Review for Exam 1.

- Exam in webwork, in CC-415, Computer Center.
- 5 grading attempts per problem.
- 5 to 6 problems, 60 minutes.
- Problems similar to webwork problems.
- Integration table provided in the handout.
- No notes, no books, no calculators, no phones.
- MLC Exam Review for MTH 235, today, 7:30pm, ANH-1281.
- MTH 340 Exam 1 covers:
  - Linear equations (1.1), (1.2).
  - Bernoulli equation (1.2).
  - Separable equations (1.3).
  - Euler homogeneous equations (1.3).
  - Exact equations (1.4).
  - Exact equations with integrating factors (1.4).
  - Applications (1.5).
  - Picard-Lindelöf iteration (1.6).

Exam overview

Remark:

- Exam problems will be: Solve this equation. We don't tell you if the equation is linear, separable, etc. You must find that out.
- If you know what type of equation is, then the equation is simple to solve.
- The difficult part in Exam 1 is to know what type of equation is the one you have to solve.

Exam overview

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)$.)
2. Bernoulli equations.
   (Just by looking at it: $y' + a(t)\, y = b(t)\, y^n$.)
3. Separable equations.
   (Few manipulations: $h(y)\, y' = g(t)$.)
4. Euler homogeneous equations.
   (Several manipulations: $y' = F(y/t)$.)
5. Exact equations.
   (Check one equation: $N\, y' + M = 0$, and $\partial_t N = \partial_y M$.)
6. Exact equation with integrating factor.
   (Could be very complicated to check.)

Review Exam 1.

Example

Find every solution y to the equation $(t^2 + y^2)(t + y\, y') + 2 = 0$.

Solution: Rewrite the equation in a more standard way:

$$(t^2 + y^2) y\, y' + (t^2 + y^2) t + 2 = 0 \quad \Leftrightarrow \quad y' = -\frac{(t^2 + y^2) t + 2}{(t^2 + y^2) y}.$$

Not linear. Not Bernoulli. Not Separable. Not Euler homogeneous. So the equation must be exact or exact with integrating factor.

$$N = t^2 y + y^3 \quad \Rightarrow \quad \partial_t N = 2ty.$$

$$M = t^3 + ty^2 + 2 \quad \Rightarrow \quad \partial_y M = 2ty.$$

The equation is exact: $\partial_t N = \partial_y M$.

Review Exam 1.

Example

Find every solution y to the equation $(t^2 + y^2)(t + y\,y') + 2 = 0$.

Solution: $\partial_t N = \partial_y M$, $\quad \big[(t^2 + y^2)y\big]\, y' + \big[(t^2 + y^2)t + 2\big] = 0$.

There exits a potential function ψ such that

$$\partial_y \psi = N, \qquad \partial_t \psi = M.$$

$$\partial_y \psi = t^2 y + y^3 \quad \Rightarrow \quad \psi = t^2\,\frac{y^2}{2} + \frac{y^4}{4} + g(t).$$

$$t y^2 + g'(t) = \partial_t \psi = M = t^3 + t y^2 + 2.$$

$$g'(t) = t^3 + 2 \quad \Rightarrow \quad g(t) = \frac{t^4}{4} + 2t.$$

$$\psi(t, y) = \frac{1}{2}\,t^2 y^2 + \frac{y^4}{4} + \frac{t^4}{4} + 2t, \qquad \psi(t, y(t)) = c. \qquad \lhd$$

Review Exam 1.

Example

Find the explicit solution y to the IVP

$$y' = \frac{t(t^2 + e^t)}{4y^3}, \qquad y(0) = -\sqrt{2}.$$

Solution: Not linear. Bernoulli with $n = -3$. Numerator depends only on t, denominator depends only on y: Separable.

$$4y^3\, y' = t^3 + te^t \quad \Rightarrow \quad \int 4y^3\, y'\, dt = \int (t^3 + te^t)\, dt + c$$

The usual substitution: $u = y(t)$ implies $du = y'(t)\, dt$,

$$\int 4u^3\, du = \int (t^3 + te^t)\, dt + c \quad \Rightarrow \quad u^4 = \frac{t^4}{4} + \int te^t\, dt + c.$$

Review Exam 1.

