Review for Exam 1.

- Exam in webwork, in CC-415, Computer Center.
- 5 grading attempts per problem.
- 5 to 6 problems, 60 minutes.
- Problems similar to webwork problems.
- Integration table provided in the handout.
- No notes, no books, no calculators, no phones.
- MLC Exam Review for MTH 235, today, 7:30pm, ANH-1281.
- MTH 340 Exam 1 covers:
  - Linear equations (1.1), (1.2).
  - Bernoulli equation (1.2).
  - Separable equations (1.3).
  - Euler homogeneous equations (1.3).
  - Exact equations (1.4).
  - Exact equations with integrating factors (1.4).
  - Applications (1.5).
  - Picard-Lindelöf iteration (1.6).

Exam overview

Remark:

- Exam problems will be: Solve this equation. We don’t tell you if the equation is linear, separable, etc. You must find that out.
- If you know what type of equation is, then the equation is simple to solve.
- The difficult part in Exam 1 is to know what type of equation is the one you have to solve.
Exam overview

Advice: In order to find out what type of equation is the one you have to solve, check from simple types to the more difficult types:

1. Linear equations.
   (Just by looking at it: \( y' + a(t) y = b(t) \).)

2. Bernoulli equations.
   (Just by looking at it: \( y' + a(t) y = b(t) y^n \).)

   (Few manipulations: \( h(y) y' = g(t) \).)

4. Euler homogeneous equations.
   (Several manipulations: \( y' = F(y/t) \).)

5. Exact equations.
   (Check one equation: \( N y' + M = 0 \), and \( \partial_t N = \partial_y M \).)

6. Exact equation with integrating factor.
   (Could be very complicated to check.)

Review Exam 1.

Example
Find every solution \( y \) to the equation \((t^2 + y^2)(t + y y') + 2 = 0\).

Solution: Rewrite the equation in a more standard way:

\[
(t^2 + y^2) y y' + (t^2 + y^2) t + 2 = 0 \iff y' = -\frac{(t^2 + y^2) t + 2}{(t^2 + y^2) y}.
\]

Not linear. Not Bernoulli. Not Separable. Not Euler homogeneous. So the equation must be exact or exact with integrating factor.

\[
N = t^2 y + y^3 \quad \Rightarrow \quad \partial_t N = 2ty.
\]
\[
M = t^3 + ty^2 + 2 \quad \Rightarrow \quad \partial_y M = 2ty.
\]

The equation is exact: \( \partial_t N = \partial_y M \).
Example

Find every solution $y$ to the equation $(t^2 + y^2)(t + yy') + 2 = 0$.

Solution: $\partial_t N = \partial_y M, \quad [(t^2 + y^2)y] y' + [(t^2 + y^2)t + 2] = 0$.

There exists a potential function $\psi$ such that

$$\partial_y \psi = N, \quad \partial_t \psi = M.$$

$$\partial_y \psi = t^2y + y^3 \quad \Rightarrow \quad \psi = t^2y^2/2 + y^4/4 + g(t).$$

$$ty^2 + g'(t) = \partial_t \psi = M = t^3 + ty^2 + 2.$$

$$g'(t) = t^3 + 2 \quad \Rightarrow \quad g(t) = \frac{t^4}{4} + 2t.$$

$$\psi(t, y) = \frac{1}{2} t^2y^2 + \frac{y^4}{4} + \frac{t^4}{4} + 2t, \quad \psi(t, y(t)) = c. \quad \triangleleft$$

Example

Find the explicit solution $y$ to the IVP

$$y' = \frac{t(t^2 + e^t)}{4y^3}, \quad y(0) = -\sqrt{2}.$$

Solution: Not linear. Bernoulli with $n = -3$. Numerator depends only on $t$, denominator depends only on $y$: Separable.

$$4y^3y' = t^3 + te^t \quad \Rightarrow \quad \int 4y^3y' \, dt = \int (t^3 + te^t) \, dt + c$$

