

Review: On solutions of $y'' + a_1 y' + a_0 y = 0$.

Summary:

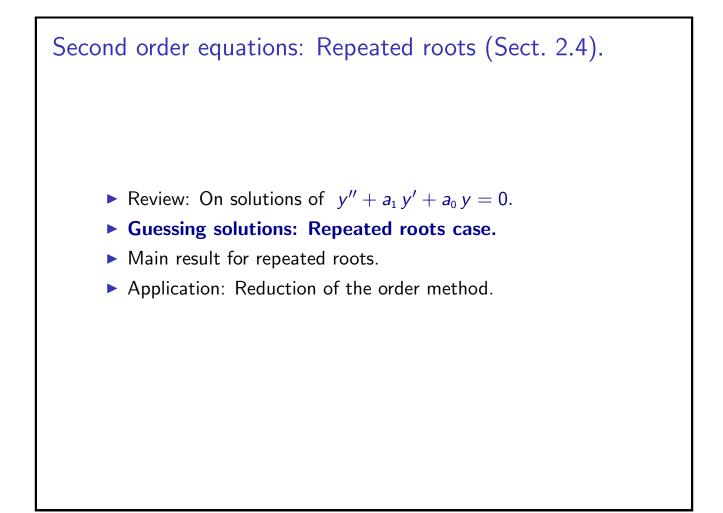
Given constants a_1 , $a_0 \in \mathbb{R}$, consider the differential equation

 $y'' + a_1 y' + a_0 y = 0$

with characteristic polynomial having roots

$$r_{\pm} = -rac{a_1}{2} \pm rac{1}{2}\sqrt{a_1^2 - 4a_0}.$$

(1) If $a_1^2 - 4a_0 > 0$, then $y_1(t) = e^{r_+ t}$ and $y_2(t) = e^{r_- t}$. (2) If $a_1^2 - 4a_0 < 0$, then introducing $\alpha = -\frac{a_1}{2}$, $\beta = \frac{1}{2}\sqrt{4a_0 - a_1^2}$, $y_1(t) = e^{\alpha t} \cos(\beta t)$, $y_2(t) = e^{\alpha t} \sin(\beta t)$. (3) If $a_1^2 - 4a_0 = 0$, then $y_1(t) = e^{-\frac{a_1}{2}t}$.



Guessing solutions: Repeated roots case.

Remark:

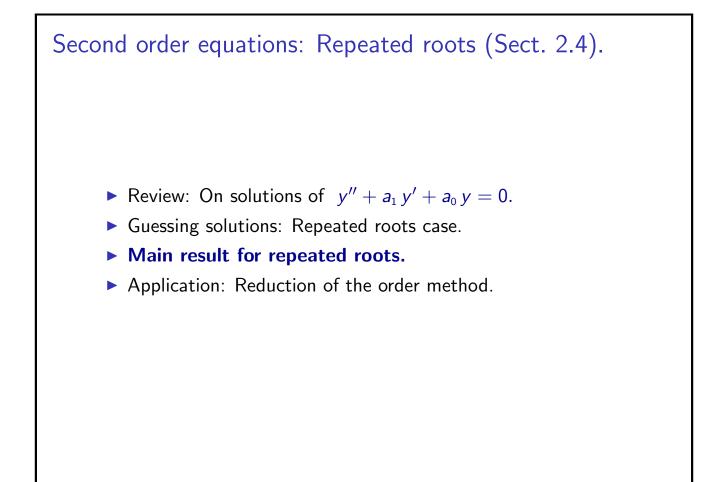
- Case (3), where $4a_0 a_1^2 = 0$ guessed as $\beta \to 0$ in case (2).
- Study solutions in the case (2) as $\beta \rightarrow 0$ for fixed t.
- Since $\cos(\beta t) \rightarrow 1$ as $\beta \rightarrow 0$, we conclude that

$$y_{1\beta}(t) = e^{-\frac{a_1}{2}t} \cos(\beta t) \to e^{-\frac{a_1}{2}t} = y_1(t).$$

• Since $\frac{\sin(\beta t)}{\beta t} \to 1$ as $\beta \to 0$, that is, $\sin(\beta t) \to \beta t$, $y_{2\beta}(t) = e^{-\frac{\vartheta_1}{2}t} \sin(\beta t) \to \beta t e^{-\frac{\vartheta_1}{2}t} \to 0$.

Guessed solution: y₂(t) = t y₁(t).
 Introducing y₂ in the differential equation one obtains: Yes.

• Hence, y_2 and y_1 are a fundamental solutions in case (3).



Main result for repeated roots.

Theorem

If a_1 , $a_0 \in R$ satisfy that $a_1^2 = 4a_0$, then the functions

$$y_1(t) = e^{-\frac{a_1}{2}t}, \qquad y_2(t) = t e^{-\frac{a_1}{2}t},$$

are a fundamental solution set for the differential equation

$$y'' + a_1 y' + a_0 y = 0.$$

Example

Find the general solution of 9y'' + 6y' + y = 0.