Example

Find the explicit solution y to the IVP

$$y' = \frac{t(t^2 + e^t)}{4y^3}, \qquad y(0) = -\sqrt{2}.$$

Solution: Recall: $u^4 = \dfrac{t^4}{4} + \displaystyle\int te^t\,dt + c$. Integration by parts:

$$\left.\begin{array}{ll} f = t, & g' = e^t, \\ f' = 1, & g = e^t, \end{array}\right\} \quad \Rightarrow \quad \int te^t\,dt = te^t - \int e^t\,dt = (t-1)e^t.$$

We obtain: $y^4(t) = \dfrac{t^4}{4} + (t-1)e^t + c$. The initial condition:

$$\left(-\sqrt{2}\right)^4 = 0 + (0-1) + c \quad \Rightarrow \quad 4 = -1 + c \quad \Rightarrow \quad c = 5.$$

We conclude: $y^4(t) = \dfrac{t^4}{4} + (t-1)e^t + 5$. Implicit form.

Review Exam 1.

Example

Find the explicit solution y to the IVP

$$y' = \frac{t(t^2 + e^t)}{4y^3}, \qquad y(0) = -\sqrt{2}.$$

Solution: Recall: $y^4(t) = \dfrac{t^4}{4} + (t-1)e^t + 5$. Implicit form.

The explicit form of the solution is one of:

$$y(t) = \pm\left[\frac{t^4}{4} + (t-1)e^t + 5\right]^{1/4}.$$

The initial condition implies $y(0) = -\sqrt{2} < 0$.

We conclude that the unique solution to the IVP is

$$y(t) = -\left[\frac{t^4}{4} + (t-1)e^t + 5\right]^{1/4}. \qquad \triangleleft$$

Review Exam 1.

Example

Find every solution y of the equation $y' = \dfrac{3y^2 - t^2}{2ty}$.

Solution: Not linear. Bernoulli $n = -1$: $y' = \dfrac{3y}{2t} - \dfrac{t}{2y}$.
Not separable. Every term on the right hand side is of the form $t^n y^m$ with $n + m = 2$. Euler homogeneous.

$$y' = \frac{3y^2 - t^2}{2ty} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{1}{t^2}\right)} \quad \Rightarrow \quad y' = \frac{3\left(\frac{y}{t}\right)^2 - 1}{2\left(\frac{y}{t}\right)}.$$

We introduce the change of unknown:

$$v = \frac{y}{t} \quad \Rightarrow \quad y = t\,v \quad \Rightarrow \quad y' = v + t\,v'.$$

Review Exam 1.

Example

Find every solution y of the equation $y' = \dfrac{3y^2 - t^2}{2ty}$.

Solution: $y' = \dfrac{3\left(\frac{y}{t}\right)^2 - 1}{2\left(\frac{y}{t}\right)}$, $\quad v = \dfrac{y}{t}$, $\quad y' = v + t\,v'$.

$$v + t\,v' = \frac{3v^2 - 1}{2v} \quad \Rightarrow \quad t\,v' = \frac{3v^2 - 1}{2v} - v = \frac{3v^2 - 1 - 2v^2}{2v}$$

$$t\,v' = \frac{v^2 - 1}{2v} \quad \Rightarrow \quad \frac{2v}{v^2 - 1}\,v' = \frac{1}{t}.$$

This is a separable equation for v: $\displaystyle\int \frac{2v}{v^2 - 1}\,v'\,dt = \int \frac{1}{t}\,dt + c.$

Review Exam 1.