The usual substitution: $u = y(t)$ implies $du = y'(t) \, dt$,

$$\int 4u^3 \, du = \int (t^3 + te^t) \, dt + c \quad \Rightarrow \quad u^4 = \frac{t^4}{4} + \int te^t \, dt + c.$$
Example

Find the explicit solution $y$ to the IVP

$$y' = \frac{t(t^2 + e^t)}{4y^3}, \quad y(0) = -\sqrt{2}.$$ 

Solution: Recall: $u^4 = \frac{t^4}{4} + \int te^t \, dt + c$. Integration by parts:

$$f = t, \quad g' = e^t, \quad f' = 1, \quad g = e^t,$$

$$\Rightarrow \int te^t \, dt = te^t - \int e^t \, dt = (t - 1)e^t.$$

We obtain: $y^4(t) = \frac{t^4}{4} + (t - 1)e^t + c$. The initial condition:

$$(-\sqrt{2})^4 = 0 + (0 - 1) + c \quad \Rightarrow \quad 4 = -1 + c \quad \Rightarrow \quad c = 5.$$

We conclude: $y^4(t) = \frac{t^4}{4} + (t - 1)e^t + 5$. Implicit form.

The explicit form of the solution is one of:

$$y(t) = \pm \left[ \frac{t^4}{4} + (t - 1)e^t + 5 \right]^{1/4}.$$

The initial condition implies $y(0) = -\sqrt{2} < 0$.

We conclude that the unique solution to the IVP is

$$y(t) = -\left[ \frac{t^4}{4} + (t - 1)e^t + 5 \right]^{1/4}. \quad \triangleleft$$
Example

Find every solution $y$ of the equation $y' = \frac{3y^2 - t^2}{2ty}$.

Solution: Not linear. Bernoulli $n = -1$: $y' = \frac{3y}{2t} - \frac{t}{2y}$.

Not separable. Every term on the right hand side is of the form $t^n y^m$ with $n + m = 2$. Euler homogeneous.

$$y' = \frac{3y^2 - t^2}{2ty} \frac{\frac{1}{t^2}}{\frac{1}{t^2}} \quad \Rightarrow \quad y' = \frac{3\left(\frac{y}{t}\right)^2 - 1}{2\left(\frac{y}{t}\right)}.$$ 

We introduce the change of unknown:

$v = \frac{y}{t} \quad \Rightarrow \quad y = t v \quad \Rightarrow \quad y' = v + t v'$.

$$v + t v' = \frac{3v^2 - 1}{2v} \quad \Rightarrow \quad t v' = \frac{3v^2 - 1}{2v} - v = \frac{3v^2 - 1 - 2v^2}{2v}$$

$$t v' = \frac{v^2 - 1}{2v} \quad \Rightarrow \quad \frac{2v}{v^2 - 1} v' = \frac{1}{t}.$$ 

This is a separable equation for $v$: $\int \frac{2v}{v^2 - 1} v' \, dt = \int \frac{1}{t} \, dt + c.$
Review Exam 1.

Example

Find every solution $y$ of the equation $y' = \frac{3y^2 - t^2}{2ty}$.

Solution: $\int \frac{2v}{v^2 - 1} v' \, dt = \int \frac{1}{t} \, dt + c$.

The substitution $u = v^2 - 1$ implies $du = 2v \, v' \, dt$. So,

$\int \frac{du}{u} = \int \frac{1}{t} \, dt + c \Rightarrow \ln(|u|) = \ln(|t|) + c \Rightarrow |u| = c_1 |t|.$

where $c_1 = e^c$. Substitute back: $|v^2 - 1| = c_1 |t|$. Finally, $v = y/t$,

$\left| \frac{y^2}{t^2} - 1 \right| = c_1 |t| \Rightarrow |y^2 - t^2| = c_1 |t|^3$. \hfill \small\checkmark$

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Example

A water tank initially has $V_0 = 100$ liters of water with $Q_0$ grams of salt. At $t_0 = 0$ fresh water is poured into the tank. The salt in the tank is always well mixed. Find the rates $r_i$ and $r_o$ such that:

(a) The tank water volume is constant.