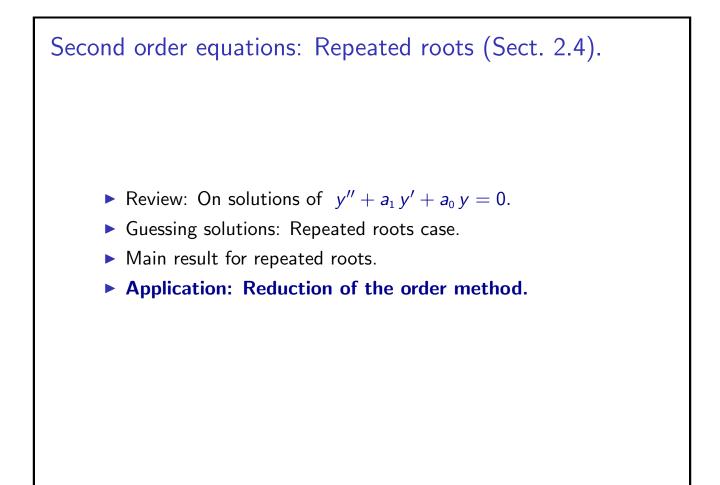
Solution: The characteristic equation is $9r^2 + 6r + 1 = 0$, so

$$r_{\pm} = rac{1}{(2)(9)} \left[-6 \pm \sqrt{36 - 36}
ight] \quad \Rightarrow \quad r_{\pm} = -rac{1}{3} \cdot r_{\pm}$$

Theorem (2.1.7) in LN implies that the general solution is

$$y(t) = c_1 e^{-t/3} + c_2 t e^{-t/3}.$$

 \triangleleft



Application: Reduction of the order method.

Proof case $a_1^2 - 4a_0 = 0$: Recall: The characteristic equation is $r^2 + a_1r + a_0 = 0$, and its solutions are $r_{\pm} = (1/2) \left[-a_1 \pm \sqrt{a_1^2 - 4a_0} \right]$. The hypothesis $a_1^2 = 4a_0$ implies $r_{\pm} = r_{-} = -a_1/2$. So, the solution r_{\pm} of the characteristic equation satisfies both

 $r_{+}^{2} + a_{1}r_{+} + a_{0} = 0,$ $2r_{+} + a_{1} = 0.$

It is clear that $y_1(t) = e^{r_t t}$ is solutions of the differential equation.

A second solution y_2 not proportional to y_1 can be found as follows: (D'Alembert ~ 1750.)

Express: $y_2(t) = v(t) y_1(t)$, and find the equation that function v satisfies from the condition $y_2'' + a_1 y_2' + a_0 y_2 = 0$.

Application: Reduction of the order method. Recall: $y_2 = vy_1$ and $y_2'' + a_1y_2' + a_0y_2 = 0$. So, $y_2 = ve^{r_+t}$ and $y_2' = v'e^{r_+t} + r_+ve^{r_+t}$, $y_2'' = v''e^{r_+t} + 2r_+v'e^{r_+t} + r_+^2ve^{r_+t}$. Introducing this information into the differential equation $[v'' + 2r_+v' + r_+^2v]e^{r_+t} + a_1[v' + r_+v]e^{r_++t} + a_0ve^{r_+t} = 0$. $[v'' + 2r_+v' + r_+^2v] + a_1[v' + r_+v] + a_0v = 0$ $v'' + (2r_+ + a_1)v' + (r_+^2 + a_1r_+ + a_0)v = 0$ Recall that r_+ satisfies: $r_+^2 + a_1r_+ + a_0 = 0$ and $2r_+ + a_1 = 0$. $v'' = 0 \implies v = (c_1 + c_2t) \implies y_2 = (c_1 + c_2t)e^{r_+t}$.

Application: Reduction of the order method.

Recall: We have obtained that $y_2(t) = (c_1 + c_2 t) e^{r_t t}$.

If $c_2 = 0$, then $y_2 = c_1 e^{r_+ t}$ and $y_1 = e^{r_+ t}$ are linearly dependent functions.

If $c_2 \neq 0$, then $y_2 = (c_1 + c_2 t) e^{r_1 t}$ and $y_1 = e^{r_1 t}$ are linearly independent functions.

Simplest choice: $c_1 = 0$ and $c_2 = 1$. Then, a fundamental solution set to the differential equation is

$$y_1(t) = e^{r_+ t}, \qquad y_2(t) = t e^{r_+ t}$$

The general solution to the differential equation is

$$y(t) = \tilde{c}_1 e^{r_+ t} + \tilde{c}_2 t e^{r_+ t}.$$

Application: Reduction of the order method.

Example

Find the solution to the initial value problem

$$9y'' + 6y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = \frac{5}{3}.$$

Solution: The solutions of $9r^2 + 6r + 1 = 0$, are $r_* = r_- = -\frac{1}{3}.$
The Theorem above says that the general solution is
 $y(t) = c_1 e^{-t/3} + c_2 t e^{-t/3} \Rightarrow y'(t) = -\frac{c_1}{3} e^{-t/3} + c_2 \left(1 - \frac{t}{3}\right) e^{-t/3}.$
The initial conditions imply that
 $1 = y(0) = c_1, \\ \frac{5}{3} = y'(0) = -\frac{c_1}{3} + c_2 \right\} \Rightarrow c_1 = 1, \qquad c_2 = 2.$
We conclude that $y(t) = (1 + 2t) e^{-t/3}.$