Example

Find every solution y of the equation $y' = \dfrac{3y^2 - t^2}{2ty}$.

Solution: $\displaystyle\int \frac{2v}{v^2-1}\, v'\, dt = \int \frac{1}{t}\, dt + c.$

The substitution $u = v^2 - 1$ implies $du = 2v\, v'\, dt$. So,

$$\int \frac{du}{u} = \int \frac{1}{t}\, dt + c \quad \Rightarrow \quad \ln(|u|) = \ln(|t|) + c \quad \Rightarrow \quad |u| = c_1\, |t|.$$

where $c_1 = e^c$. Substitute back: $|v^2 - 1| = c_1\, |t|$. Finally, $v = y/t$,

$$\left|\frac{y^2}{t^2} - 1\right| = c_1\, |t| \quad \Rightarrow \quad |y^2 - t^2| = c_1\, |t|^3.$$

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Review Exam 1.

Example

A water tank initially has $V_0 = 100$ liters of water with Q_0 grams of salt. At $t_0 = 0$ fresh water is poured into the tank. The salt in the tank is always well mixed. Find the rates r_i and r_o such that:

(a) The tank water volume is constant.

(b) The time to reduce the salt in the tank to one percent of the initial value is $t_1 = 25$ min.

Solution:

Part (a): Water volume constant implies $r_i = r_o = r$. Then $V'(t) = 0$, so $V(t) = V_0$.

Part (b): First find the salt in the tank $Q(t)$: $\dfrac{dQ}{dt} = r_i q_i - r_o q_o(t)$.
Incoming fresh water: $q_i = 0$. Mixing: $q_o(t) = Q(t)/V(t)$.

$$\frac{dQ}{dt} = -\frac{r}{V_0}\, Q(t) \quad \Rightarrow \quad Q(t) = Q_0\, e^{-rt/V_0}.$$

Review Exam 1.

Example

A water tank initially has $V_0 = 100$ liters of water with Q_0 grams of salt. At $t_0 = 0$ fresh water is poured into the tank. The salt in the tank is always well mixed. Find the rates r_i and r_o such that:

(a) The tank water volume is constant.

(b) The time to reduce the salt in the tank to one percent of the initial value is $t_1 = 25$ min.

Solution: Recall: $Q(t) = Q_0\, e^{-rt/V_0}$. Condition for r:

$$Q(t_1) = \frac{Q_0}{100} \;\Rightarrow\; Q_0\, e^{(-rt_1/V_0)} = \frac{Q_0}{100} \;\Rightarrow\; -\frac{rt_1}{V_0} = \ln\Bigl(\frac{1}{100}\Bigr).$$

$$\frac{rt_1}{V_0} = \ln(100) \;\Rightarrow\; r = \frac{V_0}{t_1}\ln(100) \;\Rightarrow\; r = 4\ln(100). \qquad \triangleleft$$

Review Exam 1.

Example

Find the solution y to the IVP

$$y' = \frac{2}{t}\,y - \frac{\sin(t)}{t}\,y^2, \qquad y(2\pi) = 2\pi, \qquad t > 0.$$

Solution: Not linear. Bernoulli for $n = 2$. Divide by y^2.

$$\frac{y'}{y^2} - \frac{2}{t}\frac{1}{y} = -\frac{\sin(t)}{t}, \qquad v = \frac{1}{y} \quad\Rightarrow\quad v' = -\frac{y'}{y^2}.$$

$$-v' - \frac{2}{t}\,v = -\frac{\sin(t)}{t} \quad\Rightarrow\quad v' + \frac{2}{t}\,v = \frac{\sin(t)}{t}.$$

We solve the linear equation with the integrating factor method.

$$A(t) = \int \frac{2}{t}\,dt = 2\ln(t) = \ln\bigl(t^2\bigr) \quad\Rightarrow\quad \mu(t) = t^2.$$

Review Exam 1.