(b) The time to reduce the salt in the tank to one percent of the initial value is $t_1 = 25$ min.

Solution:

Part (a): Water volume constant implies $r_i = r_o = r$. Then $V'(t) = 0$, so $V(t) = V_0$.

Part (b): First find the salt in the tank $Q(t)$: $\frac{dQ}{dt} = r_i q_i - r_o q_o(t)$.

Incoming fresh water: $q_i = 0$. Mixing: $q_o(t) = \frac{Q(t)}{V(t)}$.

$\frac{dQ}{dt} = -\frac{r}{V_0} Q(t) \Rightarrow Q(t) = Q_0 e^{-rt/V_0}$. \hfill \small\checkmark
Review Exam 1.

Example

A water tank initially has $V_0 = 100$ liters of water with $Q_0$ grams of salt. At $t_0 = 0$ fresh water is poured into the tank. The salt in the tank is always well mixed. Find the rates $r_i$ and $r_o$ such that:

(a) The tank water volume is constant.

(b) The time to reduce the salt in the tank to one percent of the initial value is $t_1 = 25$ min.

Solution: Recall: $Q(t) = Q_0 e^{-rt/V_0}$. Condition for $r$:

$$Q(t_1) = \frac{Q_0}{100} \Rightarrow Q_0 e^{-rt_1/V_0} = \frac{Q_0}{100} \Rightarrow -\frac{rt_1}{V_0} = \ln\left(\frac{1}{100}\right).$$

$$\frac{rt_1}{V_0} = \ln(100) \Rightarrow r = \frac{V_0}{t_1} \ln(100) \Rightarrow r = 4 \ln(100).$$

\[\hat{\triangleright}\]

Review Exam 1.

Example

Find the solution $y$ to the IVP

$$y' = \frac{2}{t} y - \frac{\sin(t)}{t} y^2, \quad y(2\pi) = 2\pi, \quad t > 0.$$


$$\frac{y'}{y^2} - \frac{2}{t} \frac{1}{y} = -\frac{\sin(t)}{t}, \quad v = \frac{1}{y} \Rightarrow v' = -\frac{y'}{y^2}.$$

$$-v' - \frac{2}{t} v = -\frac{\sin(t)}{t} \Rightarrow v' + \frac{2}{t} v = \frac{\sin(t)}{t}.$$

We solve the linear equation with the integrating factor method.

$$A(t) = \int \frac{2}{t} \, dt = 2 \ln(t) = \ln(t^2) \Rightarrow \mu(t) = t^2.$$
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Example
Find the solution \( y \) to the IVP

\[
y' = \frac{2}{t} y - \frac{\sin(t)}{t} y^2, \quad y(2\pi) = 2\pi, \quad t > 0.
\]

Solution: Recall: \( \mu(t) = t^2 \). Then,

\[
t^2 \left( v' + \frac{2}{t} v \right) = t^2 \frac{\sin(t)}{t} \quad \Rightarrow \quad (t^2 v)' = t \sin(t).
\]

Integrating: \( t^2 v = \int t \sin(t) \, dt + c \). The right hand side can be computed integrating by parts,

\[
\int t \sin(t) \, dt = -t \cos(t) + \int \cos(t) \, dt, \quad \left\{ \begin{array}{l}
 f = t, \quad g' = \sin(t), \\
 f' = 1, \quad g = -\cos(t).
\end{array} \right.
\]

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Example
Find the solution \( y \) to the IVP

\[
y' = \frac{2}{t} y - \frac{\sin(t)}{t} y^2, \quad y(2\pi) = 2\pi, \quad t > 0.
\]

