Variable coefficients second order linear ODE (Sect. 2.1).

- Second order linear ODE.
- Existence and uniqueness of solutions.
- Operator notation.
- Linear operator and Superposition property.
- Linearly dependent and independent functions.
- General and fundamental solutions.
- The Wronskian of two functions.
- Abel's theorem on the Wronskian.

Second order linear differential equations.

Definition
Given functions a_1, a_0, $b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t) \tag{1}$$

is called a *second order linear* differential equation with *variable coefficients*. The equation in (1) is called *homogeneous* iff for all $t \in \mathbb{R}$ holds

$$b(t) = 0.$$

The equation in (1) is called of *constant coefficients* iff a_1, a_0, and b are constants.

Remark: The notion of an homogeneous equation presented here is not the same as the Euler homogeneous from the previous chapter.

Second order linear differential equations.

Example

(a) A second order, linear, homogeneous, constant coefficients equation is
$$y'' + 5y' + 6 = 0.$$

(b) A second order order, linear, constant coefficients, non-homogeneous equation is
$$y'' - 3y' + y = 1.$$

(c) A second order, linear, non-homogeneous, variable coefficients equation is
$$y'' + 2t\, y' - \ln(t)\, y = e^{3t}.$$

(d) Newton's second law of motion ($ma = f$) for point particles of mass m moving in one space dimension under a force $f : \mathbb{R} \to \mathbb{R}$ is given by
$$m\, y''(t) = f(t).$$
◁

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Existence and uniqueness of solutions.

Theorem (Variable coefficients)

If the functions a_1, a_0, $b : I \to \mathbb{R}$ are continuous on I, $t_0 \in I$, and y_0, $y_1 \in \mathbb{R}$ are any constants, then there exist a unique solution $y : I \to \mathbb{R}$ to the initial value problem

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \quad y(t_0) = y_0, \quad y'(t_0) = \hat{y}_0.$$

Remarks:

- A proof of this Theorem can be constructed with a generalization of the fixed-point arguments and the Picard-Lindelöf iteration used in first order nonlinear equations.
- Two integrations must be done to find solutions to second order linear. Therefore, initial value problems with two initial conditions can have a unique solution.

Existence and uniqueness of solutions.

Example

Find the longest interval $I \in \mathbb{R}$ such that there exists a unique solution to the initial value problem

$$(t-1)y'' - 3ty' + 4y = t(t-1), \qquad y(-2) = 2, \qquad y'(-2) = 1.$$

Solution: We first write the equation above in the form given in the Theorem above,

$$y'' - \frac{3t}{t-1}\, y' + \frac{4}{t-1}\, y = t.$$

The intervals where the hypotheses in the Theorem above are satisfied, that is, where the equation coefficients are continuous, are $I_1 = (-\infty, 1)$ and $I_2 = (1, \infty)$. Since the initial condition belongs to I_1, the solution domain is

$$I_1 = (-\infty, 1).$$

$\lhd$

Existence and uniqueness of solutions.

Remarks:

- Every solution of the first order linear equation

$$y' + a(t)\, y = 0$$

is given by $y(t) = c\, e^{-A(t)}$, with $A(t) = \int a(t)\, dt$.

- All solutions above are proportional to each other:

$$y_1(t) = c_1\, e^{-A(t)}, \quad y_2(t) = c_2\, e^{-A(t)} \;\Rightarrow\; y_1(t) = \frac{c_1}{c_2}\, y_2(t)$$

Remark: The above statement is *not true* for solutions of second order, linear, homogeneous equations, $y'' + a_1(t)\, y' + a_0(t) y = 0$. Before we prove this statement we need few definitions:

- Proportional functions (linearly dependent).
- Wronskian of two functions.

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Operator notation.

Notation: Given functions a_1, a_0, denote

$$L(y) = y'' + a_1(t)\, y' + a_0(t)\, y.$$

Therefore, the differential equation

$$y'' + a_1(t)\, y' + a_(t)\, y = f(t)$$

can be written as

$$L(y) = f.$$

The homogeneous equation can be written as

$$L(y) = 0.$$

The function L acting on a function y is called an operator.

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Linear operator and superposition property.