Example

Find the solution y to the IVP

$$y' = \frac{2}{t}\,y - \frac{\sin(t)}{t}\,y^2, \qquad y(2\pi) = 2\pi, \qquad t > 0.$$

Solution: Recall: $\mu(t) = t^2$. Then,

$$t^2\Big(v' + \frac{2}{t}\,v\Big) = t^2\,\frac{\sin(t)}{t} \quad \Rightarrow \quad \big(t^2\,v\big)' = t\sin(t).$$

Integrating: $t^2\,v = \int t\sin(t)\,dt + c$. The right hand side can be computed integrating by parts,

$$\int t\sin(t)\,dt = -t\cos(t) + \int \cos(t)\,dt, \quad \begin{cases} f = t, & g' = \sin(t), \\ f' = 1, & g = -\cos(t). \end{cases}$$

Review Exam 1.

Example

Find the solution y to the IVP

$$y' = \frac{2}{t}\,y - \frac{\sin(t)}{t}\,y^2, \qquad y(2\pi) = 2\pi, \qquad t > 0.$$

Solution: $\int t\sin(t)\,dt = -t\cos(t) + \int \cos(t)\,dt$. Then,

$$t^2\,v = -t\cos(t) + \sin(t) + c \quad \Rightarrow \quad t^2\,\frac{1}{y} = -t\cos(t) + \sin(t) + c.$$

The initial condition: $4\pi^2\,\dfrac{1}{2\pi} = -2\pi\cos(2\pi) + 0 + c$, so $c = 4\pi$.

$$y = \frac{t^2}{\sin(t) - t\cos(t) + 4\pi} \qquad \triangleleft$$

Review Exam 1.

Example

Find the integrating factor that converts the equation below into an exact equation, where

$$\left(x^3 e^y + \frac{x}{y}\right) y' + (2x^2 e^y + 1) = 0.$$

Solution: We first verify if the equation is not exact.

$$N = \left(x^3 e^y + \frac{x}{y}\right) \quad \Rightarrow \quad \partial_x N = 3x^2 e^y + \frac{1}{y}.$$

$$M = (2x^2 e^y + 1) = 0 \quad \Rightarrow \quad \partial_y M = 2x^2 e^y.$$

So the equation is not exact. We now compute

$$\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \left(3x^2 e^y + \frac{1}{y}\right)}{\left(x^3 e^y + \frac{x}{y}\right)} = \frac{-x^2 e^y - \frac{1}{y}}{x\left(x^2 e^y + \frac{1}{y}\right)} = -\frac{1}{x}.$$

Review Exam 1.

Example

Find the integrating factor that converts the equation below into an exact equation, where

$$\left(x^3 e^y + \frac{x}{y}\right) y' + (2x^2 e^y + 1) = 0.$$

Solution: Recall: $\dfrac{\partial_y M - \partial_x N}{N} = -\dfrac{1}{x}$. Therefore,

$$\frac{\mu'(x)}{\mu(x)} = -\frac{1}{x} \quad \Rightarrow \quad \ln(\mu) = -\ln(x) = \ln\left(\frac{1}{x}\right) \quad \Rightarrow \quad \mu(x) = \frac{1}{x}.$$

So the equation $\left(x^2 e^y + \dfrac{1}{y}\right) y' + \left(2x e^y + \dfrac{1}{x}\right) = 0$ is exact. Indeed,

$$\left.\begin{aligned} \tilde{N} = \left(x^2 e^y + \frac{1}{y}\right) \quad &\Rightarrow \quad \partial_x \tilde{N} = 2x e^y, \\ \tilde{M} = \left(2x e^y + \frac{1}{x}\right) \quad &\Rightarrow \quad \partial_y \tilde{M} = 2x e^y, \end{aligned}\right\} \quad \Rightarrow \quad \partial_x \tilde{N} = \partial_y \tilde{M}.$$

Review Exam 1.

