm} = (1/2)\big[-a_1 \pm \sqrt{a_1^2 - 4a_0}\big]$.

The hypothesis $a_1^2 = 4a_0$ implies $r_+ = r_- = -a_1/2$.

So, the solution r_+ of the characteristic equation satisfies both

$$r_+^2 + a_1 r_+ + a_0 = 0, \qquad 2r_+ + a_1 = 0.$$

It is clear that $y_1(t) = e^{r_+ t}$ is solutions of the differential equation.

A second solution y_2 not proportional to y_1 can be found as follows: (D'Alembert $\sim$ 1750.)

Express: $y_2(t) = v(t)\, y_1(t)$, and find the equation that function v satisfies from the condition $y_2'' + a_1 y_2' + a_0 y_2 = 0$.

Application: Reduction of the order method.

Recall: $y_2 = vy_1$ and $y_2'' + a_1 y_2' + a_0 y_2 = 0$. So, $y_2 = v e^{r_+ t}$ and

$$y_2' = v' e^{r_+ t} + r_+ v e^{r_+ t}, \qquad y_2'' = v'' e^{r_+ t} + 2r_+ v' e^{r_+ t} + r_+^2 v e^{r_+ t}.$$

Introducing this information into the differential equation

$$\big[v'' + 2r_+ v' + r_+^2 v\big]\, e^{r_+ t} + a_1 \big[v' + r_+ v\big]\, e^{r_+ + t} + a_0 v\, e^{r_+ t} = 0.$$

$$\big[v'' + 2r_+ v' + r_+^2 v\big] + a_1 \big[v' + r_+ v\big] + a_0 v = 0$$

$$v'' + \big(2r_+ + a_1\big)\, v' + \big(r_+^2 + a_1 r_+ + a_0\big)\, v = 0$$

Recall that r_+ satisfies: $r_+^2 + a_1 r_+ + a_0 = 0$ and $2r_+ + a_1 = 0$.

$$v'' = 0 \quad \Rightarrow \quad v = (c_1 + c_2 t) \quad \Rightarrow \quad y_2 = (c_1 + c_2 t)\, e^{r_+ t}.$$

Application: Reduction of the order method.

Recall: We have obtained that $y_2(t) = (c_1 + c_2 t)\, e^{r_+ t}$.

If $c_2 = 0$, then $y_2 = c_1 e^{r_+ t}$ and $y_1 = e^{r_+ t}$ are linearly dependent functions.

If $c_2 \neq 0$, then $y_2 = (c_1 + c_2 t)\, e^{r_+ t}$ and $y_1 = e^{r_+ t}$ are linearly independent functions.

Simplest choice: $c_1 = 0$ and $c_2 = 1$. Then, a fundamental solution set to the differential equation is

$$y_1(t) = e^{r_+ t}, \qquad y_2(t) = t\, e^{r_+ t} \qquad \square$$

The general solution to the differential equation is

$$y(t) = \tilde{c}_1\, e^{r_+ t} + \tilde{c}_2\, t e^{r_+ t}.$$

Application: Reduction of the order method.

Example

Find the solution to the initial value problem

$$9y'' + 6y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = \frac{5}{3}.$$

Solution: The solutions of $9r^2 + 6r + 1 = 0$, are $r_+ = r_- = -\frac{1}{3}$.

The Theorem above says that the general solution is

$$y(t) = c_1 e^{-t/3} + c_2 t e^{-t/3} \;\Rightarrow\; y'(t) = -\frac{c_1}{3} e^{-t/3} + c_2 \Big(1 - \frac{t}{3}\Big)\, e^{-t/3}.$$

The initial conditions imply that

$$\left.\begin{aligned} 1 &= y(0) = c_1, \\ \frac{5}{3} &= y'(0) = -\frac{c_1}{3} + c_2 \end{aligned}\right\} \quad \Rightarrow \quad c_1 = 1, \qquad c_2 = 2.$$

We conclude that $y(t) = (1 + 2t)\, e^{-t/3}$. ◁