Remark: The operator $L(y) = y'' + a_1(t)\, y' + a_0(t)\, y$ is a linear function of y.

Theorem

For every continuously differentiable functions $y_1, y_2 : I \to \mathbb{R}$ and every $c_1, c_2 \in \mathbb{R}$ holds that

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2).$$

Proof:

$$L(c_1y_1+c_2y_2) = (c_1y_1+c_2y_2)''+a_1(t)\,(c_1y_1+c_2y_2)'+a_0(t)\,(c_1y_1+c_2y_2)$$

$$L(c_1y_1 + c_2y_2) = \big(c_1y_1'' + a_1(t)\,c_1y_1' + a_0(t)\,c_1y_1\big) + \big(c_2y_2'' + a_1(t)\,c_2y_2' + a_0(t)\,c_2y_2\big)$$

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2).$$

□

Linear operator and superposition property.

Theorem

If L is a linear operator and y_1, y_2 are solutions of the homogeneous equations $L(y_1) = 0$, $L(y_2) = 0$, then for every constants c_1, c_2 holds that $L(c_1\, y_1 + c_2\, y_2) = 0$.

Proof: Verify that the function $y = c_1y_1 + c_2y_2$ satisfies $L(y) = 0$ for every constants c_1, c_2, that is,

$$L(y) = L(c_1y_1 + c_2y_2) = c_1\, L(y_1) + c_2\, L(y_2) = 0$$

□

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Linearly dependent and independent functions.

Definition

Two continuous functions $y_1, y_2 : (t_1, t_2) \subset \mathbb{R} \to \mathbb{R}$ are called *linearly dependent, (ld),* on the interval (t_1, t_2) iff there exists a constant c such that for all $t \in I$ holds

$$y_1(t) = c\, y_2(t).$$

The two functions are called *linearly independent, (li),* on the interval (t_1, t_2) iff they are not linearly dependent.

Remarks:

- $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ are ld $\Leftrightarrow$ there exist constants c_1, c_2, not both zero, such that $c_1\, y_1(t) + c_2\, y_2(t) = 0$ for all $t \in (t_1, t_2)$.
- $y_1, y_2 : (t_1, t_2) \to \mathbb{R}$ are li $\Leftrightarrow$ the only constants c_1, c_2, solutions of $c_1\, y_1(t) + c_2\, y_2(t) = 0$ for all $t \in (t_1, t_2)$ are $c_1 = c_2 = 0$.

Linearly dependent and independent functions.

Example

(a) Show that $y_1(t) = \sin(t)$, $y_2(t) = 2\sin(t)$ are ld.

(b) Show that $y_1(t) = \sin(t)$, $y_2(t) = t\sin(t)$ are li.

Solution:

Case (a): Trivial. $y_2 = 2y_1$.

Case (b): Find constants c_1, c_2 such that for all $t \in \mathbb{R}$ holds

$$c_1 \sin(t) + c_2 t \sin(t) = 0 \quad \Leftrightarrow \quad (c_1 + c_2 t)\sin(t) = 0.$$

Evaluating at $t = \pi/2$ and $t = 3\pi/2$ we obtain

$$c_1 + \frac{\pi}{2}\, c_2 = 0, \quad c_1 + \frac{3\pi}{2}\, c_2 = 0 \quad \Rightarrow \quad c_1 = 0, \quad c_2 = 0.$$

We conclude: The functions y_1 and y_2 are li. $\triangleleft$

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General and fundamental solutions.

Theorem (General solution)

If y_1 and y_2 are linearly independent solutions of the equations $L(y_1) = 0$ and $L(y_2) = 0$, where $L(y) = y'' + a_1\, y' + a_0\, y$, and a_1, a_2 are continuous functions, then there exist unique constants c_1, c_2 such that every solution y of the differential equation $L(y) = 0$ can be written as a linear combination

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t).$$

Definition

Two linearly independent solutions y_1, y_2 of $L(y_1) = 0$, $L(y_2) = 0$, with L a linear operator, are called *fundamental solutions*.