Example

Find the integrating factor that converts the equation below into an exact equation, where

$$\left(x^3 e^y + \frac{x}{y}\right) y' + (2x^2 e^y + 1) = 0.$$

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$$M = (2x^2 e^y + 1) = 0 \quad \Rightarrow \quad \partial_y M = 2x^2 e^y.$$

So the equation is not exact. We now compute

$$\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \left(3x^2 e^y + \frac{1}{y}\right)}{\left(x^3 e^y + \frac{x}{y}\right)} = \frac{-x^2 e^y - \frac{1}{y}}{x\left(x^2 e^y + \frac{1}{y}\right)} = -\frac{1}{x}.$$

Review Exam 1.

Example

Find the integrating factor that converts the equation below into an exact equation, where

$$\left(x^3 e^y + \frac{x}{y}\right) y' + (2x^2 e^y + 1) = 0.$$

Solution: Recall: $\dfrac{\partial_y M - \partial_x N}{N} = -\dfrac{1}{x}$. Therefore,

$$\frac{\mu'(x)}{\mu(x)} = -\frac{1}{x} \quad \Rightarrow \quad \ln(\mu) = -\ln(x) = \ln\left(\frac{1}{x}\right) \quad \Rightarrow \quad \mu(x) = \frac{1}{x}.$$

So the equation $\left(x^2 e^y + \dfrac{1}{y}\right) y' + \left(2x e^y + \dfrac{1}{x}\right) = 0$ is exact. Indeed,

$$\left.\begin{aligned} \tilde{N} &= \left(x^2 e^y + \frac{1}{y}\right) \quad \Rightarrow \quad \partial_x \tilde{N} = 2x e^y, \\ \tilde{M} &= \left(2x e^y + \frac{1}{x}\right) \quad \Rightarrow \quad \partial_y \tilde{M} = 2x e^y, \end{aligned}\right\} \quad \Rightarrow \quad \partial_x \tilde{N} = \partial_y \tilde{M}.$$

Review Exam 1.

Example

Find every solution y of the equation

$$\Big(x^2 e^y + \frac{1}{y}\Big)\, y' + \Big(2x\, e^y + \frac{1}{x}\Big) = 0.$$

Solution: The equation is exact. We need to find the potential function ψ.

$$\partial_y \psi = N, \qquad \partial_x \psi = M.$$

From the first equation we get:

$$\partial_y \psi = x^2 e^y + \frac{1}{y} \quad \Rightarrow \quad \psi = x^2 e^y + \ln(y) + g(x).$$

Introduce the expression for ψ in the equation $\partial_x \psi = M$, that is,

$$2xe^y + g'(x) = \partial_x \psi = M = 2x\, e^y + \frac{1}{x} \quad \Rightarrow \quad g'(x) = \frac{1}{x}.$$

Review Exam 1.

Example

Find every solution y of the equation

$$\Big(x^2 e^y + \frac{1}{y}\Big)\, y' + \Big(2x\, e^y + \frac{1}{x}\Big) = 0.$$

Solution: Recall: $g'(x) = \dfrac{1}{x}$. Therefore $g(x) = \ln(x)$.

The potential function is $\psi = x^2 e^y + \ln(y) + \ln(x)$.

The solution y satisfies $x^2 e^{y(x)} + \ln(y(x)) + \ln(x) = c.$ ◁

Verification: Compute the implicit derivative in the equation above, and you should get the original differential equation.

$$2xe^y + x^2 e^y y' + \frac{1}{y}\, y' + \frac{1}{x} = 0.$$

Review Exam 1.

Example

Find every solution of the initial value problem

$$y' = 4x(y + \sqrt{y}), \qquad y(0) = 4.$$

Solution: The equation is: Not linear.
It is a Bernoulli equation: $y' - 4x\,y = 4x\,y^n$, with $n = 1/2$.
It is separable: $\dfrac{y'}{y + \sqrt{y}} = 4x$.
The equation is not homogeneous. It is not exact.