Solution: \( \int t \sin(t) \, dt = -t \cos(t) + \int \cos(t) \, dt \). Then,

\[
t^2 v = -t \cos(t) + \sin(t) + c \quad \Rightarrow \quad t^2 \frac{1}{y} = -t \cos(t) + \sin(t) + c.
\]

The initial condition: \( 4\pi^2 \frac{1}{2\pi} = -2\pi \cos(2\pi) + 0 + c \), so \( c = 4\pi \).

\[
y = \frac{t^2}{\sin(t) - t \cos(t) + 4\pi} \quad \triangleleft
\]
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Example
Find the integrating factor that converts the equation below into an exact equation, where
\[
\left( x^3 e^y + \frac{x}{y} \right) y' + (2x^2 e^y + 1) = 0.
\]

Solution: We first verify if the equation is not exact.
\[
N = \left( x^3 e^y + \frac{x}{y} \right) \quad \Rightarrow \quad \partial_x N = 3x^2 e^y + \frac{1}{y}.
\]
\[
M = (2x^2 e^y + 1) = 0 \quad \Rightarrow \quad \partial_y M = 2x^2 e^y.
\]
So the equation is not exact. We now compute
\[
\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \left( 3x^2 e^y + \frac{1}{y} \right)}{\left( x^3 e^y + \frac{x}{y} \right)} = \frac{-x^2 e^y - \frac{1}{y}}{x \left( x^2 e^y + \frac{1}{y} \right)} = -\frac{1}{x}.
\]

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Example
Find the integrating factor that converts the equation below into an exact equation, where
\[
\left( x^3 e^y + \frac{x}{y} \right) y' + (2x^2 e^y + 1) = 0.
\]

Solution: Recall: \( \frac{\partial_y M - \partial_x N}{N} = -\frac{1}{x} \). Therefore,
\[
\frac{\mu'(x)}{\mu(x)} = -\frac{1}{x} \quad \Rightarrow \quad \ln(\mu) = -\ln(x) = \ln \left( \frac{1}{x} \right) \quad \Rightarrow \quad \mu(x) = \frac{1}{x}.
\]
So the equation \( \left( x^2 e^y + \frac{1}{y} \right) y' + \left( 2xe^y + \frac{1}{x} \right) = 0 \) is exact. Indeed,
\[
\tilde{N} = \left( x^2 e^y + \frac{1}{y} \right) \quad \Rightarrow \quad \partial_x \tilde{N} = 2xe^y,
\]
\[
\tilde{M} = \left( 2xe^y + \frac{1}{x} \right) \quad \Rightarrow \quad \partial_y \tilde{M} = 2xe^y,
\]
\[
\Rightarrow \quad \partial_x \tilde{N} = \partial_y \tilde{M}.
\]
Example
Find the integrating factor that converts the equation below into an exact equation, where
\[(x^3 e^y + \frac{x}{y}) y' + (2x^2 e^y + 1) = 0.\]

Solution: We first verify if the equation is not exact.
\[N = \left(x^3 e^y + \frac{x}{y}\right) \Rightarrow \partial_x N = 3x^2 e^y + \frac{1}{y}.\]
\[M = (2x^2 e^y + 1) = 0 \Rightarrow \partial_y M = 2x^2 e^y.\]
So the equation is not exact. We now compute
\[
\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \left(3x^2 e^y + \frac{1}{y}\right)}{(x^3 e^y + \frac{x}{y})} = \frac{-x^2 e^y - \frac{1}{y}}{x \left(x^2 e^y + \frac{1}{y}\right)} = -\frac{1}{x}.
\]
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Example
Find every solution $y$ of the equation
\[(x^2e^y + \frac{1}{y}) y' + (2x e^y + \frac{1}{x}) = 0.\]