Given any two fundamental solutions y_1, y_2, and arbitrary constants c_1, c_2, the *general solution* of the homogeneous equation $L(y_{\text{gen}}) = 0$ is the set of all functions given by the expression

$$y_{\text{gen}}(t) = c_1\, y_1(t) + c_2\, y_2(t).$$

General and fundamental solutions.

Idea of the Proof: Given any fundamental solution pair, y_1, y_2, any other solution to $L(y) = 0$ must be

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t),$$

The existence-uniquenes theorem of IVP implies: Good parameters to label all solutions to $L(y) = 0$ are the initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = \hat{y}_0.$$

Show that the map between y_0, $\hat{y}_0$ and c_1, c_2 is invertible.

$$y_0 = c_1\, y_1(t_0) + c_2\, y_2(t_0)$$
$$\hat{y}_0 = c_1\, y_1'(t_0) + c_2\, y_2'(t_0).$$

This map is invertible iff holds

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)\, y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0.$$

Study: $W_{y_1,y_2} = y_1\, y_2' - y_1' y_2$. Conclusion: The map is invertibe. $\square$

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The Wronskian of two functions.

Remark:

(a) The Wronskian is a function computed out of two other functions.

(b) The Wronskian values provide information about the linear dependence of the two functions used to compute it.

Definition

The *Wronskian* of functions y_1, $y_2 : I \to \mathbb{R}$ is the function

$$W_{y_1 y_2}(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Remark:

- If $A(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$, then $W_{y_1 y_2}(t) = \det\big(A(t)\big)$.
- An alternative notation is: $W_{y_1 y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

The Wronskian of two functions.

Example

Find the Wronskian of the functions:

(a) $y_1(t) = \sin(t)$ and $y_2(t) = 2\sin(t)$. (ld)

(b) $y_1(t) = \sin(t)$ and $y_2(t) = t\sin(t)$. (li)

Solution:

Case (a): $W_{y_1y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin(t) & 2\sin(t) \\ \cos(t) & 2\cos(t) \end{vmatrix}$. Therefore,

$$W_{y_1y_2}(t) = \sin(t)2\cos(t) - \cos(t)2\sin(t) \quad \Rightarrow \quad W_{y_1y_2}(t) = 0.$$

Case (b): $W_{y_1y_2} = \begin{vmatrix} \sin(t) & t\sin(t) \\ \cos(t) & \sin(t) + t\cos(t) \end{vmatrix}$. Therefore,

$$W_{y_1y_2}(t) = \sin(t)\big[\sin(t) + t\cos(t)\big] - \cos(t)t\sin(t).$$

We obtain $W_{y_1y_2}(t) = \sin^2(t)$. $\triangleleft$

The Wronskian of two functions.

Remark: The Wronskian is related to linear dependence.

Theorem (Wronskian I)

If the the continuously differentiable functions y_1, y_2 on an interval I are linearly dependent, then $W_{y_1y_2}(t) = 0$ for all $t \in I$.

Remark: The converse is not true.
Example: $y_1(t) = t^2$, and $y_2(t) = |t|\, t$.

Corollary (Wronskian I)

If the Wronskian $W_{y_1y_2}(t_0) \neq 0$ at a single point $t_0 \in I$, then the functions y_1, $y_2 : I \to \mathbb{R}$ are linearly independent.

Remark: The Corollary is the negative of the Theorem.

The Wronskian of two functions.

Remark: The Wronskian is related to linear dependence.

Theorem (Wronskian II)

Let y_1, $y_2 : I \to \mathbb{R}$ be both solutions of $L(y) = 0$ on I. If there exists one point $t_0 \in I$ such that $W_{y_1 y_2}(t_0) = 0$, then y_1 y_2 are linearly dependent.

Remark: The proof of this Theorem requires Abel's Theorem.

Corollary (Wronskian II)

Let y_1, $y_2 : I \to \mathbb{R}$ be both solutions of $L(y) = 0$ on I. If y_1 y_2 are linearly independent on I, then their Wronskian $W_{y_1 y_2}(t) \neq 0$ for all $t \in I$.

Remark: This Corollary is needed to prove the Theorem on General solutions.

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Abel's theorem on the Wronskian.