Although the equation is both separable and Bernoulli, it is not simple to integrate using the separable equation method. Indeed

$$\int \frac{y'}{y + \sqrt{y}}\,dt = \int 4x\,dx + c \quad \Rightarrow \quad \int \frac{dy}{y + \sqrt{y}} = 2x^2 + c.$$

The integral on the left-hand side requires an integration table.

Review Exam 1.

Example

Find every solution of the initial value problem

$$y' = 4x(y + \sqrt{y}), \qquad y(0) = 4.$$

Solution: We find solutions using the Bernoulli method.

$$y' - 4x\,y = 4x\,y^{1/2} \quad \Rightarrow \quad \frac{y'}{y^{1/2}} - 4x\,y^{1/2} = 4x.$$

Change the unknowns: $v = 1/y^{n-1}$, with $n = 1/2$. That is,

$$v = \frac{1}{y^{-1/2}} \quad \Rightarrow \quad v = y^{1/2}, \quad \Rightarrow \quad v' = \frac{1}{2}\,\frac{y'}{y^{1/2}}.$$

$$2v' - 4xv = 4x \quad \Rightarrow \quad v' - 2xv = 2x.$$

The coefficient function is $a(x) = -2x$, so $A(x) = -x^2$, and the integrating factor is $\mu(x) = e^{-x^2}$.

Review Exam 1.

Example

Find every solution of the initial value problem

$$y' = 4x(y + \sqrt{y}), \qquad y(0) = 4.$$

Solution: Recall: $v' - 2xv = 2x$ and $\mu(x) = e^{-x^2}$.

$$e^{-x^2} v' - 2xe^{-x^2} v = 2x\, e^{-x^2} \overset{\text{Verify!}}{\Longrightarrow} \big(e^{-x^2} v\big)' = 2xe^{-x^2}.$$

$$e^{-x^2} v = \int 2xe^{-x^2}\, dx + c \quad \Rightarrow \quad e^{-x^2} v = -e^{-x^2} + c.$$

We conclude that $v = c\, e^{x^2} - 1$. The initial condition for y implies the initial condition for v, that is, $v(x) = \sqrt{y(x)}$ implies $v(0) = 2$.

$$2 = v(0) = c - 1 \quad \Rightarrow \quad c = 3 \quad \Rightarrow \quad v(x) = 3e^{x^2} - 1.$$

We finally find $y = v^2$, that is, $y(x) = (3e^{x^2} - 1)^2$. $\triangleleft$

Review Exam 1.

Example

Find the domain of the function y solution of the IVP

$$y' = -\frac{2t}{y}, \qquad y(1) = 2.$$

Solution: We first need to find the solution y.
The equation is separable.

$$y\, y' = -2t \quad \Rightarrow \quad \int y\, y'\, dt = \int -2t\, dt + c \quad \Rightarrow \quad \frac{y^2}{2} = -t^2 + c$$

$$\frac{4}{2} = \frac{y^2(1)}{2} = -1 + c \quad \Rightarrow \quad c = 3 \quad \Rightarrow \quad y(t) = \sqrt{2(3 - t^2)}.$$

The domain of the solution y is $D = (-\sqrt{3}, \sqrt{3})$.

The points $\pm\sqrt{3}$ do not belong to the domain of y, since y' and the differential equation are not defined there. $\triangleleft$

Review Exam 1.

Example

Find the domain of the function y solution of the IVP

$$y' = -\frac{2t}{y}, \qquad y(t_0) = y_0 > 0.$$

Solution: The solution y is given as above, $\dfrac{y^2}{2} = -t^2 + c$.
The initial condition implies

$$\frac{y_0^2}{2} = \frac{y^2(t_0)}{2} = -t_0^2 + c \;\Rightarrow\; c = \frac{y_0^2}{2} + t_0^2 \;\Rightarrow\; \frac{y^2}{2} = -t^2 + t_0^2 + \frac{y_0^2}{2}.$$

The solution to the IVP is $y(t) = \sqrt{2(t_0^2 - t^2) + y_0^2}$.