Solution: The equation is exact. We need to find the potential function $\psi$.
$$\partial_y \psi = N, \quad \partial_x \psi = M.$$ 
From the first equation we get:
$$\partial_y \psi = x^2e^y + \frac{1}{y} \quad \Rightarrow \quad \psi = x^2e^y + \ln(y) + g(x).$$
Introduce the expression for $\psi$ in the equation $\partial_x \psi = M$, that is,
$$2xe^y + g'(x) = \partial_x \psi = M = 2xe^y + \frac{1}{x} \quad \Rightarrow \quad g'(x) = \frac{1}{x}.$$ 

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Example
Find every solution $y$ of the equation
\[(x^2e^y + \frac{1}{y}) y' + (2x e^y + \frac{1}{x}) = 0.\]

Solution: Recall: $g'(x) = \frac{1}{x}$. Therefore $g(x) = \ln(x)$.
The potential function is $\psi = x^2e^y + \ln(y) + \ln(x)$.
The solution $y$ satisfies $x^2e^{y(x)} + \ln(y(x)) + \ln(x) = c$. ◯

Verification: Compute the implicit derivative in the equation above, and you should get the original differential equation.
$$2xe^y + x^2e^y y' + \frac{1}{y} y' + \frac{1}{x} = 0.$$
Example
Find every solution of the initial value problem
\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]

Solution: The equation is: Not linear.
It is a Bernoulli equation: \( y' - 4xy = 4xy^n, \) with \( n = 1/2. \)
It is separable: \( \frac{y'}{y + \sqrt{y}} = 4x. \)
The equation is not homogeneous. It is not exact.
Although the equation is both separable and Bernoulli, it is not simple to integrate using the separable equation method. Indeed
\[
\int \frac{y'}{y + \sqrt{y}} \, dt = \int 4x \, dx + c \quad \Rightarrow \quad \int \frac{dy}{y + \sqrt{y}} = 2x^2 + c.
\]
The integral on the left-hand side requires an integration table.

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Example
Find every solution of the initial value problem
\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]

Solution: We find solutions using the Bernoulli method.
\[ y' - 4xy = 4xy^{1/2} \quad \Rightarrow \quad \frac{y'}{y^{1/2}} - 4x y^{1/2} = 4x. \]
Change the unknowns: \( v = 1/y^{n-1}, \) with \( n = 1/2. \) That is,
\[ v = \frac{1}{y^{-1/2}} \quad \Rightarrow \quad v = y^{1/2}, \quad \Rightarrow \quad v' = \frac{1}{2} \frac{y'}{y^{1/2}}. \]
\[ 2v' - 4xv = 4x \quad \Rightarrow \quad v' - 2xv = 2x. \]
The coefficient function is \( a(x) = -2x, \) so \( A(x) = -x^2, \) and the integrating factor is \( \mu(x) = e^{-x^2}. \)
Example
Find every solution of the initial value problem

\[ y' = 4x(y + \sqrt{y}), \quad y(0) = 4. \]

Solution: Recall: \( v' - 2xv = 2x \) and \( \mu(x) = e^{-x^2} \).

\[
e^{-x^2} v' - 2xe^{-x^2} v = 2x e^{-x^2} \quad \Rightarrow \quad (e^{-x^2} v)' = 2xe^{-x^2}.
\]

\[
e^{-x^2} v = \int 2xe^{-x^2} \, dx + c \quad \Rightarrow \quad e^{-x^2} v = -e^{-x^2} + c.
\]

We conclude that \( v = c e^{x^2} - 1 \). The initial condition for \( y \) implies the initial condition for \( v \), that is, \( v(x) = \sqrt{y(x)} \) implies \( v(0) = 2 \).

\[
2 = v(0) = c - 1 \quad \Rightarrow \quad c = 3 \quad \Rightarrow \quad v(x) = 3e^{x^2} - 1.
\]

We finally find \( y = v^2 \), that is, \( y(x) = (3e^{x^2} - 1)^2 \).

\[
\square
\]

Example
Find the domain of the function \( y \) solution of the IVP

\[ y' = -\frac{2t}{y}, \quad y(1) = 2. \]

Solution: We first need to find the solution \( y \).
The equation is separable.