Theorem (Abel)

If a_1, $a_0 : (t_1, t_2) \to \mathbb{R}$ are continuous functions and y_1, y_2 are continuously differentiable solutions of the equation

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0,$$

then the Wronskian $W_{y_1 y_2}$ is a solution of the equation

$$W'_{y_1 y_2}(t) + a_1(t)\, W_{y_1 y_2}(t) = 0.$$

Therefore, for any $t_0 \in (t_1, t_2)$, the Wronskian $W_{y_1 y_2}$ is given by

$$W_{y_1 y_2}(t) = W_{y_1 y_2}(t_0)\, e^{-A(t)} \qquad A(t) = \int_{t_0}^{t} a_1(s)\, ds.$$

Remarks: If the the Wronskian of two solutions vanishes at the initial time, then it vanishes at all times.

Abel's theorem on the Wronskian.

Example

Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

Solution: Write the equation as in Abel's Theorem,

$$y'' - \Big(\frac{2}{t} + 1\Big)\, y' + \Big(\frac{2}{t^2} + \frac{1}{t}\Big)\, y = 0.$$

Abel's Theorem says that the Wronskian satisfies the equation

$$W'_{y_1 y_2}(t) - \Big(\frac{2}{t} + 1\Big)\, W_{y_1 y_2}(t) = 0.$$

This is a first order, linear equation for $W_{y_1 y_2}$. The integrating factor method implies

$$A(t) = -\int_{t_0}^{t} \Big(\frac{2}{s} + 1\Big)\, ds = -2\ln\Big(\frac{t}{t_0}\Big) - (t - t_0)$$

Abel's theorem on the Wronskian.

Example

Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

Solution: $A(t) = -2\ln\Big(\frac{t}{t_0}\Big) - (t - t_0) = \ln\Big(\frac{t_0^2}{t^2}\Big) - (t - t_0).$

The integrating factor is $\mu = \frac{t_0^2}{t^2}\, e^{-(t-t_0)}$. Therefore,

$$\Big[\mu(t) W_{y_1 y_2}(t)\Big]' = 0 \quad \Rightarrow \quad \mu(t) W_{y_1 y_2}(t) - \mu(t_0) W_{y_1 y_2}(t_0) = 0$$

so, the solution is $W_{y_1 y_2}(t) = W_{y_1 y_2}(t_0)\, \frac{t^2}{t_0^2}\, e^{(t-t_0)}$.

Denoting $c = \big(W_{y_1 y_2}(t_0)/t_0^2\big)\, e^{-t_0}$, then $W_{y_1 y_2}(t) = c\, t^2 e^t$. $\lhd$

General and fundamental solutions.

Example

Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are fundamental solutions of

$$2t^2\, y'' + 3t\, y' - y = 0.$$

Solution: First show that y_1 is a solution:

$$y_1 = t^{1/2}, \quad y_1' = \frac{1}{2}\, t^{-1/2}, \quad y_1'' = -\frac{1}{4}\, t^{-3/2},$$

$$2t^2 \left(-\frac{1}{4}\, t^{-\frac{3}{2}}\right) + 3t \left(\frac{1}{2} t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}} = -\frac{1}{2} t^{\frac{1}{2}} + \frac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}} = 0.$$

Now show that y_2 is a solution:

$$y_2 = t^{-1}, \quad y_2' = -t^{-2}, \quad y_2'' = 2t^{-3},$$

$$2t^2 \left(2t^{-3}\right) + 3t \left(-t^{-2}\right) - t^{-1} = 4t^{-1} - 3t^{-1} - t^{-1} = 0.$$

General and fundamental solutions.

Example

Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are fundamental solutions of

$$2t^2\, y'' + 3t\, y' - y = 0.$$

Solution: We show that y_1, y_2 are linearly independent.

$$W_{y_1 y_2}(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}\, t^{-1/2} & -t^{-2} \end{vmatrix}.$$

$$W_{y_1 y_2}(t) = -t^{1/2}\, t^{-2} - \frac{1}{2} t^{-1/2}\, t^{-1} = -t^{-3/2} - \frac{1}{2} t^{-3/2}$$

$$W_{y_1 y_2}(t) = -\frac{3}{3}\, t^{-3/2} \quad \Rightarrow \quad y_1,\ y_2 \text{ li.} \qquad \triangleleft$$