The domain of the solution depends on the initial condition t_0, y_0:

$$D = \Big(-\sqrt{t_0^2 + \frac{y_0^2}{2}}, +\sqrt{t_0^2 + \frac{y_0^2}{2}}\Big).$$

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Review Exam 1.

Example

Find every solution y to the equation $y' = -\dfrac{2x + 3y}{3x + 4y}$.

Solution: The equation is not linear, not Bernoulli, not separable.
It is homogeneous. (Multiply numerator and denominator on the right hand side by $(1/x)$.)
Is it exact? $(3x + 4y)\,y' + (2x + 3y) = 0$ implies $\partial_x N = 3 = \partial_y M$.
So the equation is exact.

We choose here the exact equation method. (Finding the potential function is sometimes simpler that solving homogeneous Eqs.)

We need to find the potential function ψ:

$$\partial_y \psi = N \quad \Rightarrow \quad \psi = 3xy + 2y^2 + g(x).$$

$$\partial_x \psi = M \quad \Rightarrow \quad 3y + g'(x) = 2x + 3y \quad \Rightarrow \quad g(x) = x^2.$$

We conclude: $\psi(x, y) = 3xy + 2y^2 + x^2$, and $\psi(x, y(x)) = c$. ◁

Review Exam 1.

Example

Find every solution y to the equation $y' = -\dfrac{2x+3y}{3x+4y}$.

Solution: If we solve the problem using that the equation is homogeneous, it is more complicated than the previous calculation. We just start the calculation to see the difficulty:

$$y' = -\frac{(2x+3y)}{(3x+4y)}\,\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = -\frac{2+3\left(\frac{y}{x}\right)}{3+4\left(\frac{y}{x}\right)}.$$

The change $v = y/x$ implies $y = xv$ and $y' = v + x\,v'$. Hence

$$v + x\,v' = \frac{2+3v}{3+4v} \quad \Rightarrow \quad x\,v' = \frac{2+3v}{3+4v} - v = \frac{2+3v-3v+4v^2}{3+4v}.$$

We conclude that v satisfies $\dfrac{3+4v}{2-4v^2}\,v' = \dfrac{1}{x}$.

Review Exam 1.

Example

Find every solution y to the equation $y' = -\dfrac{2x+3y}{3x+4y}$.

Solution: Recall: $\dfrac{3+4v}{2-4v^2}\,v' = \dfrac{1}{x}$.

This equation is complicated to integrate.

$$\int \frac{3\,v'}{2-4v^2}\,dx + \int \frac{4v\,v'}{2-4v^2}\,dx = \int \frac{1}{x}\,dx + c = \ln(x) + c.$$

The usual substitution $u = v(x)$ implies $du = v'\,dx$, so

$$\int \frac{3\,du}{2-4u^2} + \int \frac{4u\,du}{2-4u^2} = \ln(x) + c.$$

The first integral on the left-hand side requires integration tables.

This is why the exact method is simpler to use in this case. $\triangleleft$

Review Exam 1.

Example

Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = 2\,y + 3 \qquad y(0) = 1.$$

Solution: First notice that the equation is linear. So it is simple to find the solution following Section 1.1,

$$e^{-2t}(y' - 2y) = 3\,e^{-2t} \quad \Rightarrow \quad \big(e^{-2t}\,y\big) = -\frac{3}{2}\,e^{-2t} + c,$$

$$y(t) = c\,e^{2t} - \frac{3}{2}.$$

The initial condition implies,

$$1 = y(0) = c - \frac{3}{2} \quad \Rightarrow \quad y(t) = \frac{5}{2}\,e^{2t} - \frac{3}{2}.$$

In the next slide we use Picard-Lindelöf's idea.