\[
y y' = -2t \quad \Rightarrow \quad \int y y' \, dt = \int -2t \, dt + c \quad \Rightarrow \quad \frac{y^2}{2} = -t^2 + c
\]

\[
\frac{4}{2} = \frac{y^2(1)}{2} = -1 + c \quad \Rightarrow \quad c = 3 \quad \Rightarrow \quad y(t) = \sqrt{2(3 - t^2)}.
\]

The domain of the solution \( y \) is \( D = (-\sqrt{3}, \sqrt{3}) \).
The points \( \pm \sqrt{3} \) do not belong to the domain of \( y \), since \( y' \) and the differential equation are not defined there.

\[
\square
\]
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Example
Find the domain of the function $y$ solution of the IVP

$$y' = -\frac{2t}{y}, \quad y(t_0) = y_0 > 0.$$ 

Solution: The solution $y$ is given as above, $\frac{y^2}{2} = -t^2 + c$.
The initial condition implies

$$\frac{y_0^2}{2} = \frac{y_0^2(t_0)}{2} = -t_0^2 + c \Rightarrow c = \frac{y_0^2}{2} + t_0^2 \Rightarrow \frac{y^2}{2} = -t^2 + t_0^2 + \frac{y_0^2}{2}.$$ 

The solution to the IVP is $y(t) = \sqrt{2(t_0^2 - t^2) + y_0^2}$. 
The domain of the solution depends on the initial condition $t_0, y_0$:

$$D = \left(-\sqrt{t_0^2 + \frac{y_0^2}{2}}, +\sqrt{t_0^2 + \frac{y_0^2}{2}}\right).$$ 

\[\triangleright\]

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Example
Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: The equation is not linear, not Bernoulli, not separable. It is homogeneous. (Multiply numerator and denominator on the right hand side by $(1/x)$.)

Is it exact? $(3x + 4y) y' + (2x + 3y) = 0$ implies $\partial_x N = 3 = \partial_y M$.

So the equation is exact.

We choose here the exact equation method. (Finding the potential function is sometimes simpler that solving homogeneous Eqs.)

We need to find the potential function $\psi$:

$$\partial_y \psi = N \Rightarrow \psi = 3xy + 2y^2 + g(x).$$

$$\partial_x \psi = M \Rightarrow 3y + g'(x) = 2x + 3y \Rightarrow g(x) = x^2.$$ 

We conclude: $\psi(x, y) = 3xy + 2y^2 + x^2$, and $\psi(x, y(x)) = c$. \[\triangleright\]
Example

Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: If we solve the problem using that the equation is homogeneous, it is more complicated than the previous calculation. We just start the calculation to see the difficulty:

$$y' = -\frac{(2x + 3y)}{(3x + 4y)} \cdot \frac{1}{x} = -\frac{2 + 3\left(\frac{y}{x}\right)}{3 + 4\left(\frac{y}{x}\right)}.$$ 

The change $v = y/x$ implies $y = xv$ and $y' = v + xv'$. Hence

$$v + xv' = \frac{2 + 3v}{3 + 4v} \Rightarrow xv' = \frac{2 + 3v}{3 + 4v} - v = \frac{2 + 3v - 3v + 4v^2}{3 + 4v} \Rightarrow xv' = \frac{3 + 4v}{2 - 4v^2}.$$ 

We conclude that $v$ satisfies $\frac{3 + 4v}{2 - 4v^2} v' = \frac{1}{x}$.

Example

Find every solution $y$ to the equation $y' = -\frac{2x + 3y}{3x + 4y}$.

Solution: Recall: $\frac{3 + 4v}{2 - 4v^2} v' = \frac{1}{x}$.

This equation is complicated to integrate.

$$\int \frac{3v'}{2 - 4v^2} \, dx + \int \frac{4v v'}{2 - 4v^2} \, dx = \int \frac{1}{x} \, dx + c = \ln(x) + c.$$ 

The usual substitution $u = v(x)$ implies $du = v' \, dx$, so

$$\int \frac{3 \, du}{2 - 4u^2} + \int \frac{4u \, du}{2 - 4u^2} = \ln(x) + c.$$ 

The first integral on the left-hand side requires integration tables. This is why the exact method is simpler to use in this case. ☐
Example
Use the proof of Picard-Lindelöf’s Theorem to find the solution to
\[ y' = 2y + 3 \quad y(0) = 1. \]

Solution: First notice that the equation is linear. So it is simple to find the solution following Section 1.1,
\[ e^{-2t}(y' - 2y) = 3e^{-2t} \quad \Rightarrow \quad (e^{-2t}y) = -\frac{3}{2}e^{-2t} + c, \]
\[ y(t) = ce^{2t} - \frac{3}{2}. \]
The initial condition implies,
\[ 1 = y(0) = c - \frac{3}{2} \quad \Rightarrow \quad y(t) = \frac{5}{2}e^{2t} - \frac{3}{2}. \]
In the next slide we use Picard-Lindelöf’s idea.
Example
Use the proof of Picard-Lindelöf’s Theorem to find the solution to
\[ y' = 2y + 3 \quad y(0) = 1. \]

Solution: Integral equation: \( y(t) = 1 + \int_0^t (2y(s) + 3) \, ds. \)
We now define the sequence of approximate solutions:
\[ y_0 = y(0) = 1, \quad y_{n+1}(t) = 1 + \int_0^t (2y_n(s) + 3) \, ds, \quad n \geq 0. \]

We now compute the first elements in the sequence.
\[ n = 0, \quad y_1(t) = 1 + \int_0^t (2y_0(s) + 3) \, ds = 1 + \int_0^t 5 \, ds = 1 + 5t. \]
So \( y_0 = 1 \), and \( y_1 = 1 + 5t \).

Let’s compute \( y_2 \),
\[ y_2 = 1 + \int_0^t (2y_1(s) + 3) \, ds = 1 + \int_0^t (2(1 + 5s) + 3) \, ds \]
\[ y_2 = 1 + \int_0^t (5 + 10s) \, ds = 1 + 5t + 5t^2. \]
So we’ve got \( y_2(t) = 1 + 5t + 5t^2. \)
Example
Use the proof of Picard-Lindelöf’s Theorem to find the solution to
\[ y' = 2y + 3, \quad y(0) = 1. \]

Solution: Integral equation: \[ y(t) = 1 + \int_0^t (2y(s) + 3) \, ds. \]
And \( y_0 = 1 \), and \( y_1 = 1 + 5t \), and \( y_2 = 1 + 5t + 5t^2 \). Now \( y_3 \),
\[ y_3 = 1 + \int_0^t (2y_2(s) + 3) \, ds = 1 + \int_0^t (2(1 + 5s + 5s^2) + 3) \, ds \]
\[ y_3 = 1 + \int_0^t (5 + 10s + 10s^2) \, ds = 1 + 5t + 5t^2 + \frac{10}{3} t^3. \]
So we’ve got \( y_3(t) = 1 + 5t + 5t^2 + \frac{10}{3} t^3. \)
Rewrite: \[ y_3(t) = 1 + \frac{5}{2} \left[ (2t) + \frac{(2t)^2}{2} + \frac{(2t)^3}{3!} \right]. \]

By computing few more terms one finds
\[ y_n(t) = 1 + \frac{5}{2} \sum_{k=1}^{n} \frac{(2t)^k}{k!} \]
Hence the limit \( n \to \infty \) is given by
\[ y(t) = \lim_{n \to \infty} y_n(t) = 1 + \frac{5}{2} \sum_{k=1}^{\infty} \frac{(2t)^k}{k!} = 1 + \frac{5}{2} \left( e^{2t} - 1 \right) \]
since \( e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \). We conclude, \[ y(t) = \frac{5}{2} e^{2t} - \frac{3}{2}. \] \( \triangleq \)