Review Exam 1.

Example

Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = 2\,y + 3 \qquad y(0) = 1.$$

Solution: We first transform the differential equation into an integral equation.

$$\int_0^t y'(s)\,ds = \int_0^t (2\,y(s) + 3)\,ds$$

$$y(t) - y(0) = \int_0^t (2\,y(s) + 3)\,ds.$$

Using the initial condition, $y(0) = 1$,

$$y(t) = 1 + \int_0^t (2\,y(s) + 3)\,ds.$$

This is the integral equation.

Review Exam 1.

Example

Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = 2\,y + 3 \qquad y(0) = 1.$$

Solution: Integral equation: $y(t) = 1 + \int_0^t (2\,y(s) + 3)\,ds$.

We now define the sequence of approximate solutions:

$$y_0 = y(0) = 1, \quad y_{n+1}(t) = 1 + \int_0^t (2\,y_n(s) + 3)\,ds, \quad n \geqslant 0.$$

We now compute the first elements in the sequence.

$$n = 0, \quad y_1(t) = 1 + \int_0^t (2\,y_0(s) + 3)\,ds = 1 + \int_0^t 5\,ds = 1 + 5t.$$

So $y_0 = 1$, and $y_1 = 1 + 5t$.

Review Exam 1.

Example

Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = 2\,y + 3 \qquad y(0) = 1.$$

Solution: Integral equation: $y(t) = 1 + \int_0^t (2\,y(s) + 3)\,ds$.

And $y_0 = 1$, and $y_1 = 1 + 5t$. Let's compute y_2,

$$y_2 = 1 + \int_0^t (2\,y_1(s) + 3)\,ds = 1 + \int_0^t \big(2(1 + 5s) + 3\big)\,ds$$

$$y_2 = 1 + \int_0^t \big(5 + 10s\big)\,ds = 1 + 5t + 5t^2.$$

So we've got $y_2(t) = 1 + 5t + 5t^2$.

Review Exam 1.

Example

Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = 2\,y + 3 \qquad y(0) = 1.$$

Solution: Integral equation: $y(t) = 1 + \int_0^t \big(2\,y(s) + 3\big)\,ds$.

And $y_0 = 1$, and $y_1 = 1 + 5t$, and $y_2 = 1 + 5t + 5t^2$. Now y_3,

$$y_3 = 1 + \int_0^t \big(2\,y_2(s) + 3\big)\,ds = 1 + \int_0^t \Big(2(1 + 5s + 5s^2) + 3\Big)\,ds$$

$$y_3 = 1 + \int_0^t \big(5 + 10s + 10s^2\big)\,ds = 1 + 5t + 5t^2 + \frac{10}{3}\,t^3.$$

So we've got $y_3(t) = 1 + 5t + 5t^2 + \frac{10}{3}\,t^3$.

Rewrite: $y_3(t) = 1 + \frac{5}{2}\Big[(2t) + \frac{(2t)^2}{2} + \frac{(2t)^3}{3!}\Big]$.

Review Exam 1.

Example

Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = 2\,y + 3, \qquad y(0) = 1.$$

Solution: $y_3(t) = 1 + \frac{5}{2}\Big[(2t) + \frac{(2t)^2}{2} + \frac{(2t)^3}{3!}\Big]$.

By computing few more terms one finds

$$y_n(t) = 1 + \frac{5}{2}\sum_{k=1}^{n} \frac{(2t)^k}{k!}$$

Hence the limit $n \to \infty$ is given by

$$y(t) = \lim_{n\to\infty} y_n(t) = 1 + \frac{5}{2}\sum_{k=1}^{\infty} \frac{(2t)^k}{k!} = 1 + \frac{5}{2}\big(e^{2t} - 1\big)$$

since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. We conclude, $y(t) = \frac{5}{2}e^{2t} - \frac{3}{2}$. $\triangleleft$